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THOM POLYNOMIALS WITH INTEGER COEFFICIENTS

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ABSTRACT. The Thom polynomial of a singularity is a universal characteristic class that expresses the cohomology class represented by a singularity locus. In this paper we compute the Thom polynomials of real singularities up to codimension 8 in cohomology with integer coefficients.

1. Introduction

Thom polynomials of real singularities have been calculated mainly with $\mathbb{Z}_2$-coefficients. The exceptions are the Σ^i singularities (see [Ron71]) and some $\Sigma^{i,j}$ Thom-Boardman singularities (see [And82]). In this paper we demonstrate how the methods of the theory of Thom polynomials for group actions (see [FR]) can be used in this case. We concentrate on the case of (contact class) singularities between manifolds of equal dimension which have been studied the most. We calculate the Thom polynomials up to codimension 8.

It turned out that the difficult part is not to calculate these Thom polynomials, but to find out "who" has a Thom polynomial. Vassiliev [Vas88, §8] defined a cochain complex where the cochains are linear combinations of cooriented orbits (singularities in our case) and showed that exactly the cocycles admit Thom polynomials (see also [Kaz97] and [FR]). In cases where every orbit is cooriented and even codimensional—e.g., the case of complex singularities—the differential of the Vassiliev complex is trivial. Such Thom polynomials are calculated in, e.g., [Rim01]; see also the references therein.

Calculation of Thom polynomials of real singularities with $\mathbb{Z}_2$-coefficients is easier due to a result of Borel and Haefliger [BH61]. This result implies that we can get the Thom polynomial of a real singularity η by replacing Chern classes by the corresponding Stiefel-Whitney classes in the Thom polynomial of the complexification of η. So it also gives the answer if the integer Thom

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polynomial is of order two. Consequently the calculation of the Vassiliev complex presented below and the Borel-Haefliger theorem is enough to calculate all but two Thom polynomials (see Theorem 2.7). The Thom polynomial of $I_{2,2} + II_{2,2}$ was previously known. For the remaining case we applied the method of restriction equations established by the second author in [Rim01]. This method calculates the Thom polynomial by solving a system of linear equations. We will see that these equations are *not* enough in the case of real singularities. However, knowing also their $\mathbb{Z}_2$-reductions and finding an additional equation similar to the *incidence* calculations in [Rim02a] we can calculate them. We are grateful for the referee for suggesting a way to correct a mistake in the first version of the manuscript.

In Section 2 we calculate the Vassiliev complex. These calculations are fairly complicated. An additional difficulty compared to the $\mathbb{Z}_2$-case (which was considered in [Ohm94]) is the determination of the signs in the differential. In Section 3 we calculate the Thom polynomials. In Section 4 we study the connection between the real and complex case, which leads us to finding obstructions to avoid certain singularities.

2. The Vassiliev complex

The n-cochains in the Vassiliev complex are linear combinations of the n-codimensional cooriented orbits. (We assume here that every orbit is simple in the codimension range we are interested in.) Following Vassiliev we can calculate the coefficients of the differential d_1 as follows: Let ξ be an n-codimensional cooriented orbit and

$$d_1(\xi) = \sum_{\operatorname{codim} \eta = n+1} c(\eta, \xi)\eta.$$

Then $c(\eta, \xi)$ is "the number of ξ-curves leaving η" counted with sign. More precisely, if we take a normal slice N_η to the stratum η, then the intersection $N_\eta \cap \xi$ is one dimensional, i.e., a disjoint union of curves L_i. For every curve L_i we calculate its sign as follows: Choose a point $x_i \in L_i$ and a normal slice N_i. In other words,

$$(*) \qquad N_\eta = T_{x_i} L_i \oplus N_i.$$

Notice that N_i is also a normal slice to ξ, and therefore oriented (with the orientation defined by the coorientation of ξ). N_η is also oriented and we can give an orientation to $T_{x_i} L_i$ by choosing a vector pointing out of the origin (the "η-point"). If the three orientations agree, we give plus sign to L_i; otherwise we give it a minus sign.

Thus, we first have to find the coorientable singularities, then the curves L_i, and finally we have to calculate the signs.

