Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 567–570
S 0019-2082

A NOTE ON THE FINITE IMAGES OF PROFINITE GROUPS

JOHN S. WILSON

Dedicated to the memory of Reinhold Baer

In [2] and a number of other papers, Reinhold Baer studied the deduction of properties of infinite groups from properties of their finite images. Here we consider one aspect of the inverse problem. Associated naturally with each profinite group G there are two families of finite groups: the family $\mathcal{F}_{\mathrm{o}}(G)$ of quotient groups of G modulo open normal subgroups and the family $\mathcal{F}_{\mathrm{a}}(G)$ of quotients modulo arbitrary subgroups of finite index. The two classes coincide if and only if $\mathcal{F}_{\mathrm{a}}(G)$ is countable (see [7]). While the relationship between G and $\mathcal{F}_{\mathrm{o}}(G)$ is fairly clear, the relationship of $\mathcal{F}_{\mathrm{a}}(G)$ to G and to $\mathcal{F}_{\mathrm{o}}(G)$ is not well understood.

If each group in $\mathcal{F}_{\mathrm{o}}(G)$ satisfies an abstract group law v, then clearly each group in $\mathcal{F}_{\mathrm{a}}(G)$ satisfies v. It was proved by Anderson [1, Theorem 1] that if for some set of primes π each group in $\mathcal{F}_{\mathrm{o}}(G)$ is a π-group then each group in $\mathcal{F}_{\mathrm{a}}(G)$ is a π-group. Moreover if $\mathcal{F}_{\mathrm{o}}(G)$ consists of soluble (resp. nilpotent) groups, then so does $\mathcal{F}_{\mathrm{a}}(G)$; see [1, Proposition 6] for the soluble case. The proofs of these two latter statements are indirect: they rely on Lemma 1 below and the fact that prosolubility (resp. pronilpotency) can be characterized in terms of the existence (resp. existence and normality) of p-complements. Beyond these results, little seems to be known about the influence of $\mathcal{F}_{\mathrm{o}}(G)$ on $\mathcal{F}_{\mathrm{a}}(G)$.

It has been conjectured by Serre that if H is a finitely generated profinite group then each subgroup of finite index is open. If true, this conjecture would imply that each group in $\mathcal{F}_{\mathrm{a}}(G)$ is an epimorphic image of a subgroup of a group in $\mathcal{F}_{\mathrm{o}}(G)$. Indeed, let K be a normal subgroup of finite index in G; let T be a transversal to K in G and let H be the closed subgroup of G generated by T; thus H is a finitely generated profinite group having $H \cap K$ as a normal subgroup of finite index. The conjecture would imply that $H \cap K$ is open in H, so that $H \cap K \geqslant H \cap N$ for some open normal subgroup N of

Received June 4, 2002.
2000 *Mathematics Subject Classification.* 20E15, 20E18.

G, and one would conclude that G/K is an image of the subgroup HN/N of G/N.

Unfortunately Serre's conjecture remains unproven, but Segal [6] has made some striking progress by proving that subgroups of finite index in finitely generated prosoluble groups are open. The argument above gives the following easy consequence of Segal's result.

PROPOSITION. *Let G be a prosoluble group. Then every group in $\mathcal{F}_{\mathrm{a}}(G)$ is an epimorphic image of a subgroup of a group in $\mathcal{F}_{\mathrm{o}}(G)$.*

Here we shall prove the following result.

THEOREM. *Let G be a profinite group. If the non-abelian composition factors of the groups in $\mathcal{F}_{\mathrm{o}}(G)$ fall into finitely many isomorphism classes then so do the non-abelian composition factors of the groups in $\mathcal{F}_{\mathrm{a}}(G)$.*

The proof is indirect: it is hard to use directly the fact that a given simple group does or does not arise as a composition factor of a quotient group. It seems possible that each non-abelian composition factor of a group in $\mathcal{F}_{\mathrm{a}}(G)$ must also be a composition factor of some group in $\mathcal{F}_{\mathrm{o}}(G)$. It is an easy consequence of the main result of Wilson and Saxl [5] that this conclusion holds if G happens to have a finite series of closed normal subgroups each of whose factors is prosoluble or a Cartesian product of simple groups.

We shall deduce the theorem from the following three lemmas, of which the first is well known. In the third lemma we use a result of Guralnick [4] which, on the basis of the classification of the finite simple groups, identifies the finite simple groups having p-complements.

LEMMA 1. *Let K be a normal subgroup of finite index in a profinite group G and let σ be a set of primes.*

(a) *If Q is a pro-σ subgroup of G then KQ/K is a σ-subgroup of G/K.*
(b) *If Q is a Hall pro-σ subgroup of G and G has a Hall pro-σ' subgroup R, then KQ/K is a Hall σ-subgroup of G/K.*

Proof. (a) Write $n = |G/K|$ and let C be a procyclic subgroup of Q. Then all finite continuous images of C are cyclic σ-groups, and so C/L is a cyclic σ-group, where L is the closed subgroup $\{c^n \mid c \in C\}$. But clearly the kernel of the map from C to G/K contains L and so the image of C in G/K is a σ-group.

