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ON THE STRUCTURE OF THE GROUP OF AUTOPROJECTIVITIES OF A LOCALLY FINITE MODULAR p-GROUP OF FINITE EXPONENT

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Dedicated in memory of Reinhold Baer on the occasion of his 100th birthday

Abstract. In the description of the group of lattice automorphisms of modular groups, certain locally finite modular p-groups of finite exponent play a basic role. In the present paper significant structural properties of the group of autoprojectivities of such groups are investigated and placed in evidence.

1. Introduction

Given a group G, let $P(G)$ be the group of autoprojectivities of G and $PA(G)$ be the subgroup of autoprojectivities induced by group automorphisms. In two seminal papers on projectivities of abelian groups, R. Baer [B], [B1] proved the following basic facts: (1) Every modular locally finite non-Hamiltonian p-group is projective to an abelian group. (2) $P(G) = PA(G)$ if G is either a non-periodic abelian group of torsion free rank greater than 1, or an abelian torsion group where each primary component G_p has the following property: if G_p contains an element of order p^n, then it contains at least three independent elements of this order. On the other hand, simple examples show that if these conditions are not satisfied, we may have $P(G) \neq PA(G)$.

In a series of more recent papers ([GM], [Ho], [C], [CHZ], [CZ] and [CZ1]), the rather complex problem of describing the structure of $P(G)$, with G a modular group, has been investigated, covering also the cases left open by Baer's work. As a result of these studies, it turns out that a fundamental role is played by a certain subgroup of the group of autoprojectivities of an (n, s)-*group* M, i.e., of an abelian p-group $M = H \oplus C$, where $H = \langle a \rangle \oplus \langle b \rangle$ with $|\,a\,| = |\,b\,| = p^n$ and $\exp C = p^s$, $0 < s < n$.

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The structural properties we are interested in are mainly those of the following subgroup of $P(M)$:

$$\Gamma(M) = \{\rho \in R(M) \mid \rho|\Omega_{s+1}(M)/p^s\Omega_{s+1}(M) = 1\},$$

where $R(M) = \{\rho \in P(M) \mid H^\rho = H,\ \rho|\Omega_s(M) = 1\}$, with $P(M) = PA(M) \cdot R(M)$. Given (a,b), we know [CHZ] that there exists a well defined monomorphism j of $R(M)$ into $L = PR(R_n) \times PR(pR_n) \times \mathcal{U}(R_n/p^sR_n)$, where $R_n \cong \mathbb{Z}/p^n\mathbb{Z}$ and $PR(X)$ denotes the group of automorphisms of the partially ordered set $\mathcal{R}(X)$ of all cosets of the group X (see [S, 9.4]). More precisely,

$$\Psi_{n,s} = \Gamma(M)^j \leq \Phi_{n,s} = R(M)^j,$$

where $R(M)^j$ is the subgroup of elements $(\sigma, \tau, [\mu])$ in L satisfying the following conditions:

(a) $i\sigma \equiv i$, $i\tau \equiv i \bmod p^sR_n$.
(b) $j \equiv i \bmod p^fR_n \Rightarrow j\sigma - i\sigma \equiv (j-i)\mu^f, j\tau - i\tau \equiv (j-i)\mu^f \bmod p^{s+f}R_n$, for $0 \leq f \leq n-s$, with $\mu \in \mathcal{U}(R_n)$, $\mu \equiv 1 \bmod p^{s-1}R_n$.

We shall freely make use of these identifications via j.

This paper is divided into five sections. In Section 2 we collect, for easy reference, several results established in [CHZ] and [CZ] with regard to the groups $R(M)$ and $\Gamma(M)$. In Section 3 we determine the center of $\Gamma(M)$ relative to an (n,s)-group M, while in Section 4 the derived and the Frattini subgroups of $\Gamma(M)$ are characterized. In Section 5 we give a recursive construction of the elements of $R(M)$ and we study the action of $R(M)$ on $\Gamma(M)$. Finally, in Section 6 we give the exact nilpotent class of $\Gamma(M)$, even in the more general situation of a proper (n,m,s)-group (see Section 6 for the definition), and obtain bounds for the class of $R(M)$, a p-group when $s \geq 2$ or $s = 1$ and $p = 2$ (see [CZ, Theorem A and Proposition 1.3]).

For notation and terminology we shall refer mainly to [R], [S], [CHZ] and [CZ]. We denote by $\operatorname{cl} X$ the class of a nilpotent group X, while C_{p^n} stands for a cyclic group of order p^n. Whenever convenient, we shall identify R_n with the interval $0 \leq t < p^n$ of the ordered set $\mathbb{N}$, and pR_n with the interval $[0, p^n)$ of $p\mathbb{N}$. For $\xi \in R_n$ and $0 \leq t \leq n-1$, the coset $\xi + p^{t+1}R_n$ of R_n will be denoted by $\overline{\xi}_t$.

2. Preliminaries

Given the (n,s)-group $M = H \oplus C$, for $0 \leq i < p$, set

$$\tilde{S}_{i,n} = \{\sigma|i + pR_n \mid (\sigma, \tau, [\mu]) \in \Phi_{n,s}\},$$
$$S_{i,n} = \{\sigma|i + pR_n \mid (\sigma, \tau, [1]) \in \Psi_{n,s}\}.$$

Then

$$\begin{aligned}(2.1)\qquad &\Phi_{n,s} \cong D(\tilde{S}_{i,n}^{p+1})\Psi_{n,s},\ |\,\tilde{S}_{i,n} : S_{i,n}\,| = p,\\ &\Psi_{n,s} \cong S_{i,n}^{p+1},\ \tilde{S}_{i,n} \trianglelefteq \Phi_{n,s};\end{aligned}$$

moreover,

$$|\,\tilde{S}_{i,n} : S_{i,n}\,| = \begin{cases} p-1 & \text{if } s = 1,\\ p & \text{if } s \geq 2.\end{cases}$$

(See [CZ, Section 2].) Geometrically the group $S_{0,n}$ may be viewed as a group of automorphisms of a tree, with root in $\langle p^{n-1}a\rangle$, that is dual-isomorphic to the partially ordered set $\mathcal{R}(pR_n) = \{\overline{\xi}_t \mid \xi \in pR_n,\ 0 \leq t \leq n-1,\ \subseteq\}$.

An element $\sigma \in S_{i,n}$ is called an *elementary transformation* on $i + pR_n$ if there exists ξ in R_n, an integer t with $0 \leq t \leq n-s-1$ and z in p^tR_n such that

$$\sigma|\overline{\xi}_t : x \mapsto x + zp^s, \quad \sigma|i + pR_n \setminus \overline{\xi}_t = 1.$$

We shall denote σ by $\sigma_{\xi,z,t}$. Given $z = i_0 + i_1p + \cdots + i_\gamma p^\gamma$ in R_n, define $v(z) = \gamma$ if $i_\gamma \neq 0$, $v(0) = 0$ and, for $z \neq 0$, $w(z) = \max\{\ell \mid z \in p^\ell R_n\}$; set $\sigma_{\xi,t} := \sigma_{\xi,p^t,t}$ and $\sigma_\xi := \sigma_{\xi,v(\xi)}$. Assume $\sigma_{\xi,z,t} \neq 1$. Then:

(2.2) $\sigma_{\xi,z,t} = \sigma_{\xi',t',z'} \iff \xi' \equiv \xi \quad p^{t+1}R_n, z' \equiv z \bmod p^{n-s}R_n, t' = t$;
$|\,\sigma_{\xi,z,t}\,| = p^{n-s-w(z)} \leq p^{n-s-t} = |\,\sigma_{\xi,t}\,|$;
$\sigma_{\xi,z,t}^{-1}\sigma_{\xi',z',t'}\sigma_{\xi,z,t} = \sigma_{\xi'\sigma_{\xi,z,t},z',t'}$ if either $\overline{\xi} \cap \overline{\xi'} = \emptyset$ or $\overline{\xi'}_{t'} \subseteq \overline{\xi}_t$;
$[\sigma_{\xi',z',t'}, \sigma_{\xi,z,t}] = 1$ if $\overline{\xi'}_{t'} \cap \overline{\xi}_t = \emptyset$, or $\overline{\xi'}_{t'} \subseteq \overline{\xi}_t$ and $t' - w(z) < s$;
for $\sigma_{\xi',z',t'} \neq 1$, $1 \neq [\sigma_{\xi',z',t'}, \sigma_{\xi,z,t}] = [\sigma_{\xi',z',t'}, \sigma_0^z]$ if $\overline{\xi'}_{t'} \subseteq \overline{\xi}_t$ and $t' - w(z) \geq s$.

