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THE EXPONENTIAL MAP OF A WEAK RIEMANNIAN HILBERT MANIFOLD

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Abstract. We prove the Focal Index Lemma and the Rauch–Berger Comparison Theorems on a weak Riemannian Hilbert manifold with a smooth Levi-Civita connection and we apply these results to the free loop space $\Omega(M^n)$ with the L^2 (weak) Riemannian structure.

1. Introduction

As a first step towards understanding the global geometry of a Riemannian Hilbert manifold M one can study the singularities of its exponential map. Singular values of exp are the conjugate points in M. In infinite dimension, there exist two types of conjugate points: when the differential of the exponential map fails to be injective (a monoconjugate point) or when the differential of the exponential map fails to be surjective (an epiconjugate point). More generally, let N be a submanifold of M such that for all $p \in N$ the tangent space at p of N, T_pN, is a closed subspace of T_pM. Singular values of the map $\mathrm{Exp}^\perp : T^\perp N \longrightarrow M$, defined by $\mathrm{Exp}^\perp(X) = \exp(X)$, where exp is the exponential map of M and $T^\perp N$ is the normal bundle of N, are called focal points. A focal point is a *monofocal point*, when the differential fails to be injective, and an *epifocal point*, when the differential fails to be surjective. Clearly, the map $\mathrm{Exp}^\perp$ is defined a priori only in an open subset which contains the "zero section", i.e., the subset $\{0_p \in T_p^\perp N : \ p \in N\} \subseteq T^\perp N$.

Let now $(M, \langle\cdot,\cdot\rangle)$ be a *weak* Riemannian Hilbert manifold with a smooth Levi-Civita connection ∇, whose existence is not guaranteed a priori. It defines parallel transport, the curvature tensor R, the geodesics and a smooth exponential map. These manifolds have been extensively studied and they have found many diverse applications, in particular in geometry, the calculus of variations and mathematical physics (see [3], [5], [7], [15], [18], [17], [19],

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[20]). For example (see [3], [5], [18], [19]), any motion of a perfect fluid corresponds to a geodesic on the group of volume-preserving diffeomorphisms of a compact manifold M, namely the region filled with fluid, with respect to the weak Riemannian metric given by the L^2 inner product on each tangent space. Moreover, the existence of conjugate points is related to the stability of the fluid flows of M.

Another important example is the free loop space $\Omega(M^n)$ of a compact manifold M^n, which is among the simplest Hilbert manifolds. This Hilbert manifold has been extensively studied and it has many diverse applications (see [7], [17], [20]). It has an L^2 metric that is a weak Riemannian structure, which induces a smooth Levi-Civita connection and a smooth exponential map.

One motivation for the results presented here was the paper of Misiolek [17], where it was proved that for every $s > 0$ the exponential map of the H^s metric on $H^{s_o}(S^1, G)$, i.e., the set of Sobolev H^{s_o} maps from the unit circle S^1 into a compact, connected Lie group G, is a nonlinear Fredholm map of zero index, while the exponential map of $\Omega(\mathrm{SU}(2))$ with respect to the L^2 weak Riemannanian metric is not of this type.

Since the model space $\mathbb{H}$ on which M is modeled is a Hilbert space, it is possible to transport to the tangent space the structure of a topological vector space, which we will denote by τ, given by the chart, and this topology can be induced by a scalar product (see [13, p. 26]). We assume that the curvature tensor R is a trilinear continuous operator with respect to the topology τ.

Let N be a submanifold of M such that for some $p \in N$, T_pN is a closed subspace of (T_pM, τ) and $T_pM = T_pN \oplus T_p^{\perp}N$. In this context, we shall define the notion of focal point along a normal geodesic starting from p, which is equivalent to the usual one in Riemannian geometry. We shall prove the Focal Index Lemma when there exist a finite number of epifocal points which are not monofocal along a geodesic of finite length. This generalizes the Index Lemmas (see [4, p. 24]) in finite dimensional Riemannian geometry. As immediate corollaries we obtain the Rauch–Berger Comparison Theorems.

After formulating and proving the Focal Index Lemma and its corollaries, we apply this result, in Section 4, to the loop group $\Omega(M^n)$. We prove that a geodesic $c : [0, b] \longrightarrow \Omega(M^n)$ with large enough length has conjugate points and that its index is infinite. A similar result can be proved for focal points of $c(0)$ along c with respect to the geodesic submanifold defined by $\dot{c}(0)$. Then we analyze the case when $M^n = G$ is a non-abelian compact Lie group and we prove that its exponential map fails to be Fredholm. Moreover, we give an example of a submanifold N of $\Omega(G)$ such that $\mathrm{Exp}^{\perp} : T^{\perp}N \longrightarrow M$ fails to be Fredholm as well.

2. The exponential map on Hilbert manifolds

In this section we recall some general results and well known facts. Our basic references are [1], [11] and [13].

Let M be a Hilbert manifold modeled on an infinite dimensional Hilbert space $\mathbb{H}$. Recall that a weak Riemannian metric on M is a smooth assignment to each point $p \in M$ of a continuous, positive definite, symmetric bilinear form $p \longrightarrow \langle\cdot,\cdot\rangle(p)$ on the tangent space T_pM. Note that $T_pM \cong \mathbb{H}$ need not be complete as a metric space under the distance induced by $\langle\cdot,\cdot\rangle(p)$. Consequently, the existence of a smooth Levi-Civita connection ∇ associated with a weak Riemannian metric is not immediately guaranteed. If, however, such a connection exists, it is necessarily unique.

Throughout this paper we assume that M is a Hilbert manifold endowed with a weak Riemannian metric $\langle\cdot,\cdot\rangle$, and M will be called a weak Riemannian Hilbert manifold. We further assume that M admits a Levi-Civita connection ∇, whose curvature tensor R is a continuous trilinear operator of the tangent space with respect to the topology τ.