2.1. Coorientable singularities. The top (up to codimension 9) of the classification of stable singularities between equal dimensional spaces is as follows (see, e.g., [PW95]):

codim	0	1	2	3	4	5
	A_0	A_1	A_2	A_3	A_4	A_5
					$I_{2,2}$	$I_{2,3}$
					$II_{2,2}$	

codim	6	7	8	9
	A_6	A_7	A_8	A_9
	$I_{2,4}$	$I_{2,5}$	$I_{2,6}$	$I_{2,7}$
	$II_{2,4}$		$II_{2,6}$	
	$I_{3,3}$	$I_{3,4}$	$I_{3,5}$	$I_{3,6}$
			$I_{4,4}$	$I_{4,5}$
			$II_{4,4}$	
	IV_3		IV_4	
		(x^2, y^3)	(x^2+y^3, xy^2)	(x^2+y^3, y^4)
				(x^2+y^4, xy^2)
				Σ^3

Here by 'singularity' we mean a stratum (satisfying the Vassiliev conditions [Vas88], [FR]) of the following group action: The group $\mathrm{Diff}(\mathbb{R}^\infty, 0) \times \mathrm{Diff}(\mathbb{R}^\infty, 0)$ (diffeomorphism germs at 0) acts on $\mathcal{E} := \{$ stable $(\mathbb{R}^\infty, 0) \to (\mathbb{R}^\infty, 0)$ germs$\}$ by $(\psi, \varphi) \cdot f := \varphi \circ f \circ \psi^{-1}$. In fact, all but Σ^3 is an orbit, and the latter is a 1-parameter family of orbits.

Orbits of this group action are characterized by their local algebras [Mat69]. So the above symbols encode local algebras as follows: A_i stands for the singularity with local algebra $\mathbb{R}[[x]]/(x^{i+1})$. The symbols I–IV stand for algebras corresponding to $\Sigma^{2,0}$ singularities as in [Mat71]. In the other cases we indicated the ideal in $\mathbb{R}[[x, y]]$ which is to be factored out to get the algebra. The stratum Σ^3 corresponds to the 1-parameter family of algebras

$$\mathbb{R}[[x, y]]/(x^3 + \lambda yz, y^3 + \lambda xz, z^3 + \lambda xy), \quad \lambda(\lambda^3 - 1)(8\lambda^3 + 1) \neq 0.$$

Definition 2.1. A stable germ $\kappa : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is called a *prototype* of the singularity η if the infinite trivial unfolding of κ is in η and n is minimal.

Let η be a singularity and κ a prototype of η. Then we can consider the right-left symmetry group of κ,

$$\{(\varphi,\psi)\in\mathcal{A}=\mathrm{Diff}(\mathbb{R}^n,0)\times\mathrm{Diff}(\mathbb{R}^n,0):\psi\circ\kappa\circ\varphi^{-1}=\kappa\},$$

or its maximal compact subgroup G_η. (For more details see [Rim02b], [FR].) If κ is well chosen, then by the definition of a maximal compact subgroup for subgroups of $\mathcal{A}$, the group G_η acts linearly on the source and target spaces. We denote these representations by λ_0 and λ_1, respectively. If $\lambda_0(G_\eta)\subset GL^+(n)$ then we call η *coorientable*. If the singularity is *simple*, i.e., not a member in a continuous family, then this condition is equivalent to the coorientability of the stratum in $\mathcal{E}$ since the source space of a prototype can be identified with a normal slice to the orbit η. Geometrically this means that if $\eta(f)\subset N$ is the set of η-points of a stable map $f:N\to P$ then the normal bundle of $\eta(f)$ in N is orientable.

In case of non-simple singularities (i.e., in our case for Σ^3) the representation $\lambda_0(G_\eta)$ decomposes to summands tangent and normal to the stratum. In this case coorientability of the stratum means orientability of the representation on the normal slice.

THEOREM 2.2. *Among the above singularities exactly the following are coorientable:*

codim	0	1	2	3	4	5	6	7	8	9
Σ^0	A_0									
Σ^1				A_3	A_4			A_7	A_8	
$\Sigma^{2,0}$					$I_{2,2}$	$I_{2,3}$		$I_{2,5}$	$I_{2,6}$	$I_{2,7}$
					$II_{2,2}$				$II_{2,6}$	
										$I_{3,6}$
									IV_4	
$\Sigma^{2,1}$									(x^2+y^3,xy^2)	(x^2+y^3,y^4)

The problem of calculating the maximal compact symmetry group and its representation λ_0 was solved in [Rim96] and [Rim02b]. Here we give an example. For a more detailed discussion see [Rim00].