(b) We have $G = QR$ (since, for example, QR is closed, and NQ/N, NR/N have coprime indices in G/N so that $G/N = (NQ/N)(NR/N)$ and hence $G = N(QR)$ for each N). Therefore $G/K = (KQ/K)(KR/K)$, so that the index of KQ/K is a product of primes in σ' and KQ/K is a Hall σ-subgroup. □

LEMMA 2. *Let σ be a set of primes with $2 \in \sigma$, and suppose that S is a finite non-abelian simple group which has Hall σ'-subgroups and Hall $(\sigma \cup \{p\})$-subgroups for all $p \in \sigma'$. Then $|\pi(S) \setminus \sigma| \leqslant 1$.*

Proof. Let Q be a Hall σ'-subgroup of S. We may assume that $Q \neq 1$. By the Feit–Thompson theorem, Q is soluble. Let A be a minimal normal subgroup of Q; thus A is a p-group for some $p \in \sigma'$. Let R be a Hall $\sigma \cup \{p\}$-subgroup of S. Replacing R by a conjugate if necessary, we can suppose that $A \leqslant R$, by Sylow's theorem. Since $|S : Q|$, $|S : R|$ are coprime we have $S = QR$, and so

$$1 < \langle A^S \rangle = \langle A^{QR} \rangle = \langle A^R \rangle \leqslant R.$$

However S is simple, and hence $R = S$ and $\pi(G) = \sigma \cup \{p\}$. □

It is well known that for each finite set σ of primes there are only finitely many isomorphism types of finite non-abelian simple σ-groups; this is an easy consequence of the classification of the finite simple groups together with Zsigmondy's theorem [9], which asserts that if l is a prime and s a positive integer then, unless either (a) $(l, s) = (2, 6)$ or (b) l is a Mersenne prime and $s = 2$, there is a prime divisor of $l^s - 1$ which does not divide $l^t - 1$ for any $t < s$. We need a slight extension of this result on simple groups.

LEMMA 3. *Let σ be a finite set of primes. Then there are only finitely many (isomorphism types of) finite simple groups S having Hall σ-subgroups and satisfying $|\pi(S) \setminus \sigma| \leqslant 1$.*

Proof. Let $\sigma^+ = \sigma \cup \{11, 23\}$; the groups S with $\pi(S) \subseteq \sigma^+$ fall into finitely many isomorphism classes, and each of the remaining groups S has a proper Hall σ-subgroup H of prime-power index p^r with $p \notin \{11, 23\}$. From Guralnick [4, Theorem 1], each of the latter groups S is isomorphic to one of the following:

(a) A_p, for some prime $p \geqslant 5$;
(b) $\mathrm{PSL}_n(q)$, where n is a prime and H is a parabolic subgroup whose preimage in $\mathrm{SL}_n(q)$ stabilizes a line or hyperplane in its natural representation.

Clearly there is a bound on the numbers n such that $\pi(A_n) \subseteq \sigma$ and so since A_n is the Weyl group of $\mathrm{PSL}_n(q)$ the integers p, n in (a), (b) are bounded. Moreover in case (b), since H is a parabolic subgroup, q must be a power l^s of a prime l in σ, and also $q - 1$ divides $|H|$, so that by Zsigmondy's theorem there are only finitely many choices for s. □

Now suppose that G satisfies the hypothesis of the theorem. Let σ be the finite set of prime divisors of orders of non-abelian composition factors of groups G/N in $\mathcal{F}_{\mathrm{o}}(G)$. Then each G/N is σ-separable, and so G/N has Hall π-subgroups for $\pi = \sigma$, $\pi = \sigma'$ and $\pi = \sigma \cup \{p\}$ for each $p \in \sigma'$; see [3, Theorem 6.3.4]. Since the Hall σ'-subgroups are soluble, each G/N also has Hall π-subgroups for $\pi = \sigma' \setminus \{p\}$ for each $p \in \sigma'$. Therefore G has Hall pro-π subgroups for all of the above sets π (see [8, 2.7.7 or the proof of 2.2.2 (a)]), and hence, by Lemma 1, each finite quotient G/K has Hall π-subgroups for all of these sets π. Since the property of having Hall π-subgroups is inherited by subnormal subgroups and quotient groups, each non-abelian composition factor S of a quotient group G/K has Hall π-subgroups for all of these sets π. By Lemmas 2, 3, it now follows that such groups S fall into finitely many isomorphism classes.

Acknowledgement. The author thanks the Swiss National Science Foundation for financial support during the preparation of this paper.

References

[1] M. P. Anderson, *Subgroups of finite index in profinite groups*, Pacific J. Math. **62** (1976), 19–28.

[2] R. Baer, *Einbettungseigenschaften von Normalteilern: der Schluss vom Endlichen aufs Unendliche*, Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973), Lecture Notes in Math., vol. 372, Springer, Berlin, 1974, pp. 13–62.

[3] D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.

[4] R. M. Guralnick, *Subgroups of prime power index in a simple group*, J. Algebra **81** (1983), 304–311.

[5] J. Saxl and J. S. Wilson, *A note on powers in simple groups*, Math. Proc. Cambridge Philos. Soc. **122** (1997), 91–94.

[6] D. Segal, *Closed subgroups of profinite groups*, Proc. London Math. Soc. (3) **81** (2000), 29–54.

[7] M. G. Smith and J. S. Wilson, *On subgroups of finite index in compact Hausdorff groups*, Arch. Math. (Basel) **80** (2003), 123–129.

[8] J. S. Wilson, *Profinite groups*, The Clarendon Press, Oxford Univ. Press, New York, 1998.

[9] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. Phys. **3** (1892), 265–284.

School of Mathematics and Statistics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom

E-mail address: J.S.Wilson@bham.ac.uk