For any $p \in M$ the exponential map $\exp_p : T_pM \longrightarrow M$ is a local diffeomorphism in a neighborhood of the origin in T_pM. The differential $d(\exp_p)$ can be computed using the Jacobi equation, that is, the linearized version of the geodesic equation.

Let $c : [0,b] \longrightarrow M$ be a geodesic. A vector field J along c is called Jacobi field if it satisfies the Jacobi differential equation

$$\nabla_{\partial/\partial t}\nabla_{\partial/\partial t}J(t) + R(J(t),\dot{c}(t))\dot{c}(t) = 0,$$

where $\nabla_{\partial/\partial t}$ denotes the covariant derivation along c.

It is well known that if $c(t) = \exp_p(tv)$ is the geodesic starting at p in the direction v, then the vector field $Y(t) = d(\exp_p)_{tv}(tw)$ satisfies the Jacobi differential equations with initial values $Y(0) = 0$ and $\nabla_{\partial/\partial t}Y(0) = w$.

Let N be a submanifold of M and let $c : [0,b] \longrightarrow M$ be a geodesic such that $c(0) = p \in N$ and $\xi = \dot{c}(0) \in T_p^{\perp}N$, i.e., c is a normal geodesic of N. Suppose also that T_pN is a closed subspace of (T_pM,τ) and $T_pM = T_pN \oplus T_p^{\perp}N$. This happens when N is a submanifold of M defined by some vector $v \in T_pM$, i.e., $N = \exp_p(B_\epsilon(0_p) \cap \langle v\rangle^{\perp})$, where ϵ is sufficiently small such that the map $\exp_p : B_\epsilon(0_p) \longrightarrow M$ is a diffeomorphism onto the image.

As in the Riemannian case, the Weingarten operator is given by $A_\xi(X) = -P(\nabla_X\xi(p))$, where P is the projection of T_pM onto T_pN. Of course, the Weingarten operator is a linear continuous map of (T_pM,τ) and symmetric with respect to $\langle\cdot,\cdot\rangle$. In finite dimensional Riemannian geometry, the Jacobi fields along c with initial values

$$J(0) \in T_pN,\ \nabla_{\partial/\partial t}J(0) + A_\xi(J(0)) \in T_p^{\perp}N,$$

which are called $N-$Jacobi fields, describe completely the differential of the map $\mathrm{Exp}^{\perp}$.

Now, let $\tau_t^s : T_{c(t)}M \longrightarrow T_{c(s)}M$ be the isomorphism between the tangent spaces given by the parallel transport along a geodesic c. Since the parallel transport along c and $\nabla_{\partial/\partial t}$ commute, we can rewrite the Jacobi equation relative to N as an initial value problem on T_pM as follows:

$$\begin{cases} T''(t) \ + \ R_t(T(t)) = 0, \\ T(0)(v,w) = (v,0), \ T'(0)(v,w) = (-A_\xi(v), w), \end{cases}$$

where

$$R_t : T_pM \longrightarrow T_pM, \quad R_t(X) = \tau_t^0(R(\tau_0^t(X), \dot{c}(t))\dot{c}(t))$$

is a family of symmetric operators of T_pM. We will call the above differential equation *Jacobi flow* of c relative to N. Note also that all maps $\Phi(t)$ defined by

$$\begin{aligned} T_pM \times T_pM &\xrightarrow{\Phi(t)} \mathbb{R} \\ (u,v) &\longrightarrow \langle T(t)(u), T'(t)(v)\rangle \end{aligned}$$

are symmetric; indeed, $\Phi(0)$ is symmetric since A_ξ is a symmetric operator, and

$$(\langle T(t)(u), T'(t)(w)\rangle \ - \ \langle T(t)(w), T'(t)(u)\rangle)' = 0.$$

A point $q = c(t_o)$ is called a *monofocal point*, respectively an *epifocal point*, of $p = c(0)$ along c if $T(t_o)$ fails to be injective, respectively fails to be surjective. In general, we call a point $q = c(t_o)$ a *focal point* of p along c when $T(t_o)$ is not an isomorphism. Note that this definition is equivalent to the Riemannian one.

Now, let $\mathbb{E}_1$ and $\mathbb{E}_2$ be Banach spaces. A bounded linear operator $T : \mathbb{E}_1 \longrightarrow \mathbb{E}_2$ is called Fredholm if it has a closed range and its kernel and cokernel ($\operatorname{coker} T = \mathbb{E}_2/T(\mathbb{E}_1)$) are finite dimensional. The index of T is the number $\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{coker} T$.

A smooth map between Banach manifolds $f : M \longrightarrow S$ is called Fredholm if for each $p \in M$ the derivative $d(f)_p : T_pM \longrightarrow T_{f(p)}N$ is a Fredholm operator. If M is connected, then $\operatorname{ind}(df)_p$ is independent of p, and one defines the index of f by setting $\operatorname{ind}(f) = \operatorname{ind}(df)_p$ (see [6], [23]). Note that if the map $\mathrm{Exp}^{\perp}$ is well defined, then $\mathrm{Exp}^{\perp}$ is a nonlinear Fredholm map if the Jacobi flow along any normal geodesic of N describes a curve in the Fredholm operators for every $t > 0$.

We now describe the adjoint operator of $T(b)$ in order to understand the behavior of the focal points of $c(0)$ along the geodesic c.