Since the groups $S_{i,n}$ for $0 \leq i < p$ are all isomorphic, we usually deal only with $S_{0,n}$. One has:

(2.3) $S_{0,n} = \langle \sigma_{\xi,t} \mid \xi \in J_0 = [0, p^{n-s}),\ 0 \leq t \leq n-s-1\rangle$, $\exp S_{0,n} = p^{n-s}$, $|\,S_{0,n}\,| = p^{p^{n-s-1}+\cdots+p+1}$ and $S_{0,n} = \prod_{\xi \in J_0} \Delta_\xi$, with ξ in increasing (or decreasing) order, where $\Delta_\xi = \langle \sigma_\xi \rangle$. For $\sigma \in S_{0,n}$, its components in Δ_ξ are uniquely determined. The derived length of $S_{0,n}$ is q, where $sq < n \leq (q+1)s$.

(2.4) For ξ, $\eta \in J_0$, if $\overline{\eta}_{v(\eta)} \subseteq \overline{\xi}_{v(\xi)}$ and $v(\eta) - v(\xi) \geq s$, then $|\,\sigma_\eta^{\Delta_\xi}\,| = p^{v(\eta)-v(\xi)-s+1}$ and $1 \neq \sigma_\xi^{p^{n-s-v(\xi)-1}} \in \mathcal{C}(\sigma_\eta)$; $\xi < \eta$ implies $\xi\sigma_\eta = \xi$, $\eta\sigma_\xi = \eta + p^{s+v(\xi)}$.

From (2.3) and (2.4) it follows that $S_{0,n}$ acts transitively on pR_n only if $s = 1$; otherwise its action splits into p^{s-1} orbits $\{\xi + p^sR_n \mid \xi \in [0, p^s)\}$, each of length p^{n-s}. Since for $\xi \in pR_n$ and $t < t'$, $\xi + p^{t+1}R_n = \dot{\bigcup}_{0 \leq k < p^{t'-t}} \xi +$

$kp^{t+1} + p^{t'+1}R_n$, we get

$$\sigma_{\xi,t}^{p^{t'-t}} = \prod_{0\le k<p^{t'-t}} \sigma_{\xi+kp^{t+1},t'}. \tag{2.5}$$

We recall from [CZ, 1.2] that, in view of the restriction map from $\Gamma(M)$ to $\Gamma(\Omega_k(M))$, we have:

(2.6) There exists an epimorphism $\varphi : S_{0,n} \to S_{0,k}$ such that if $\rho : R_n \to R_k$ is the canonical epimorphism, then $\sigma_\xi^\varphi = \sigma_{\xi\rho}$ for $\xi \in pR_n$.

3. The center of $\Gamma(M)$

We may restrict ourselves to $G := S_{0,n}$. Since, by (2.2), G is abelian for $n \le 2s$, we shall assume $n > 2s$. By (2.3) and (2.4), for $\xi \in J_0$ the set $\prod_{\xi<\eta} \Delta_\eta$ is the pointwise stabilizer $G_{[0,\xi]}$ of the points of the closed interval $[0, \xi]$ in J_0; hence

$$G = G_{[0,\xi]}(\prod_{\eta\in[0,\xi]} \Delta_\eta) \quad \text{with } \eta \text{ in decreasing order.} \tag{3.1}$$

Take $\eta \in pR_n$, so $\eta = \xi + kp^s$, $\xi \in [0, p^s)$, and for $\rho \in G_{[0,p^s-p]} \cap \mathcal{C}(\sigma_0)$ we get $\eta\rho = ((\eta\sigma_0^{-k})\rho)\sigma_0^k = \eta$, i.e., $\rho = 1$. Therefore from (3.1) and (2.4) it follows that $Z(G) \le \mathcal{C}(\sigma_0) \le \Delta_0 \times \Delta_p \times \cdots \times \Delta_{p^s-p}$.

Let $\xi \in [0, p^s)$. We note that $\sigma_\xi^{p^r} \in \Omega_s(\Delta_\xi)$ if and only if $n - 2s - v(\xi) \le r$. Now take $\eta \in pR_n$. If $\overline{\eta}_{v(\eta)} \cap \overline{\xi}_{v(\xi)} = \emptyset$ or $\overline{\xi}_{v(\xi)}$, then $[\sigma_\eta, \sigma_\xi] = 1$; if $\overline{\eta}_{v(\eta)} \cap \overline{\xi}_{v(\xi)} = \overline{\eta}_{v(\eta)}$ then

$$\sigma_\xi^{-p^{n-2s-v(\xi)}}\sigma_\eta\sigma_\xi^{p^{n-2s-v(\xi)}} = \sigma_{\eta+p^{n-2s-v(\xi)+s-v(\xi)}} = \sigma_{\eta+p^{n-s}} = \sigma_\eta.$$

In conclusion we have

$$\operatorname*{Dr}_{\xi\in[0,p^s)} \Omega_s(\Delta_\xi) \le Z(G) \le \operatorname*{Dr}_{\xi\in[0,p^s)} \Delta_\xi. \tag{3.2}$$

Proposition 3.1. *$Z(G) = \operatorname*{Dr}_{\xi\in[0,p^s)} \Omega_s(\Delta_\xi)$, $Z(\Gamma(M)) \cong (Z(G))^{p+1}$.*

Proof. Let z be in $Z(G)$. By (3.2) $z = \prod_{\xi\in[0,p^s)} \sigma_\xi^{z_\xi}$. Assume that for $\xi_0 \in [0, p^s)$, $\sigma_{\xi_0}^{z_{\xi_0}} \notin \Omega_s(\Delta_{\xi_0})$, while for $\xi < \xi_0$, $\sigma_\xi^{z_\xi} \in \Omega_s(\Delta_\xi)$. By (3.2), $\prod_{\xi<\xi_0} \sigma_\xi^{z_\xi} \in Z(G)$; for $\xi > \xi_0$, $\overline{\xi 0}_{n-s-1} \cap \overline{\xi}_{v(\xi)} = \emptyset$, and hence $[\sigma_{\xi_0,n-s-1}, \sigma_{\xi_0}^{z_{\xi_0}}] = 1$ which means $\xi_0 + z_{\xi_0}p^{s+v(\xi_0)} \equiv \xi_0 \bmod p^{n-s}R_n$, i.e., $\sigma_{\xi_0}^{z_{\xi_0}} \in \Omega_s(\Delta_{\xi_0})$, a contradiction. Using (2.1) one obtains the result. □

From Proposition 3.1 and (2.2) it follows that

$$Z(G) \cong \begin{cases} C_{p^s}^{p^{s-1}} & \text{if } n \ge 3s-1, \\ C_{p^s}^{p^{n-2s}} \times C_{p^{s-1}}^{(p^{n-2s+1}-p^{n-2s})} \times \cdots \times C_{p^{n-2s+1}}^{(p^s-p^{s-1})} & \text{if } 2s+1 \le n \le 3s-2. \end{cases}$$

One of our aims is to determine the nilpotent class of G. For $s = 1$ we can already give an answer to this question:

Proposition 3.2. *If $s = 1$, the nilpotent class of $\Gamma(M)$ is p^{n-2}, with the factors of the lower central series all of exponent p.*

Proof. For $s = 1$, G is a transitive permutation group on pR_n. Now (2.3) shows that the order of G equals that of a Sylow p-subgroup of $\operatorname{Sym} p^{n-1}$. It is well known that such a group is isomorphic to $\underbrace{C_p \wr C_p \wr \cdots \wr C_p}_{n-1}$, which has nilpotent class p^{n-2}, with the factors of the lower central series all of exponent p (see [K], [Hu, III.15.3]). $\square$