Let $u \in T_pM$ and let J be the Jacobi field along the geodesic c such that $J(b) = 0$, $\nabla_{\partial/\partial t} J(b) = \tau_0^b(u)$. By a lemma of Ambrose (see [2] or [13, Lemma

3.4 p. 243]) we have

$$\langle T(b)(v,w),u\rangle = \langle T(0)(v,w), \quad \nabla_{\partial/\partial t}J(0)\rangle \;-\; \langle T'(0)(v,w),J(0)\rangle. \tag{1}$$

Let $\overline{c}(t) = c(b-t)$. Let

$$\begin{cases} \tilde{T}''(t) \;+\; R_t(\tilde{T}(t)) = 0, \\ \tilde{T}(0) = 0,\; \tilde{T}'(0) = \mathrm{id}, \end{cases}$$

be the Jacobi flow of $\overline{c}$ relative to the submanifold $\overline{N} = \{\overline{c}(0)\}$. It is easy to check that if J is a Jacobi field along c, then $\overline{J}(t) = J(b-t)$ is the Jacobi field along $\overline{c}$ such that $\nabla_{\partial/\partial t}\overline{J}(b) = -\nabla_{\partial/\partial t}J(0)$. Then (1) becomes

$$\begin{aligned}\langle T(b)(v,w),u\rangle = \langle (v,0), \tau_b^0(\tilde{T}'(b)(-\tau_0^b(u)))\rangle \\ - \langle(-A_\xi(v),w),\tau_b^0(\tilde{T}(b)(-\tau_0^b(u)))\rangle,\end{aligned}$$

so the adjoint operator is given by

$$\begin{aligned}\langle T^*(b)(u),(v,0)\rangle &= -\langle \tau_b^0(\tilde{T}'(b)(\tau_0^b(u))),(v,0)\rangle \\ &\quad + \langle A_\xi(P(\tau_b^0(\tilde{T}(b)(\tau_0^b(u))))),(v,0)\rangle, \\ \langle T^*(b)(u),(0,w)\rangle &= \;\langle \tau_b^0(\tilde{T}(b)(\tau_0^b(u))),(0,w)\rangle.\end{aligned}$$

PROPOSITION 2.1. *The kernel of $T(b)$ and the kernel of $T^*(b)$ are isomorphic.*

Proof. Let $w \in T_pM$ be such that $T(b)(w) = 0$. The Jacobi field $Y(t) = \tau_0^t(T(t)(w))$ vanishes at $t = b$. Therefore there exists a unique $w \in T_{c(b)}M$ such that

$$Y(b-t) = \tau_b^{b-t}(\tilde{T}(t)(\overline{w})).$$

Using the boundary conditions of $Y(t)$ we get $T^*(b)(\tau_b^0(\overline{w})) = 0$. In particular, the map

$$\begin{aligned} f_1 : \operatorname{Ker} T(b) &\longrightarrow \operatorname{Ker} T^*(b) \\ w &\longrightarrow \tau_b^0(\overline{w}) \end{aligned}$$

is an injective linear map.

Conversely, let $v \in \operatorname{Ker} T^*(b)$. We set $\overline{v} = \tau_0^b(v)$ and consider the Jacobi field $Y(t) = \tau_b^{b-t}(\tilde{T}(t)(\overline{v}))$ along $\overline{c}$. Since $T^*(b)(v) = 0$, there exists a unique $\theta \in T_pM$ such that $Y(b-t) = \tau_0^t(T(t)(\theta))$. Hence $T(b)(\theta) = 0$ since $Y(0) = 0$. Moreover, the map

$$\begin{aligned} f_2 : \operatorname{Ker} T^*(b) &\longrightarrow \operatorname{Ker} T(b) \\ v &\longrightarrow \theta \end{aligned}$$

is injective. Noting that $f_2 \circ f_1 = \mathrm{Id}$, we conclude our proof. $\square$

Proposition 2.2. *Let $(M, \langle \cdot, \cdot \rangle)$ be a weak Riemannian Hilbert manifold with a smooth Levi-Civita connection ∇. Let N be a submanifold of M and let $c : [0, b] \longrightarrow M$ be a normal geodesic of N, i.e., $c(0) = p \in N$ and $\xi = \dot{c}(0) \in T_p^{\perp} N$. Assume also that $T_p N$ is a closed subspace of $(T_p M, \tau)$ such that $T_p M = T_p N \oplus T_p^{\perp} N$. Then we have:*

(1) *If $c(t_o)$ is not a monofocal of p along c, then the image of $T(t_o)$ is a dense subspace relative to the topology τ induced by the metric $\langle \cdot, \cdot \rangle(p)$.*
(2) *If $c(t_o)$ is a monofocal of p along c, then $c(t_o)$ is an epifocal of p along c.*
(3) *When $N = \{p\}$, a point $q = c(t_o)$ is a monofocal of p along c if and only if p is a monofocal point of q along $\overline{c}(t) = c(t_o - t)$.*
(4) *When $N = \{p\}$, if a point $q = c(t_o)$ is an epifocal of p along c and the image of $T(t_o)$ is a closed subspace of $(T_p M, \tau)$, then p is a monofocal point of q along $\overline{c}(t) = c(t_o - t)$;*

Proof.

(1) Since $\operatorname{Ker} T^*(t_o) = 0$, by Proposition 2.1, the closure of $\operatorname{Im} T(t_o)$ with respect to τ satisfies $\overline{\operatorname{Im} T(t_o)}^{\perp} = 0$. Now, noting that τ makes $T_p M$ into a locally convex space, and applying Theorems 3.10 and 3.5 in [21], we obtain the statement.
(2) This easily follows from (1).
(3) Note that the adjoint of the Jacobi flow of c is the Jacobi flow of $\overline{c}$.
(4) Since the image of $T(t_o)$ is closed, q is a monofocal point of p along c as well. Now, the statement follows from the above item. $\square$

3. The Focal Index Lemma and the Rauch–Berger Comparison Theorems in weak Riemannian geometry

In the infinite dimensional case, the distribution of singular points of the exponential map along a geodesic of finite length is different from that in the finite dimensional case. Indeed, Grossman [8] showed that the set of monoconjugate points can have cluster points. The following example proves that the same situation may occur in the case of focal points along a geodesic of finite length.

Example 3.1. Let $M = \{x \in l_2 : \; x_1^2 \; + \; x_2^2 \; + \; \sum_{i=3}^{\infty} a_i x_i^2 = 1\}$, where $(a_i)_{i \in \mathbb{N}}$ is a positive sequence of real numbers. M is a Riemannian Hilbert manifold and it is easy to check that

$$\gamma(s) = \sin(s) e_1 \; + \; \cos(s) e_2$$

is a geodesic and $T_{\gamma(s)} M = \langle \dot{\gamma}(s), e_3, e_4, \ldots \rangle$. Let N be a submanifold defined by $\dot{\gamma}(0)$. We shall restrict ourselves to the normal $N-$Jacobi fields, i.e., the

Jacobi fields which satisfy $\langle J(0), \dot{c}(0)\rangle = 0$. Since for $k \geq 3$

$$E_k := \{x_1^2 \ + \ x_2^2 \ + \ a_k x_k^2 = 1 \ \} \hookrightarrow M$$

is a closed totally geodesic submanifold of M and the sectional curvature of the plane $\langle \dot{\gamma}(s), e_k\rangle$ is given by $K(\dot{\gamma}(s), e_k) = a_k$. Consequently the Jacobi fields with boundary conditions $J_k(0) = e_k$, $\nabla_{\partial/\partial t} J_k(0) = 0$, are given by $J_k(t) = \cos(\sqrt{a_k} t) e_k$. Hence

$$d(\mathrm{Exp}^{\perp})_{s\dot{\gamma}(0)} \left(\sum_{k=3}^{\infty} b_k e_k \right) = \sum_{k=3}^{\infty} b_k \cos(\sqrt{a_k} s) e_k.$$

Clearly, the points $\gamma(r_k^m)$, where $r_k^m = m\pi/(2\sqrt{a_k})$, $m \in \mathbb{N}$, are monofocal of e_2 along γ. Specifically, let $a_k = (1 - 1/k)^2$. The points $\gamma(s_k)$, where $s_k = k\pi/(2(k-1))$, are monofocal of e_2 along γ, $s_k \to \pi/2$ and

$$d(\mathrm{Exp}^{\perp})_{\frac{\pi}{2}\dot{\gamma}(0)} \left(\sum_{k=3}^{\infty} b_k e_k \right) = \sum_{k=3}^{\infty} b_k \cos\left(\frac{k-1}{k} \frac{\pi}{2} \right) e_k.$$

Hence $\gamma(\pi/2)$ is not monofocal of e_2 along γ. On the other hand, if

$$\sum_{k=3}^{\infty} \frac{1}{k} e_k = d(\mathrm{Exp}^{\perp})_{\frac{\pi}{2}\dot{\gamma}(0)} \left(\sum_{k=3}^{\infty} b_k e_k \right),$$

then $\sin(\pi/(2k)) b_k = 1/k$, so we have

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{\pi}{2k} \frac{1}{\sin(\frac{\pi}{2k})} \frac{2}{\pi} = \frac{2}{\pi}.$$

This means that $\gamma(\pi/2)$ is an epifocal point of e_2 along γ.

This example shows that there exist epifocal points which are not monofocal. We call them *pathological points*. Clearly, if the exponential map is a non-linear Fredholm map, and therefore necessarily of zero index, monoconjugate points and epiconjugate points along geodesics coincide. This holds for the Hilbert manifold $\Omega(M^n)$ with the H^1 Riemannian structure (see [16]).

Now we shall prove the Focal Index Lemma.

Let N be a submanifold of M and let $c : [0, b] \longrightarrow M$ be a geodesic of M. Assume that $c(0) = p \in N$, $\xi = \dot{c}(0) \in T_p^{\perp} N$, $T_p N$ is a closed subspace of $(T_p M, \tau)$ and finally $T_p M = T_p N \oplus T_p^{\perp} N$.

Let $X : [0, b] \longrightarrow T_p M$ be such that $X(0) \in T_p N$. We define the *focal index form* of X as follows:

$$\begin{aligned} I^N(X, X) = \int_0^b \langle \dot{X}(t), \dot{X}(t)\rangle - \langle R_t(X(t)), X(t)\rangle dt \\ - \langle A_\xi(X(0)), X(0)\rangle. \end{aligned}$$

Note that any vector field along c is the parallel transport of a unique map $X : [0,b] \longrightarrow T_pM$. We will denote by $\overline{X}(t) = \tau_0^t(X)$ the vector field along c starting from X.

LEMMA 3.2. *We have $I^N(X,X) = D^2E(c)(\overline{X},\overline{X})$, where $D^2E(c)$ is the index form of $B = N \times M \hookrightarrow M \times M$.*

Proof. We recall that (see [22])

$$\begin{aligned} D^2E(c)(\overline{X},\overline{X}) = \int_0^b \parallel \nabla_{\partial/\partial t}\overline{X}(t) \parallel^2 \; - \langle \overline{X}(t), R(\overline{X}(t), \dot{c}(t))\dot{c}(t)\rangle dt \\ - \langle\langle A_{(\dot{c}(0),-\dot{c}(b))}(\overline{X}(0),\overline{X}(b)), (\overline{X}(0),\overline{X}(b))\rangle\rangle, \end{aligned}$$

where A is the Weingarten operator of $N \times M \hookrightarrow M \times M$ and $\langle\langle\cdot,\cdot\rangle\rangle$ is the natural weak Riemannian structure on $M \times M$ induced by $\langle\cdot,\cdot\rangle$. Hence, it is enough to prove that $\nabla_{\partial/\partial t}\overline{X}(t) = \tau_0^t(\dot{X}(t))$. Let $Z(t)$ be a parallel transport of a vector $Z \in T_pM$. Then

$$\begin{aligned} \langle \nabla_{\partial/\partial t}\overline{X}(t), Z(t)\rangle &= \langle \overline{X}(t), Z(t)\rangle' \\ &= \langle \dot{X}(t), Z\rangle \\ &= \langle \tau_0^t(\dot{X}(t)), Z(t)\rangle. \end{aligned}$$