4. The derived and the Frattini subgroups of $\Gamma(M)$

Since G is abelian for $n \leq 2s$, unless otherwise stated, we shall assume $n > 2s$. If ξ, η are different elements in $[0, p^{t+1})$, $0 \leq t \leq n-s-1$, then $\overline{\xi}_t \cap \overline{\eta}_t = \emptyset$; it follows from (2.2) and (2.3) that

$$\begin{aligned} X_t &:= \langle \sigma_{\xi,t} \mid \xi \in pR_n \rangle = \prod_{\xi \in [0,p^{t+1})} \langle \sigma_{\xi,t} \rangle \cong (C_{p^{n-s-t}})^{p^t}, \\ G &= X_0 X_1 \cdots X_{n-s-1}. \end{aligned} \tag{4.1}$$

Using (2.2), for $s \leq t' - t$, we get $1 \neq [X_{t'}, X_t] \leq [X_{t'}, \sigma_0] \leq X_{t'}$; hence

$$X_{t'} \cdots X_t \trianglelefteq X_{t'} \cdots X_t \cdots X_0; \tag{4.2}$$

in particular, $X_{n-s-1} \cdots X_t \trianglelefteq G$. Set $Y_t := [X_t, \prod_{k=t}^{0} X_k]$; then by (4.1) and [Hu, III.1.10a], $[X_t, \sigma_0] \leq Y_t = \prod_{k=t}^{0} [X_t, X_k] \leq [X_t, \sigma_0]$, i.e.,

(4.3) $Y_t = [X_t, \sigma_0] = \langle X_t, \sigma_0 \rangle'$ and is different from 1 if $t \geq s$; also $\mathcal{N}(Y_t) \geq X_t \cdots X_0$; in particular, $\mathcal{N}(Y_t) \geq Y_t \cdots Y_s$.

Let $s \leq t \leq t'$ and for $\sigma_{\eta,t'}$, $\sigma_{\xi,t}$ assume that $[\sigma_{\xi,t}, \sigma_0]^{\sigma_{\eta,t'}} \neq [\sigma_{\xi,t}, \sigma_0]$, so that $t' - t \geq s$. Since $\overline{\xi}_t \cap \overline{\xi + p^s}_t = \emptyset$ (because $s \leq t$), either $\overline{\eta}_t = \overline{\xi}_t$ or $\overline{\eta}_t = \overline{\xi + p^s}_t$. In the first case,

$$[\sigma_{\eta,t}, \sigma_0]^{\sigma_{\eta,t'}} = (\sigma_{\eta,t}^{-1})^{\sigma_{\eta,t'}} \sigma_{\xi+p^s,t} = [\sigma_{\eta,t'}, \sigma_{\eta,t}] \sigma_{\eta,t}^{-1} \sigma_{\eta+p^s,t} \in Y_{t'} Y_t \leq [G, \sigma_0],$$

while in the second case

$$[\sigma_{\xi+p^s,t}, \sigma_0]^{\sigma_{\xi+p^s,t'}} = [\sigma_{\eta,t'}, \sigma_{\eta,t}] \sigma_{\eta,t}^{-1} \sigma_{\eta+p^s,t} \in Y_{t'} Y_t \leq [G, \sigma_0].$$

Hence, with (4.3), one concludes:

(4.4) $\mathcal{N}(Y_{t'} Y_{t'-1} \cdots Y_t) \geq X_{t'} X_{t'-1} \cdots X_0$ for $s \leq t \leq t' \leq n-s-1$; in particular, $Y_{n-s-1} \cdots Y_t \trianglelefteq G$.

We may now prove:

Proposition 4.1. $G' = Y_{n-s-1} \cdots Y_{s+1} Y_s = [G, \sigma_0]$.

Proof. We have $S := Y_{n-s-1} \cdots Y_{s+1} Y_s \leq G'$ and $S \trianglelefteq G$ by (4.4). But the group G/S is abelian since $[\sigma_{\xi',z',t'}, \sigma_{\xi,z,t}] \in S$, so that $S = G'$. Moreover, by (4.3), $G' = \prod_{k=s}^{n-s-1} [X_k, \sigma_0] \leq [G, \sigma_0] \leq G'$. $\square$

Since $\langle \sigma_0 \rangle \leq \mathcal{N}(X_t)$ for any $0 \leq t \leq n-s-1$, we may consider X_t as a $\langle \sigma_0 \rangle$-module, which is non-trivial as soon as $t \geq s$. One has

$$\begin{aligned} X_{\xi,t} &:= \langle \sigma_{\xi,t} \rangle^{\langle \sigma_0 \rangle} = \prod_{0 \leq k < p^{t-s+1}} \langle \sigma_{\xi+kp^s,t} \rangle \cong (C_{p^{n-s-t}})^{p^{t-s+1}}, \\ X_t &= \operatorname*{Dr}_{\xi \in [0,p^s)} X_{\xi,t}, \quad \mathrm{cl}\langle X_t, \sigma_0 \rangle = \mathrm{cl}\langle X_{\xi,t}, \sigma_0 \rangle. \end{aligned} \tag{4.5}$$

If $0 \leq t < t' \leq n-s-1$, using (2.5), we have

$$\begin{aligned} X_t \cap X_{t'} &= X_t^{p^{t'-t}} = X_t \cap (X_{t'} \cdots X_{n-s-1}), \\ X_t \cap \langle \sigma_0 \rangle &= \langle \sigma_0^{p^t} \rangle, \quad X_{\xi,t} \cap \langle \sigma_0 \rangle = \begin{cases} \langle \sigma_0^{p^t} \rangle & \text{if } s = 1, \\ 1 & \text{if } s \geq 2. \end{cases} \end{aligned} \tag{4.6}$$

From (4.6) and (2.3) it follows that

$$| X_t \cdots X_{n-s-1} | = \prod_{k=t}^{n-s-1} | X_k |/| X_k^p | = \prod_{k=t}^{n-s-1} p^{p^k} = p^{p^{n-s-1} + \cdots + p^t}. \tag{4.7}$$

Set $H = \langle X_{\xi,t}, \sigma_0 \rangle$, $t \geq s$ and $x_k = \sigma_{\xi+kp^s,t}$, $0 \leq k < p^{t-s+1}$. Then $N := X_{\xi,t} = \operatorname*{Dr}_k \langle x_k \rangle = \langle x_k \rangle^{\langle \sigma_0 \rangle}$, $x_k^{\sigma_0} = x_{k+1}$, $\mathcal{C}_{\langle \sigma_0 \rangle}(x_k) = \langle \sigma_0^r \rangle$, $r = p^{t-s+1}$ and $H = \langle x_k, \sigma_0 \rangle$. Hence:

(4.8) $H' = [N, \sigma_0] = \langle [x_0, \sigma_0], \ldots, [x_r, \sigma_0] \rangle = \{ x_0^{m_0} \cdots x_r^{m_r} \mid m_0 + \cdots + m_r \equiv 0 \bmod p^{n-s-t} R_n \}$, $H/\langle \sigma_0^r \rangle \cong C_{p^{n-s-t}} \wr C_r$, $\sigma_0^r \in Z(H)$, $N = H' \times \langle x_0, \ldots, x_r \rangle$, $H' \cong (C_{p^{n-s-t}})^{r-1}$, and $| Y_t | = | \prod_{\xi \in [0,p^s)} [X_{\xi,t}, \sigma_0] | = (p^{n-s-t})^{p^t - p^{s-1}}$.

Moreover:

(4.9) For $t \geq s \geq 2$, $\mathrm{cl}\langle X_t, \sigma_0 \rangle = (n-s+t)(p^{t-s+1} - p^{t-s})$; in particular, $\mathrm{cl}\langle X_{n-s-1}, \sigma_0 \rangle = p^{n-2s}$.