□

LEMMA 3.3 (Focal Index Lemma). *Let $c : [0,b] \longrightarrow M$ be a geodesic with a finite number of pathological points on its interior. Then for every vector field Z along c with $Z(0) \in T_pN$, the index form of X relative to the submanifold $N \times M \hookrightarrow M \times M$ satisfies $D^2E(c)(Z,Z) \geq D^2E(c)(J,J)$, where J is the N-Jacobi field such that $J(b) = X(b)$.*

Proof. We first assume that there are no focal points of $c(0)$ along c.

We know that $Z = \overline{X}$, where $X : [0,b] \longrightarrow T_pM$. Since $T(t)$ is invertible, there exists a piecewise differentiable map $Y : [0,b] \longrightarrow T_pM$ such that $Y(0) = X(0) \in T_pN$ and $X(t) = T(t)(Y(t))$. Hence

$$\dot{X}(t) = T'(t)(Y(t)) \; + \; T(t)(\dot{Y}(t)) = A(t) \; + \; B(t).$$

The focal index form of X is given by

$$\begin{aligned} I^N(X,X) &= \int_0^b \langle A(t), A(t)\rangle \; + \; 2\langle A(t), B(t)\rangle \; + \; \langle B(t), B(t)\rangle dt \\ &= \int_0^b \langle R_t(T(t)(Y(t))), T(t)(Y(t))\rangle dt \; - \; \langle A_\xi(X(0)), X(0)\rangle. \end{aligned}$$

One can prove that

$$\begin{aligned}\langle A(t), A(t)\rangle = \langle T(t)(Y(t)), T'(t)(Y(t)\rangle' \\ - 2\langle B(t), A(t)\rangle \\ + \langle T(t)(Y(t)), R_t(T(t)(Y(t)))\rangle,\end{aligned}$$

since the bilinear form $\Phi(t)$ is symmetric. Hence, the focal index form of X is given by

$$I^N(X,X) \;=\; \langle T(1)(u), T'(1)(u)\rangle \;+\; \int_0^b \| \, T(t)(\dot{Y}(t)) \, \|^2 \, dt.$$

This proves the Focal Index Lemma in this case. Moreover, if there are no focal points along c, the focal index of a vector field Z, with $Z(0) \in T_pN$, along c is equal to the focal index of the $N-$Jacobi field J along c such that $Z(b) = J(b)$ if and only if $Z = J$.

Now, assume that there exists a pathological point in the interior of c. This means that the Jacobi flow is an isomorphism for every $t \neq t_o$ in $(0, b)$, and when $t = t_o$, by Proposition 2.2 (1), $T(t_o)$ is a linear operator whose image is a dense subspace. Let $X : [0, b] \longrightarrow T_pM$ be a piecewise differentiable map with $X(0) \in T_pN$. Given $\epsilon > 0$, there exist X_n^ϵ, $n = 1, 2$, such that

$$\begin{aligned}\| \, T(t_o)(X_1^\epsilon)) - X(t_o)) \, \| \leq \frac{\epsilon}{4}, \\ \| \, T(t_o)(X_2^\epsilon)) - \dot{X}(t_o)) \, \| \leq \frac{\epsilon}{4}.\end{aligned}$$

Choose Y^ϵ such that

$$\| \, T(t_o)(Y^\epsilon) - T'(t_o)(X_1^\epsilon) \, \| \leq \frac{\epsilon}{4}.$$

Hence there exists $\eta(\epsilon) \leq \epsilon/2$ such that for $t \in (\eta(\epsilon) - t_o, \eta(\epsilon) + t_o)$ we have

$$\| \, T(t)(X_1^\epsilon + (t - t_o)(X_2^\epsilon - Y^\epsilon)) - X(t) \, \| \leq \epsilon, \tag{1}$$

$$\| \, \frac{d}{dt}(T(t)(X_1^\epsilon + (t - t_o)(X_2^\epsilon - Y^\epsilon))) - \dot{X}(t) \, \| \leq \epsilon. \tag{2}$$

We denote by X^ϵ the map

$$X^\epsilon(t) = \begin{cases} X(t) & \text{if } \; 0 \leq t \leq t_o \; - \; \eta(\epsilon), \\ T(t)(X_1^\epsilon + (t - t_o)(X_2^\epsilon - Y^\epsilon)) & \text{if } t_o - \eta(\epsilon) < t < t_o + \eta(\epsilon), \\ X(t) & \text{if } t_o - \eta(\epsilon) \leq t \leq b. \end{cases}$$

Since $X^\epsilon = T(t)(Y(t))$, where $Y(t)$ is piecewise differentiable map, except at the points $t = t_o + (\eta(\epsilon)$ and $t = t_o - \eta(\epsilon)$, we have

$$I^N(X^\epsilon, X^\epsilon) \geq I^N(T(t)(u), T(t)(u)).$$

On the other hand, the Focal Index of X is given by

$$\begin{aligned} I(X,X) = I(X^\epsilon, X^\epsilon) & \\ & - \int_{t_o-\eta(\epsilon)}^{t_o+\eta(\epsilon)} \langle \dot{X}^\epsilon(t), \dot{X}^\epsilon(t)\rangle \ - \ \langle R(X^\epsilon(t), \dot{c}(t))\dot{c}(t), X^\epsilon(t)\rangle dt \\ & + \int_{t_o-\eta(\epsilon)}^{t_o+\eta(\epsilon)} \langle \dot{X}(t), \dot{X}(t)\rangle \ - \ \langle R(X(t), \dot{c}(t))\dot{c}(t), X(t)\rangle dt. \end{aligned}$$