In fact, $[X_t, \sigma_0] = \prod_{\xi \in [0,p^s)} [X_{\xi,t}, \sigma_0]$, so $\mathrm{cl}\langle X_t, \sigma_0 \rangle = \mathrm{cl}\, H$. We have $H' \cap \langle \sigma_0 \rangle = 1$; hence if $g_i \in H$, $[g_1, \ldots, g_m] \in H'$ for $m \geq 2$, so that $[g_1, \ldots, g_m] = 1$ if and only if $[g_1, \ldots, g_m] \in \langle \sigma_0^r \rangle$. Therefore the class of H equals that of $H/\langle \sigma_0^r \rangle$, i.e., the class of $C_{p^{n-s-t}} \wr C_{p^{t-s+1}}$. Using now [L, 5.1], one gets (4.9).

We are going to evaluate the order of G'. By (4.3),

$$Y_t = \langle [\sigma_{\xi,t}, \sigma_0] = \sigma_{\xi,t}^{-1} \sigma_{\xi+p^s,t} \mid \xi \in [0, p^{t+1}) \rangle,$$

and since by (2.5)

$$\sigma_{\xi,t}^{-p}\sigma_{\xi+p^s,t}^{p}=\prod_{k=0}^{p-1}\sigma_{\xi+kp^{t+1},t+1}^{-1}\sigma_{\xi+p^s+kp^{t+1},t+1},$$

we have, similarly to (4.6), that $Y_t^p = Y_t \cap Y_{t+1} = Y_t \cap (Y_{t+1}\cdots Y_{n-s-1})$. But now, with the help of Proposition 4.1 and (4.8), we obtain

$$\begin{aligned}|\,G'\,| &= \left|\prod_{t=s}^{n-s-1} Y_t\right| = \prod_{t=s}^{n-s-1} |\,Y_t\,|/|\,Y_t^p\,| \\ &= \prod_{t=s}^{n-s-1} p^{p^t-p^{s-1}} = p^{p^s+\cdots+p^{n-s-1}-(n-2s)p^{s-1}}.\end{aligned} \tag{4.10}$$

Proposition 4.2. *Given $\xi\in[p^s,p^{n-s})$, set $\Lambda_\xi=\langle[\sigma_\xi,\sigma_0]\rangle$. Then*

$$G'=\prod_{\xi\in[p^s,p^{n-s})}\Lambda_\xi,$$

with ξ in increasing or decreasing order. For a given element $g\in G'$, its components in Λ_ξ are uniquely determined.

Proof. Set $\rho_\xi=[\sigma_\xi,\sigma_0]$; then $\Lambda_\xi=\langle\rho_\xi\rangle$ and for $\xi\in[p^t,p^{t+1})$ we have $|\,\rho_\xi\,|=p^{n-s-t}$. Using (4.10) we get

$$\begin{aligned}\prod_{\xi\in[p^s,p^{n-s})}|\,\Lambda_\xi\,| &= \prod_{s\le t\le n-s-1} p^{(n-s-t)(p^t-p^{t-1})} \\ &= p^{p^s+\cdots+p^{n-s-1}-(n-2s)p^{s-1}} = |\,G'\,|.\end{aligned} \tag{$*$}$$

Note that for η, ξ in $[p^s,p^{n-s})$ and $\eta<\xi$ we have $\eta[\sigma_\xi^{z_\xi},\sigma_0]=\eta$ and $\xi[\sigma_\xi^{z_\xi},\sigma_0]=\xi-z_\xi p^{s+v(\xi)}$, so that from $\prod_\xi\rho_\xi^{z_\xi}=\prod_\xi\rho_\xi^{z'_\xi}$ one has $z_\xi\equiv z'_\xi \bmod p^{n-s}R_n$, i.e., $\rho_\xi^{z_\xi}=\rho_\xi^{z'_\xi}$. This and $(*)$ imply $G'=\prod_{\xi\in[p^s,p^{n-s})}\Lambda_\xi$, with ξ in increasing or decreasing order. □

Our next goal will be to determine the structure of G/G'.

Lemma 4.3. *For any ξ in pR_n the following holds:*

(i) *If $s-1\le t\le n-s-1$, then $\sigma_{\xi,t}^p\in G'$.*

(ii) *If $0\le t\le s-1$, then $\sigma_{\xi,t}^{p^{s-t}}\in G'$.*

Proof. (i) Since $|\,\sigma_{\xi,n-s-1}\,|=p$, $\sigma_{\xi,n-s-1}^p=1\in G'$. Now we use induction on t. By (2.5), $\sigma_{\xi,t-1}^p=\sigma_{\xi,t}\cdots\sigma_{\xi+(p-1)p^t,t}$ and, by (4.8),

$$\tau=\sigma_{\xi,t}^{1-p}\sigma_{\xi+p^t,t},\cdots\sigma_{\xi+(p-1)p^t,t}\in(\langle\sigma_{\xi,t},\sigma_0^{p^{t-s}}\rangle)',$$

and hence also $\sigma_{\xi,t-1}^p=\sigma_{\xi,t}^p\tau\in G'$.

(ii) By (i), $\sigma^p_{\xi,s-1} \in G'$. By induction on t,

$$\sigma^{p^{s-(t-1)}}_{\xi,t-1} = (\sigma^p_{\xi,t-1})^{p^{s-t}} = \sigma^{p^{s-t}}_{\xi,t} \cdots \sigma^{p^{s-t}}_{\xi+(p-1)p^t,t} \in G'. \qquad \square$$

By (2.6) we have the epimorphism φ of G onto the abelian group $S_{0,2s}$. Thus, if $S = \ker \varphi$, we have $G/S = \langle \sigma_0 S\rangle \times \cdots \times \langle \sigma_{p^s-p} S\rangle$, where $|\,\sigma_\xi S\,| = p^{s-t}$ for $\xi \in [p^t, p^{t+1})$, $0 \leq t < s$. Since, by Lemma 4.3, $\sigma_\xi^{p^{s-v(\xi)}} \in G'$ if $\xi \in [0, p^s)$ and $\sigma^p_\xi \in G'$ if $\xi \in [p^s, p^{n-s})$, we conclude that

$$G/G' = \operatorname*{Dr}_{\xi \in [0,p^s)} \langle \sigma_\xi G'\rangle \times E, \tag{4.11}$$

where $|\,\sigma_\xi G'\,| = p^{s-v(\xi)}$ and E is an elementary abelian p-group. In particular, $\exp G/G' = p^s$. We can now describe the structure of G/G'.

Theorem 4.4. *Given the group $G = S_{0,n}$, we have:*

(i) *If $n \leq 2s$, $G = \operatorname*{Dr}_{\xi\in[0,p^{n-s})} \langle \sigma_\xi \rangle \cong C_{p^{n-s}} \times \operatorname*{Dr}_{1\leq t<n-s} (C_{p^{n-s-t}})^{p^t - t^{t-1}}$.*

(ii) *If $n > 2s$, $G/G' \cong C_{p^s} \times \operatorname*{Dr}_{1\leq t<s} (C_{p^{s-t}})^{p^t-t^{t-1}} \times (C_p)^{(n-2s)p^{s-1}}$.*

Proof. (i) This is the case when G is abelian, and the conclusion follows from (2.2) and (2.3).

(ii) By (4.11), $|\,G/G'\,| = |\,E\,|\; p^s \prod_{1\leq t<s} p^{(s-t)(p^t-p^{t-1})}$. Since for $\xi \in [p^s, p^{n-s})$, $|\,\Lambda_\xi\,| = |\,\Delta_\xi\,|$, from Proposition 4.2 it follows that $|\,G/G'\,| = \prod_{\xi\in[0,p^s)} |\,\Delta_\xi\,|$. We also know that $|\,\Delta_\xi\,| = p^{n-s-t}$ for $\xi \in [p^t, p^{t+1})$. Hence

$$|\,G/G'\,| = p^{n-s} \prod_{1\leq t<s} p^{(n-s-t)(p^t-p^{t-1})},$$

and thus $|\,E\,| = p^{(n-2s)p^{s-1}}$ and $E \cong C_p^{(n-2s)p^{s-1}}$. $\square$

Finally, we deal with the Frattini subgroup of G. From Theorem 4.4 it follows that

$$G/\Phi(G) \cong \begin{cases} (C_p)^{p^{n-s-1}} & \text{if } n \leq 2s, \\ (C_p)^{(n-2s+1)p^{s-1}} & \text{if } n > 2s. \end{cases} \tag{4.12}$$

Next we determine a minimal generating set, the situation being clear in the case $n \leq 2s$, for $\Phi(G) = G^p = \operatorname*{Dr}_{\xi\in[0,p^{n-s-1})} \langle\sigma_\xi\rangle^p$ and the set $\{\sigma_\xi \mid \xi \in [0, p^{n-s})\}$ is what we are looking for. In the case $n > 2s$ we introduce

$$I = \{(\xi, t) \mid \xi \in [0, p^s),\ s \leq t < n-s\},$$
$$X = \{\sigma_\xi \mid \xi \in [0, p^s)\} \,\dot\cup\, \{\sigma_{\xi,t} \mid (\xi, t) \in I\}.$$

Proposition 4.5. *If $n > 2s$, then:*

(i) *X is a minimal generating set for G.*

(ii) $G/G' = \mathop{\mathrm{Dr}}_{\xi\in[0,p^s)} \langle \sigma_\xi G'\rangle \times \mathop{\mathrm{Dr}}_{(\xi,t)\in I} \langle \sigma_{\xi,t} G'\rangle$.
(iii) $G/\Phi(G) = \mathop{\mathrm{Dr}}_{\xi\in[0,p^s)} \langle \sigma_\xi \Phi\rangle \times \mathop{\mathrm{Dr}}_{(\xi,t)\in I} \langle \sigma_{\xi,t} \Phi\rangle$.
(iv) $\Phi(G)/G' = \mathop{\mathrm{Dr}}_{\xi\in[0,p^{s-1})} \langle \sigma_\xi^p G'\rangle$.

Proof. (i) We have

$$X^{\langle\sigma_0\rangle} = \{\sigma_{\xi,t} \mid \xi\in pR_n,\ 0\le t\le n-s-1\},$$

and hence $\langle X\rangle = G$. But $|X| = (n-2s+1)p^{s-1}$, so, by (4.12), X is a minimal generating set of G.

(ii) By (i) and (4.11), it is enough to show that $\prod_{(\xi,t)\in I} |\sigma_{\xi,t}G'| \le |E|$. This follows from $|\sigma_{\xi,t}G'|\le p$ for $(\xi,t)\in I$, by 4.3 (i), and $|E| = p^{(n-2s)p^{s-1}}$.

(iii) The statements follow from (4.11) and (4.12).

(iv) Since $\Phi(G) = G^pG'$, this is a consequence of (4.11). $\square$

5. Construction of the elements of $R(M)$ and their action on $\Gamma(M)$

We have $\Phi_{n,s} = \dot{\bigcup}_{[\mu]}(\sigma,\tau,[\mu])\Psi_{n,s}$, with $\Psi_{n,s}\cong (S_{0,n})^{p+1}$. According to Section 2 in [CZ], to construct an element of $\Phi_{n,s}$ it is enough to construct an element of $\tilde S_{0,n}$. Select $\mu\in\mathcal{U}(R_n)$, with $\mu\equiv 1 \bmod p^{s-1}R_n$, and define $\tilde\sigma$ in the following way. Let $i = i_1p+\cdots+i_{n-s-1}p^{n-s-1}$ be an element of $[0,p^{n-s})$ and let j in pR_n be such that $j-i\in p^{n-s}R_n$. Set

$$(*)\qquad \begin{cases} i\tilde\sigma = i_1\mu p + i_2\mu^2p^2+\cdots+i_{n-s-1}\mu^{n-s-1}p^{n-s-1},\\ j\tilde\sigma = i\tilde\sigma + (j-i)\mu^{n-s}.\end{cases}$$

It is not difficult to check that $\tilde\sigma\in PR(pR_n)$, that $j\tilde\sigma\equiv j \bmod p^sR_n$ and that if j, i are in pR_n with $j\equiv i \bmod p^fR_n$, $0\le f\le n-s$, then $j\tilde\sigma - i\tilde\sigma\equiv (j-i)\mu^f \bmod p^{f+s}R_n$. Therefore $\tilde\sigma\in\tilde S_{0,n}$, and $\tilde\sigma\notin S_{0,n}$ as soon as $[\mu]\ne[1]$.

Considering the elements of the form $\tilde\sigma\prod_{\xi\in J_0}\sigma_{\xi,z_\xi}$, $\xi\in pR_n$, $z_\xi\in p^{v(\xi)}R_n$, one gets all the elements of $\tilde S_{0,n}$ relative to $[\mu]$.

A recursive procedure to assign, for a given μ, an element $\tilde\rho$ of $\tilde S_{0,n}$, goes as follows: for $i\in pR_n$ set

$$i\tilde\rho = \begin{cases} k_0p^s & \text{if } i=0, \text{ with } 0\le k_0<p^{n-s},\\ (i-p^t)\tilde\rho + \mu^tp^t + k_ip^{s+t} & \text{if } i\in[p^t,p^{t+1}),\ 1\le t\le n-s-1,\\ & \qquad \text{with } 0\le k_i<p^{n-s-t}.\end{cases}$$

Finally, if $j\in pR_n$ and $j-i\in p^{n-s}R_n$ with $i\in[0,p^{n-s})$, set

$$j\tilde\rho = i\tilde\rho + (j-i)\mu^{n-s}.$$

Again one may check that $\tilde\rho\in\tilde S_{0,n}$. We remark that $\tilde\sigma$ as in $(*)$ is obtained from the construction of $\tilde\rho$ by choosing $k_i=0$ for all $i\in[0,p^{n-s})$.

In the remaining part of this section we shall investigate the action of $\Phi_{n,s}$ upon $\Psi_{n,s}$. Again we shall study the action by conjugation of $\tilde{S}_{0,n}$ on $S_{0,n}$.

Let $\tilde{\rho} \in \tilde{S}_{0,n}$ be relative to $\mu \in \mathcal{U}(R_n)$. Take $\sigma_{\xi,t} \in X_t$, $0 \leq t \leq n-s-1$, and consider $\tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho}$. For a given $i \in pR_n$, if $i\tilde{\rho}^{-1} \notin \overline{\xi}_t$, then $i\tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho} = i$; but $i\tilde{\rho}^{-1} \notin \overline{\xi}_t$ is equivalent to $i \notin (\overline{\xi\tilde{\rho}})_t$, so we also have $i\sigma_{\xi\tilde{\rho},\mu^{s+t}p^t,t} = i$.

Assume now that $i\tilde{\rho}^{-1} \in \overline{\xi}_t$; then $i\tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho} = (i\tilde{\rho}^{-1} + p^{s+t})\tilde{\rho} \equiv i + \mu^{s+t}p^{s+t} \bmod p^{2s+t}R_n$. On the other hand, $i\sigma_{\xi\tilde{\rho},\mu^{s+t}p^t,t} = i + \mu^{s+t}p^{s+t}$. It follows that for $\chi = \sigma^{-1}_{\xi\tilde{\rho},\mu^{s+t}p^{s+t},t}\tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho}$ and $i \in pR_n$ we have $i\chi \equiv i \bmod p^{2s+t}R_n$, i.e., $\chi \in S_{0,n} \cap \ker\varphi_{n-2s-t} = X_{s+t}\cdots X_{n-s-1}$ if $t < n-2s$, and $\chi = 1$ if $n-2s \leq t \leq n-s-1$. Equivalently,

$$\begin{cases} \tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho} \equiv \sigma^{\mu^{s+t}}_{\xi\tilde{\rho},t} \bmod X_{s+t}\cdots X_{n-s-1} & \text{for } 0 \leq t \leq n-2s-1, \\ \tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho} = \sigma^{\mu^{s+t}}_{\xi\tilde{\rho},t} & \text{for } n-2s \leq t \leq n-s-1. \end{cases}$$

Recall that if $n-2s+1 \leq t \leq n-s-1$, $s \geq 2$, then $\mu^{s+t}p^{s+t} = p^{s+t}$, so that

$$\tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho} = \sigma_{\xi\tilde{\rho},t} \qquad \text{for } n-2s+1 \leq t \leq n-s-1,\ s \geq 2.$$

From this it follows that if $n < 2s$ then $\tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho} = \sigma_{\xi\tilde{\rho},t}$ for $0 \leq t \leq n-s-1$, while, for $n = 2s$,

$$\begin{cases} \tilde{\rho}^{-1}\sigma_{\xi,t}\tilde{\rho} = \sigma_{\xi\tilde{\rho},t}, & 1 \leq t \leq n-s-1, \\ \tilde{\rho}^{-1}\sigma_{\xi,0}\tilde{\rho} = \sigma^{\mu^s}_{\xi\tilde{\rho},0}. \end{cases} \tag{5.1}$$

We are now in the position to determine in which cases $R(M)$ is abelian.