Now, using (1) and (2) it is easy to check that

$$\lim_{\epsilon\longrightarrow 0} I^N(X^\epsilon, X^\epsilon) = I^N(X,X) \geq I^N(J,J),$$

where $J(t) = T(t)(u)$. This proves the Focal Index Lemma if there is only one pathological point. However, one can generalize easily the above proof to a finite number of pathological points. □

COROLLARY 3.4. *Let $(M, \langle\cdot,\cdot\rangle)$ be a weak Riemannian Hilbert manifold and let S and Σ be two submanifolds of codimension 1. Assume that there exists $p \in S\cap\Sigma$ such that $T_p\Sigma = T_pS$ is a closed subspace of (T_pM, τ). We denote by N and $\overline{N}$ the normal vector fields to S and Σ, respectively. Suppose also that*

$$\langle \nabla_X N, X\rangle < \langle \nabla_X \overline{N}, X\rangle$$

for every $X \in T_p\Sigma = T_pS$. Then, if the Jacobi flow T of S is invertible in $(0,b)$, the Jacobi flow of Σ must be injective in $(0,b)$. Moreover, if M is a Riemannian Hilbert manifold, assuming $A - \overline{A}$ is invertible, where A and $\overline{A}$ are the Weingarten operators at p of S and Σ, respectively, then the Jacobi flow of Σ is also invertible in $(0,b)$.

Proof. Let $s \in (0,b)$ and let $Y(t)$ be a Σ-Jacobi field. Since $T(t)$ is invertible in $(0,b)$, there exists a piecewise differentiable map $X : [0,s] \longrightarrow T_pM$ with $X(0) \in T_pS$ such that $Y(t) = T(t)(X(t))$. Hence

$$\begin{aligned} Y(0) &= T(0)(X(0)), \\ \dot{Y}(0) &= T'(0)(X(0)) \ + \ T(0)(\dot{X}(0)), \\ (-\overline{A}(Y(0)), P_n(\dot{Y}(0))) &= (-A(X(0)) \ + \ P(\dot{X}(0)), 0), \end{aligned}$$

where P_n is the projection of T_pM onto $T_p^\perp\Sigma$. Therefore, $Y(0) = X(0)$ and the tangent component of $\dot{X}(0)$ is given by $(A \ - \ \overline{A})(X(0))$. Then,

$$\begin{aligned} \langle Y(s), \nabla_{\partial/\partial s} Y(s)\rangle &= I^S(Y,Y) \\ &= \langle (A-\overline{A})(X(0)), X(0)\rangle + \int_0^s \parallel T(t)(\dot{X}(t)) \parallel^2 dt \\ &> 0. \end{aligned}$$

In particular, the Jacobi flow of Σ is injective in $(0,b)$. If $A-\overline{A}$ is invertible, then $\frac{d}{ds}\langle Y(s),Y(s)\rangle \geq \| (A-\overline{A})^{1/2} \|^{-1} \| Y(0) \|^2$, so $\| T(s)(w) \|^2 \geq C \| w \|^2$. In the Riemannian context, one can prove that the image of the Jacobi flow relative to Σ is a closed subspace for every $s \in (0,b)$, so the Jacobi flow must be invertible in $(0,b)$ since $(\operatorname{Im} T(s))^{\perp} = \operatorname{Ker} T^*(s) = 0$, by Proposition 2.1. □

THEOREM 3.5. *Let $(M,\langle\cdot,\cdot\rangle)$, $(N,\langle\cdot,\cdot\rangle^*)$ be weak Riemannian Hilbert manifolds with Levi-Civita connections, modeled on $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively, with $\mathbb{H}_1$ isometric to a closed subspace of $\mathbb{H}_2$. We denote by $K^M(X,Y)$ (respectively $K^N(Z,W)$), the sectional curvature of M relative to the plane generated by X,Y (respectively the sectional curvature of N relative to the plane generated by Z,W). Let*

$$c:[0,a]\longrightarrow M, \quad c^*:[0,a]\longrightarrow N$$

be geodesics of equal length. Suppose that for every $t\in[0,a]$ and for every $X\in T_{c(t)}M$, $X_o\in T_{c^*(t)}N$ we have

$$K^N(X_o,\dot{c}^*(t)) \geq K^M(X,\dot{c}(t)).$$

Then we have:

(1) *(Rauch) Assume that c^* has at most a finite number of pathological points of $c^*(0)$ along c^*. Let J and J^* be Jacobi fields along c and c^* such that $J(0)$ and $J^*(0)$ are tangent to c and c^*, respectively, and*
   - $\| J(0) \| = \| J^*(0) \|^*$,
   - $\langle \dot{c}(0), \nabla_{\partial/\partial t} J(0)\rangle = \langle \dot{c}^*(0), \nabla_{\partial/\partial t} J^*(0)\rangle^*$,
   - $\| \nabla_{\partial/\partial t} J(0) \| = \| \nabla_{\partial/\partial t} J^*(0) \|^*$.

   Then, for every $t\in[0,a]$,
   $$\| J(t) \| \geq \| J^*(t) \|^* .$$
(2) *(Berger) Assume c^* has at most a finite number of pathological focal points of $c^*(0)$ along c^*, with respect to the submanifold N defined by $\dot{c}^*(0)$. Let J and J^* be Jacobi fields along c and c^* such that $\nabla_{\partial/\partial t}J(0)$ and $\nabla_{\partial/\partial t}J^*(0)$ are tangent to $\dot{c}(0)$ and $\dot{c}^*(0)$ and*
   - $\| \nabla_{\partial/\partial t} J(0) \| = \| \nabla_{\partial/\partial t} J^*(0) \|^*$,
   - $\langle \dot{c}(0), J(0)\rangle = \langle \dot{c}^*(0), J^*(0)\rangle^*$, $\| J(0) \| = \| J^*(0) \|^*$.