Proposition 5.1. *The group $R(M)$ is abelian precisely in the following cases:*

(i) $n < 2s$,
(ii) $n = 2s$, $s \geq 2$, $p \,|\, s$,
(iii) $n = 2$, $s = 1$, $p = 2$.

Proof. If $n > 2s$, then even $\Gamma(M)$ is non-abelian. If $n < 2s$, $R(M)$ is abelian (see [CZ, 3.2]). So assume $n = 2s$. Then $\Gamma(M)$ is abelian. Suppose first $s \geq 2$. From (5.1) it follows that $R(M)$ is abelian if and only if $(1+p^{s-1})^s \equiv 1 \bmod p^sR_n$, that is, if and only if $p \,|\, s$. Finally, assume $s = 1$. By [CZ, 1.3], $R(M)$ is abelian if and only if $p = 2$, i.e., only when $R(M) = \Gamma(M)$. ☐

6. The nilpotent class of $\Gamma(M)$[1]

An abelian p-group M is called a *proper* (n,m,s)-group if $M = H \oplus C$ with $H = \langle a \rangle \oplus \langle b \rangle$, where $p^n = |\,a\,| \geq |\,b\,| = p^m$, $\exp C = p^s$ and $1 \leq s < m$. In what follows we are mainly concerned with determining the nilpotent class of $\Gamma(M)$. To this end we embed M in an (n,s)-group $\tilde{M} = \langle a \rangle \oplus \langle \tilde{b} \rangle \oplus C$, so that $b = p^{n-m}\tilde{b}$; we denote by $S(M)$ the stabilizer of M in $\Gamma(\tilde{M})$. By

[1]We are grateful to M. Newell for stimulating discussions on this topic.

[CZ, Theorem A] we know that the restriction map $\varphi \mapsto \varphi|M$ defines an epimorphism of $S(M)$ onto $\Gamma(M)$, and hence, via j, an epimorphism ρ of $S(M)^j \le \Psi_{n,s} = S_{0,n} \times \cdots \times S_{p-1,n} \times S_{\infty,n}$ onto $\Gamma(M)$, so that $\operatorname{cl}\Gamma(M) = \operatorname{cl} S(M)^j/\ker\rho$. If R is any subgroup of $S(M)^j$, we shall call $\operatorname{cl}(R/R \cap \ker\rho)$ the *class of the action of R on M*.

We note that

$$(\sigma, \tau, [1]) \in S(M)^j \iff \langle a + (0\sigma)b\rangle \le H \iff 0\sigma \in p^{n-m}R_n.$$

In particular, we get

$$S(M)^j = (S(M)^j \cap S_{0,n}) \times S_{1,n} \times \cdots \times S_{p-1,n} \times S_{\infty,n}. \tag{6.1}$$

Lemma 6.1. *Let σ be in $S_{0,n}$ and write, in accordance with* (2.3)*, $\sigma = \prod_{\xi \in J_0} \sigma_\xi^{z_\xi}$, with ξ in decreasing order. Then σ lies in $S(M)^j$ if and only if $z_0 p^s \in p^{n-m}R_n$.*

Proof. This follows from the fact that $0\sigma = 0\sigma_0^{z_0}$. □

Remark 6.1. Using Lemma 6.1 and (2.5), one concludes that $S(M)^j$ can be generated by convenient elementary transformations of the form $\sigma_{\xi,t}$, with $\xi \in pR_n$ and $t \ge v(\xi)$.

We know that $G = X_0 X_1 \cdots X_{n-s-1}$ and $X_{t'} \le \mathcal{N}(X_t)$, $0 \le t' \le t$. Let us define

$$T_i = \begin{cases} X_i X_{i+1} \cdots X_{n-s-1} & \text{if } i < n-s, \\ 1 & \text{if } i = n-s, \end{cases}$$
$$H_{i,k} = \langle \sigma_{\xi,t} \mid \xi \in kp + p^{t+1}R_n, i \le t \le n-s-1\rangle.$$

Given $0 \le k, k' \le p^i - 1$, the translation $\tau_{k'-k} : x \mapsto x + (k'-k)p$ on pR_n induces the isomorphism $\tau_{(k,k')} : H_{i,k} \to H_{i,k'}$, $\sigma \mapsto \tau_{k'-k}^{-1}\sigma\tau_{k'-k}$. With the help of (4.2) and (2.2), we have

$$T_i \trianglelefteq G, \quad T_i = H_{i,0} \times \cdots \times H_{i,p^i-1}, \quad G = \langle \sigma_0, T_1\rangle, \quad \sigma_0^p \in T_1. \tag{6.2}$$

We claim that

$$H_{i,k} \cong S_{0,n-i}, \quad 0 \le i \le n-s-1. \tag{6.3}$$

In fact, via obvious identifications, $\gamma_i : x \mapsto p^i x$ defines an isomorphism of R_{n-i}^+ onto $p^i R_n^+$. Now the monomorphism given by $\rho_i : \sigma_{\xi,t} \mapsto \gamma_i^{-1}\sigma_{\xi,t}\gamma_i$ defines an isomorphism ρ_i of $H_{i,0}$ onto $S_{0,n-i}$.

It follows from (6.2) that $\operatorname{cl} T_i = \operatorname{cl} S_{0,n-i}$. For our computations with elements in $S_{0,n}$ the following formula, established in [L, 3.2], turns out to be useful:

(6.4) Set $\sigma_h := \sigma_0^{p^h}$, $0 \le h \le n-s-1$. Then $[\sigma_{\xi,t}^{-1}, {}_r\sigma_h] = \prod_{k=0}^{r} \sigma_{\xi+kp^{s+h},t}^{(-1)^k\binom{r}{k}}$.

Lemma 6.2. *The nilpotent class of G/T_1' is less than or equal to p.*

Proof. It suffices to show that for $x_i \in \{\sigma_0, \sigma_{\xi,t} \mid \xi \in pR_n,\ 1 \le t \le n-s-1\}$, $[x_1, x_2, \dots, x_{p+1}] \in T_1'$. Since for $x_i \neq \sigma_0$ we have $x_i \in T_1$, it is enough to show that $[\sigma_{\xi,t}^{-1}, {}_p\sigma_0] \in T_1'$ as soon as $\xi \in pR_n$, $s \le t \le n-s-1$. We have

$$[\sigma_{\xi,t}^{-1}, {}_p\sigma_0] = \prod_{k=0}^{p} \sigma_{\xi+kp^s,t}^{(-1)^k\binom{p}{k}},$$

by (6.4). We claim that $\sigma_{\xi+kp^s,t}^{(-1)^k\binom{p}{k}} \in T_1'$ for $1 \le k \le p-1$. In fact, by (6.2) and (6.3), $T_1 \cong (S_{0,n-1})^p$, so the claim follows using Lemma 4.3 for $n-1$ and observing that $p \,|\binom{p}{k}$ for $1 \le k \le p-1$.

If p is odd, then $\sigma_{\xi,t}^{-1}\sigma_{\xi+p^{s+1},t} \in T_1'$ by (2.2), and we obtain the result. For $p=2$, we have

$$\sigma_{\xi,t}\sigma_{\xi+2^{s+1},t} = \sigma_{\xi,t}^2\sigma_{\xi,t}^{-1}\sigma_{\xi+2^{s+1},t} \in T_1',$$

by (2.2) and Lemma 4.3 (i). □

We remark that the proof shows that $\gamma_2(T_1) = [G, {}_p\sigma_0] = \gamma_{p+1}(G)$.