   Then
   $$\| J(t) \| \geq \| J^*(t) \|^*,$$
   for every $t\in[0,a]$.
(3) *The index of $D^2E(c^*)$ is greater than the index plus the nullity of the index form $D^2E(c)$.*

Proof. We shall briefly outline the proof of the Rauch Theorem (the Berger Theorem can be proved similarly), and we shall discuss item (3).

We can assume that the Jacobi fields satisfy

$$\| J(0) \| = \langle \dot{c}(0), \nabla_{\partial/\partial t} J(0) \rangle = \| J^*(0) \|^* = \langle \dot{c}^*(0), \nabla_{\partial/\partial t} J^*(0) \rangle^* = 0,$$

since the first and the second conditions imply $\langle J(t), \dot{c}(t) \rangle = \langle J^*(t), \dot{c}^*(t) \rangle^*$, which means that the norm of the component of J along $\dot{c}$ is equal to the norm of the component of J^* along $\dot{c}^*$. We note also that, by assumption, $J^*(t) \neq 0$ for every $t \in (0, a]$. Let $t_o \in (0, a]$ and let F be an isometry which satisfies

$$\begin{aligned} F &: T_{c(0)}M \longrightarrow T_{c^*(0)}N, \\ F&(\dot{c}(0)) = \dot{c}^*(0), \\ F(\tau_{t_o}^0(J(t_o))) &= \chi_{t_o}^0(J^*(t_o)) \frac{\| J(t_o) \|}{\| J^*(t_o) \|^*}, \end{aligned}$$

where χ_s^t is the parallel transport from $c^*(s)$ to $c^*(t)$ along c^*. We consider the following curve of isometries:

$$\begin{aligned} i_t &: T_{c(t)}M \longrightarrow T_{c^*(t)}N \\ i_t &= \chi_0^t \circ F \circ \tau_t^0, \end{aligned}$$

for $0 < t \leq t_o$. Let $W(t) = i_t(J(t))$. Put $c_o = c_{|[0,t_o]}$ and $c_o^* = c^*_{|[0,t_o]}$. Then

$$\begin{aligned} D^2E(c_o^*)(W, W) &= \int_0^{t_o} \| \nabla_{\partial/\partial t} W(t) \|^{*2} - \langle R^N(\dot{c}^*(t), W(t))\dot{c}^*(t), W(t) \rangle^* dt \\ &\leq \int_0^{t_o} \| \nabla_{\partial/\partial t} J(t) \|^2 - \langle R^M(\dot{c}(t), J(t))\dot{c}(t), J(t) \rangle dt \\ &= D^2E(c_o)(J, J). \end{aligned}$$

In particular, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} |_{t=t_o} \langle J(t), J(t) \rangle &= \langle J(t_o), \nabla_{\partial/\partial t} J(t_o) \rangle \\ &= D^2E(c_o)(J, J) \\ &\geq D^2E(c_o^*)(W, W) \\ &\geq D^2E(c_o^*) \left(J^* \frac{\| J(t_o) \|}{\| J^*(t_o) \|^*}, J^* \frac{\| J(t_o) \|}{\| J^*(t_o) \|^*} \right), \end{aligned}$$

where the last inequality is given by the Focal Index Lemma. Since $J^*(t)$ is a Jacobi field, we have $D^2E(c_o^*) = \langle J^*(t_o), \nabla_{\partial/\partial t} J^*(t_o) \rangle^*$, so

$$\begin{aligned} \frac{d}{dt}\Big|_{t=t_o} \langle J(t), J(t) \rangle &\geq D^2E(c_o^*) \left(J^* \frac{\| J(t_o) \|}{\| J^*(t_o) \|^*}, J^* \frac{\| J(t_o) \|}{\| J^*(t_o) \|^*} \right) \\ &= \frac{d}{dt}\Big|_{t=t_o} \langle J^*(t), J^*(t) \rangle^* \frac{\| J(t_o) \|^2}{\| J^*(t_o) \|^{*2}}. \end{aligned}$$

Hence, given $\epsilon > 0$, for every $t \geq \epsilon$ we have

$$\frac{d}{dt}\log(\| J(t) \|^2) \geq \frac{d}{dt}\log(\| J^*(t) \|^{*2}).$$

Integration over $[\epsilon, t]$, yields

$$\frac{\| J(t) \|^2}{\| J(\epsilon) \|^2} \geq \frac{\| J^*(t) \|^{*2}}{\| J^*(\epsilon) \|^{*2}}.$$

Since $\| \nabla_{\partial/\partial t} J(0) \| = \| \nabla_{\partial/\partial t} J^*(0) \|^*$, we get the desired inequality.

What does it mean that the index of $D^2E(c^*)$ is greater than the index plus the nullity of $D^2E(c)$?

Let $i_o : T_{c(0)}M \longrightarrow T_{c^*(0)}N$ be an isometry such that $i_o(\dot{c}(0)) = \dot{c}^*(0)$. We define, for each $t \in [0, b]$, the isometry

$$i_t = \tau_0^t \circ i_o \circ \chi_t^0 : T_{c(t)}M \longrightarrow T_{c^*(t)}N.$$

Let X be a vector field along c. We may consider the vector field $i(X)(t) = i_t(X(t))$ along c^*. One can prove that

$$D^2E(c)(X, X) \geq D^2E(c^*)(i(X), i(X)),$$

by the assumption on the sectional curvatures. Thus, if U is a subspace on which $D^2E(c) \leq 0$, then $D^2E(c^*)_{i(U)} \leq 0$. □

4. The free loop space of a finite dimensional Riemannian manifold

Let $(M^n, \langle \cdot, \cdot \rangle)$ be a compact Riemannian manifold of dimension n. We recall that $\Omega(M^n) = H^1(S^1, M^n)$ is the set of maps of Sobolev class H^1 from S^1 into M^n. It can be given the structure of an infinite dimensional Hilbert manifold, and the tangent space $T_\sigma \Omega(M^n)$ at a point $\sigma \in \Omega(M^n)$ consists of periodic H^1 vector fields along γ. One defines the L^2 weak Riemannian structure on $\Omega(M^n)$ by setting

$$\langle X, Y \rangle(\sigma) = \int_{S^1} \langle X(t), Y(t) \rangle dt,$$

where $X, Y \in T_\sigma \Omega(M^n)$. It is well known (see [7] or [18]) that the L^2 metric has a Levi-Civita connection which is determined pointwise by the Levi-Civita connection of M^n. Moreover, the L^2 curvature R is given pointwise by the tensor curvature of M^n, so the sectional curvature is given by

$$K(X, Y) = \int_{S^1} K^{M^n}(X(t), Y(t)) dt.$$

If M^n has positive sectional curvature, i.e., $K^{M^n} \geq K_o > 0$, then $\Omega(M)$ has positive sectional curvature since $K^{\Omega(M^n)} \geq K_o 2\pi = K_1$. In particular, there exists at least one conjugate point along any geodesic of length greater than $\pi/\sqrt{K_1}$, and its index is infinite. Indeed, let $\gamma : [0, 1] \longrightarrow \Omega(M^n)$ be a geodesic with length $l > \pi/\sqrt{K_1}$. Let $v \in T_{\gamma(0)}\Omega(M^n)$ be a unit vector such that $\langle v, w\rangle(\gamma(0)) = 0$. Let $W(t) = \sin(t\pi)V(t)$, where V is the parallel transport along γ of v. One can verify that $D^2E(\gamma)(W,W) < 0$, so, by the Focal Index Lemma, we have at least one singularity of the exponential map and it cannot be an isolated pathological point. The fact that the index is infinite follows by comparing $\Omega(M^n)$ with the manifold

$$S_{\frac{1}{\sqrt{K_1}}} := \left\{(x,Y) \in \mathbb{R} \times T_{\gamma(0)}\Omega(M^n) :\ x^2 + \langle Y, Y\rangle(\gamma(0)) = \frac{1}{\sqrt{K_1}}\right\},$$

which is a weak Riemannian Hilbert manifold of a constant sectional curvature K_1. Note that the same argument works if we consider the submanifold N defined by $\dot{\gamma}(0)$. Indeed, one verifies that $D^2E(\gamma)(W,W) < 0$, where $W(t) = \cos(t\frac{\pi}{2})V(t)$. This is in contrast with the Riemannian point of view, i.e., $\Omega(M^n)$ endowed by the H^1 metric, since Misiolek [16] proved that the exponential map is a non-linear Fredholm map and any geodesic of finite length has finite index.

Suppose now that $M^n = G$ is a non-abelian compact Lie group. In this case we get a simple expression for the Levi-Civita connection,

$$\nabla_X Y = \frac{1}{2}[X,Y],$$

and therefore for the curvature tensor

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z].$$

Consequently, any one-parameter subgroup of $\Omega(G)$ is a geodesic of the L^2 metric and the exponential map is defined on the whole tangent space. Moreover, if X, Y and Z are parallel vector fields along a geodesic c, then $R(X,Y)Z$ is parallel along c as well (see [14] and [15]).

It is well known (see [9]) that Lie $(G)= \mathfrak{z} \oplus \mathfrak{g}_s$, where $\mathfrak{g}_s$ is the maximal semisimple ideal of Lie(G) and $\mathfrak{z}$ is the Lie algebra of the center of G. Since $\mathfrak{g}_s$ is semisimple, it has a subalgebra $\mathfrak{h}_\alpha$ isomorphic to $\mathfrak{su}(2)$. We denote by A_α, B_α and C_α the standard generators of $\mathfrak{su}(2)$. Then

$$[A_\alpha, B_\alpha] = 2C_\alpha, [C_\alpha, A_\alpha] = 2B_\alpha, [C_\alpha, B_\alpha] = -2A_\alpha.$$

Let c be the one-parameter subgroup of $\Omega(G)$ generated by $(1/\sqrt{2\pi})B_\alpha$. Now, as in [17, p. 2480–2481], one can prove that the vector fields $Y_k(t) = \sin(t/\sqrt{2\pi})\tau_0^t(w_k)$, where $w_k(x) = (1/\sqrt{\pi})\sin kxA_\alpha$ is an eigenvector of $R(\cdot,(1/\sqrt{2\pi})B_\alpha)(1/2\pi)B_\alpha$ with eigenvalue $\lambda = 1/\sqrt{\pi}$, are linearly independent Jacobi fields along c. Thus the kernel of $\mathrm{d}(\exp_{\mathrm{e}})_{\pi\sqrt{2\pi}\mathrm{B}_\alpha}$ is infinite dimensional. In particular, $\Omega(G)$ has at least one monoconjugate point along c and the exponential map cannot be Fredholm. Moreover, the vectors fields

$L_k(t) = \cos(t/\sqrt{2\pi})\tau_0^t(w_k)$, are $N-$Jacobi fields along c, where N is the submanifold defined by $\dot{c}(0)$, so the kernel $\mathrm{d}(\mathrm{Exp}^{\perp})_{(\pi\sqrt{2\pi}/2)\mathrm{B}_\alpha}$ is infinite dimensional. Hence there exists a monofocal point along c and the map $\mathrm{Exp}^{\perp} : T^{\perp}N \longrightarrow M$, which is well-defined in this case, fails to be Fredholm.

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