Theorem 6.3. *Let $M = H \oplus C$ be an (n,s)-group relative to the prime p. If $s < n \le 2s$, then $\operatorname{cl}\Gamma(M) = 1$ and $\exp\Gamma(M) = p^{n-s}$. If $2s < n$, then $\operatorname{cl}\Gamma(M) = p^{n-2s}$, $\exp\Gamma(M)/\gamma_2(\Gamma(M)) = p^s$ and $\exp\gamma_i(\Gamma(M))/\gamma_{i+1}(\Gamma(M)) = p$ for all $i \ge 2$.*

Proof. Since $\Gamma(M) \cong (S_{0,n})^{p+1}$, we may restrict our considerations to the group $G = S_{0,n}$ and, by (2.3), to the case $2s < n$. Finally, by Proposition 3.2 we may assume $s \ge 2$. Let us begin with $n = 2s+1$. Then $G = X_0X_1\cdots X_s$ with $X_s \trianglelefteq G$ by (4.2), and $X_0X_1\cdots X_{s-1}$ is abelian by (2.2). By (4.1), X_s is elementary abelian, and $\exp G/G' = p^s$ by Theorem 4.4 (i).

We shall now use induction on $n \ge 2s+2$. For $T = T_1$, let us consider the series

$$(*) \qquad G = \gamma_1(G) > \gamma_2(G) > \gamma_2(T) > \gamma_3(T) > \cdots > \gamma_{c'}(T) > 1$$

with $c' = \operatorname{cl} T$. Using (6.2) and (6.3) one sees that $(*)$ is a normal series of G and that $T \cong S_{0,n-1}^p$. By induction $c' = p^{n-1-2s}$, $\exp T/\gamma_2(T) = p^s$ and $\exp\gamma_i(T)/\gamma_{i+1}(T) = p$ for $i \ge 2$. By Theorem 4.4 (i), $\exp G/\gamma_2(G) = p^s$, and since $\gamma_2(T) \le \gamma_2(G) < T$, $\gamma_2(G)/\gamma_2(T)$ is abelian. We have $\sigma_{\xi,t} \in T$ for $1 \le t \le n-s-1$ and $\sigma_0^p \in T$. Applying Lemma 4.3 with n and $n-1$ one gets $\exp\gamma_2(G)/\gamma_2(T) = p$. Consider now the normal series $(*)$ as a $\langle\sigma_0\rangle$-series. By Lemma 6.2 we may refine the group G/T' in at most p steps to a lower $\langle\sigma_0\rangle$-central series with $\gamma_2(G/\gamma_2(T)) = G'/\gamma_2(T)$, because $G = \langle\sigma_0, T\rangle$. Since $\sigma_0^p \in T$, the elementary abelian p-group $\gamma_i(T)/\gamma_{i+1}(T)$, for $i \ge 2$, can be refined in at most p steps to a lower $\langle\sigma_0\rangle$-central series (see [L, 5.1]). In conclusion the normal series $(*)$ can be refined in at most $p\cdot p^{(n-1)-2s} = p^{n-2s}$

steps to a $\langle\sigma_0\rangle$-central series of G; call this series $(**)$. Since for $g \in G$ we have $g = \sigma_0^r x$, $x \in T$, $(**)$ turns out to be a central series of G. But each term of this series is generated by simple commutators of proper weight. Hence $(**)$ is the lower central series of G. In it, besides $\exp G/\gamma_2(G) = p^s$, all other factors are of exponent p. Since $G \geq \langle\sigma_0, X_{n-s-1}\rangle$ and, by (4.9), $\mathrm{cl}\langle\sigma_0, X_{n-s-1}\rangle = p^{n-2s}$, the conclusion follows. $\square$

We remark that the proof shows that $\gamma_{i+1}(T_1) = \gamma_{pi+1}(G)$ for $i = 1, \dots, p^{n-2s-1}$.

We describe the last non-trivial term of the lower central series of $\Gamma(M)$.

Corollary 6.4. *Let M be an (n,s)-group. Then $\gamma_c\Gamma(M) = \Omega(Z(\Gamma(M)))$, where $c = p^{n-2s}$.*

Proof. Again we may restrict ourselves to $G = S_{0,n}$. We already know that $\gamma_c(G) \leq \Omega(Z(G))$. In the other direction, by Proposition 4.5

$$X_{n-s-1} = \operatorname*{Dr}_{\xi\in[0,p^s)} X_{\xi,n-s-1},$$

where

$$X_{\xi,n-s-1} = \langle\sigma_{\xi,n-s-1}\rangle^G = \langle\sigma_{\xi,n-s-1}\rangle^{\langle\sigma_0\rangle} \cong C_p^{p^{n-2s}}.$$

It follows that

$$1 \neq g_\xi := \prod_{0\leq k<p^{n-2s}} \sigma_{\xi,n-s-1}^{\sigma_0^k} = \prod_{0\leq k<p^{n-2s}} \sigma_{\xi+kp^s,n-s-1} \in \Omega(Z(G)).$$

By order considerations we get $\Omega(Z(G)) = \operatorname*{Dr}_{\xi\in[0,p^s)} \langle g_\xi\rangle$ and, by (6.4),

$$[\sigma_{\xi,n-s-1}^{-1}, {}_{c-1}\sigma_0] = \prod_{k=0}^{c-1} \sigma_{\xi+kp^s}^{(-1)^k\binom{c-1}{k}} = g_\xi$$

since $(-1)^k\binom{c-1}{k} \equiv 1 \bmod p$ for $0 \leq k \leq c-1$. Hence $g_\xi \in \gamma_c(G)$, and we are done. $\square$

Given an (n,s)-group $M = H \oplus C$ and a basis (a,b) of H, we introduced in [CHZ] the frame $\mathcal{A} = (\langle a\rangle, \langle b\rangle)$, the unit point $u = \langle a+b\rangle$, and the subgroups

$$\Gamma_{\mathcal{A}}(M) = \{\rho \in \Gamma(M) \mid A^\rho = A\}, \quad \Gamma_{\mathcal{A},u}(M) = \{\rho \in \Gamma_{\mathcal{A}}(M) \mid u^\rho = u\}.$$

We are going to prove:

Corollary 6.5. *With the above notation we have*

$$\mathrm{cl}\,\Gamma_{\mathcal{A}}(M) = p^{n-2s}, \quad \mathrm{cl}\,\Gamma_{\mathcal{A},u}(M) = \begin{cases} p^{n-2s} & \text{if } p \text{ is odd,} \\ p^{n-1-2s} & \text{if } p = 2. \end{cases}$$

Proof. First we observe that $S_{1,n} \leq \Gamma_{\mathcal{A}}(M)$, and hence $\operatorname{cl}\Gamma_{\mathcal{A}}(M) = p^{n-2s}$. If p is odd, we have $S_{2,n} \leq \Gamma_{\mathcal{A},u}(M)$, and hence $\operatorname{cl}\Gamma_{\mathcal{A},u}(M) = p^{n-2s}$. Now assume $p = 2$. In this case $T_1 = H_0 \times H_1$, and $H_1 \leq \Gamma_{\mathcal{A},u}(M)$, so that $\operatorname{cl}\Gamma_{\mathcal{A},u}(M) \geq p^{n-1-2s}$. On the other hand, if we write $\Gamma(M) = S_{0,n} \times S_{1,n} \times S_{\infty,n}$, then it is clear that $\Gamma_{\mathcal{A},u}(M) = S_{0,n}(0) \times S_{1,n}(1) \times S_{\infty,n}(\infty)$, where $S_{k,n}(k)$ is the stabilizer of k in $S_{k,n}$. Hence $\operatorname{cl}\Gamma_{\mathcal{A},u}(M) = \operatorname{cl} S_{0,n}(0)$. Finally, $S_{0,n}(0) = \prod_{\eta \in J_0, \eta > 0} \Delta_\eta$ is contained in T_1, so that $\operatorname{cl}\Gamma_{\mathcal{A},u}(M) \leq \operatorname{cl} T_1 = p^{n-1-2s}$, and we are done. □

We finally give a bound for the nilpotent class of $R(M)$.

Corollary 6.6. *Let M be an (n,s)-group with $s \geq 2$. Then*

$$\operatorname{cl} R(M) \leq p^{n-2s}(s(p-1)+1).$$

Proof. By a result of P. Hall ([H, Theorem 7]) we have $\operatorname{cl} R(M) \leq \operatorname{cl}(R(M)/\Gamma(M)')p^{n-2s}$. Now $R(M)/\Gamma(M)'$ embeds in $A \wr C_p$, where A is abelian of exponent p^s. By [L, 5.1] we get $\operatorname{cl}(R(M)/\Gamma(M)') \leq s(p-1)+1$, and the proof is complete. □

We will now determine the nilpotent class of $\Gamma(M)$, when M is a proper (n,m,s)-group. Recall from (6.1) that $S(M)^j = (S(M)^j \cap S_{0,n}) \times S_{1,n} \times \cdots S_{p-1,n} \times S_{\infty,n}$. Set $\rho' : \Gamma(\tilde{M})^j \to \Gamma(\Omega_m(\tilde{M}))$, $\varphi^j \mapsto \varphi|\Omega_m(\tilde{M})$. Then, for $k = 1, \dots, p-1, \infty$, we have $\ker\rho \cap S_{k,n} = \ker\rho' \cap S_{k,n}$, so that, by Theorem 6.3, the class of action of $S_{k,n}$ on M is p^{m-2s}. Here, and in the following, we are using the convention that $p^h = 1$ if $h < 0$.

We note that with the help of (6.4) one has

$$[\sigma_{0,n-s-1}^{-1}, {}_{(p^{n-2s}-1)}\sigma_0]|p^{n-s}R_n \neq 1, \tag{6.5a}$$

$$[\sigma_{0,n-s-1}^{-1}, {}_{(p^{m-s}-1)}\sigma_0^{p^{n-m-s}}]|p^{n-s}R_n \neq 1 \quad \text{if } n-m > s. \tag{6.5b}$$

Proposition 6.7. *Let M be a proper (n,m,s)-group relative to the prime p. If $n-m \leq s$, then $\operatorname{cl}\Gamma(M) = p^{n-2s}$.*

Proof. Since $n-m \leq s$, we are in the case $S_{0,n} \leq S(M)^j$. If $n \leq 2s$, $\Gamma(\tilde{M})' = 1$ by (2.2), so $\Gamma(M)$ is abelian. Assume now that $n > 2s$. Since $\operatorname{cl}\Gamma(M) \leq \operatorname{cl}\Gamma(\tilde{M}) = p^{n-2s}$, the conclusion follows from (6.5a). □

It remains to deal with the case $s < n-m$. Here we already observed that $S(M) \cap \langle\sigma_0\rangle = \langle\sigma_0^{p^{n-m-s}}\rangle$. In particular, $G \cap S(M)^j \leq T_1$ and, since $H_{1,k}$ stabilizes M for every $k = 1, \dots, p-1$, we have more precisely

$$G \cap S(M)^j = (H_{1,0} \cap S(M)^j) \times H_{1,1} \times \cdots \times H_{1,p-1}. \tag{6.6}$$

Proposition 6.8. *Assume $0 \leq i < k \leq n-s$. Then $\operatorname{cl} T_i/T_k = p^{k-i-s}$.*

Proof. Set $r = n - i$, and let $0 \le j \le r$. The restriction map $\Gamma(\Omega_r(\tilde{M})) \to \Gamma(\Omega_{r-j}(\tilde{M}))$ induces an epimorphism $\varphi_j : S_{0,r} \to S_{0,r-j}$. Consider the sequence

$$H_{i,0} \xrightarrow{\rho_i} S_{0,r} \xrightarrow{\varphi_j} S_{0,r-j}$$

Then, by Theorem 6.3, we get

$$\operatorname{cl} H_{i,0}/\ker \rho_i\varphi_j = \operatorname{cl} S_{0,r}/\ker \varphi_j = \operatorname{cl} S_{0,r-j} = p^{r-j-2s} = p^{n-i-j-2s}.$$

With the help of the relation $\sigma^{\rho_i}_{p^i\eta,t} = \sigma_{\eta,t-i}$ one checks that

$$(H_{i,0}/\ker \rho_i\varphi_j)^{p^i} \cong \operatorname*{Dr}_{0\le k<p^i} H_{i,k}/(\ker \rho_i\varphi_j)^{\tau(0,k)} = T_i/T_{n-j-s},$$

so that $\operatorname{cl} T_i/T_{n-j-s} = p^{n-i-j-2s}$. So for $k = n - j - s$ we have $\operatorname{cl} T_i/T_k = p^{k-i-s}$. □

We are now in a position to prove the main result of this section.

Theorem 6.9. *Let $M = H \oplus C$ be a proper (n, m, s)-group relative to the prime p. If $n \le 2s$, then $\Gamma(M)$ is abelian. If $n > 2s$ the nilpotent class of $\Gamma(M)$ is given by*

$$\operatorname{cl}\Gamma(M) = \begin{cases} p^{n-2s} & \text{if } n - m \le s,\\ p^{m-s} & \text{if } n - m > s.\end{cases}$$

Proof. By our previous results, it remains to deal with the case $n - m > s$ (which implies $n > 2s$). Since $\operatorname{cl}\Gamma(M)$ is determined by the action of $S(M)^j$ on M, by (6.6) we may consider the action of $A := H_{1,1} \times \cdots \times H_{1,p-1}$ and that of $B := H_{1,0} \cap S(M)^j$ separately. As already pointed out, we have

$$\operatorname{cl} A/\ker\rho \cap A \le \operatorname{cl}\Gamma(\Omega_m(\tilde{M})) = p^{m-2s} < p^{m-s}. \tag{6.7}$$

It remains to work out the nilpotent class of the action of B on M. To generate B, according to Remark 6.1, we may restrict ourselves to those $\sigma_{\xi,t} \in H_{1,0}$ with $\xi \in p^2R_n$ and $v(\xi) \le t$. Assume $\xi \in p^iR_n \setminus p^{i+1}R_n$, where $2 \le i \le n - m - s$. Then $0\sigma_{\xi,t} = 0$, and hence $\sigma_{\xi,t} \in B$, i.e.,

$$R_i := \langle \sigma_{\xi,t} \mid \xi \in p^iR_n \setminus p^{i+1}R_n,\ t \ge i\rangle \ \le\ B.$$

Finally, if $\xi \in p^{n-m-s+1}R_n$ and $t \ge n - m - s$, then $0\sigma_{\xi,t} \in p^{n-m}R_n$, so that

$$R_0 := \langle \sigma_{\xi,t} \mid \xi \in p^{n-m-s+1}R_n,\ t \ge n - m - s\rangle \le B \cap T_{n-m-s};$$

in particular, $\operatorname{cl} R_0 \le p^{m-s}$ by Proposition 6.8.

We obtained $B = \langle R_i \mid i = 2, \ldots, n - m - s, 0\rangle$, which is the direct product $R_2 \times \cdots \times R_{n-m-s} \times R_0$ since if $\sigma_{\xi,t} \in R_i$ and $\sigma_{\xi',t'} \in R_j$ with $i \ne j$, then $\xi + p^{t+1}R_n \cap \xi' + p^{t'+1}R_n = \emptyset$. For $i = 2, \ldots, n - m - s$ set

$$V_i = \langle \sigma_{\xi,t} \mid \xi \in p^iP_n \setminus p^{i+1}R_n,\ t \ge m - s + i\rangle.$$

Then $V_i \leq \ker\rho \cap R_i$, and since R_i/V_i embeds in T_i/T_{m-s+i}, it follows by Proposition 6.8 that $\operatorname{cl} R_i/\ker\rho \cap R_i \leq p^{m-2s}$. Thus the nilpotent class of the action of B on M is $\leq p^{m-s}$, from which it follows by (6.7) that $\operatorname{cl}\Gamma(M) \leq p^{m-s}$. But then we conclude that $\operatorname{cl}\Gamma(M) = p^{m-s}$ by (6.5b). □

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