A prototype κ of (x^2,y^3) is the miniversal unfolding of $(x,y)\mapsto(x^2,y^3)$,

$$\kappa:(x,y,\underline{v})\mapsto(x^2+v_1y+v_2y^2,y^3+v_3x+v_4y+v_5xy,\underline{v}),$$

where $\underline{v}=v_1,\dots,v_5$. Its maximal compact symmetry group is $G_{(x^2,y^3)}\cong\mathbb{Z}_2\times\mathbb{Z}_2$ and $\lambda_0=\alpha\oplus\beta\oplus\beta\oplus1\oplus\alpha\beta\oplus1\oplus\alpha$, where α and β are the nontrivial irreducible representations of the first and second $\mathbb{Z}_2$-factor. So (x^2,y^3) is not coorientable.

From now on the symbols of coorientable singularities will denote the given singularity with a chosen coorientation. We do not specify these coorientations. This leaves some sign indeterminacy in our final results. The calculations of the coorientations would be a tedious job presenting no theoretical novelties, so we decided to omit them.

2.2. Computation of the differentials. Some of these calculations (Theorem 2.3) are standard, based on results of Lander and the notion of multiplicity. For Theorem 2.5 we use our knowledge of the symmetry group of these singularities calculated in [Rim02b]. In Theorem 2.6 we apply a method using Hilbert functions.

The easiest case in the computation of the Vassiliev coefficients $c(\eta,\xi)$ is when *near an η-point there are no ξ-points* at all, i.e., when the germ of the set $\xi(\kappa)$ is empty for the prototype κ of η.

THEOREM 2.3. *The following Vassiliev coefficients are all* 0*:*

(1) $c(A_8, I_{2,5})$, $c(I_{2,7}, (x^2{+}y^3, xy^2))$, $c(I_{3,6}, (x^2{+}y^3, xy^2))$;
(2) $c(II_{2,2}, A_3)$, $c(II_{2,6}, A_7)$, $c(IV_4, I_{2,5})$, $c(I_{2,7}, IV_4)$, $c(I_{3,6}, IV_4)$;
(3) $c((x^2{+}y^3, xy^2), A_7)$, $c((x^2{+}y^3, y^4), A_8)$.

Proof. Case (1) holds because of Thom-Boardman symbols; i.e., if J, I are Thom-Boardman symbols and $J < I$ in the lexicographic order, then near a J-point there are no I points. Case (2) follows from the work of Lander [Lan76] saying that the appropriate set germs are empty. Case (3) follows from the notion of multiplicity (see, e.g., [AVGL91, p. 161]). The multiplicity of $(x^2{+}y^3, xy^2)$ is 7 since it is the dimension of its local algebra. Geometrically this means that the preimage (at the complexified map) of a general point near 0 in the target consists of 7 points. The multiplicity of A_7 is 8, so it cannot be near 0 of a $(x^2{+}y^3, xy^2)$-germ. Similarly the multiplicities for $(x^2{+}y^3, y^4)$ and A_8 are 8 and 9, respectively. □

Now let us consider $c(A_4, A_3)$. A prototype of A_4 is

$$\kappa : (x, y_3, y_2, y_1) \mapsto (x^5 + y_3x^3 + y_2x^2 + y_1x, y_3, y_2, y_1)$$

with maximal compact symmetry group $\langle g\rangle \cong \mathbb{Z}_2$ acting as $\alpha \oplus 1 \oplus \alpha \oplus 1$ on the source. Calculating the partial derivatives shows that the A_3-points of κ are parameterized as $(t, -10t^2, 20t^3, -15t^4)$, which is a non-singular curve and thus has two intersections with a sphere centered at the origin. To determine the signs associated with these intersection points we would need a clear definition of the coorientation of A_4 and A_3. Although we have not specified the chosen coorientations we can still see that the signs associated with the two intersection points must coincide in view of the following lemma, because an orientation preserving diffeomorphism germ, namely $\alpha(g)$, interchanges them:

LEMMA 2.4. *If the symmetry group G_η of the orbit η interchanges the curves L_i and L_j then they have the same sign.*

Proof. Suppose that $gL_i = L_j$ for some $g \in G$. Then by choosing $x_j := gx_i$ we have $gN_i = N_j$. Since ξ and η are cooriented and $g\eta = \eta$, the symmetry g preserves the orientations in the decomposition $(*)$. □

So we can state that $c(A_4, A_3) = \pm 2$. A similar computation shows that we also have $c(A_8, A_7) = \pm 2$. The key in these computations was our ability to write down the equations of the 'nearby singularity types' and the luck that the obtained points on the sphere are permuted by the symmetry group of the singularity at 0. The equations of the singularities near $\Sigma^{2,0}$-points are described in [Lan76]. Their symmetry groups have been computed in [Rim02b]. Luckily enough, in the following cases Lemma 2.4 applies, so—as above—we can determine the absolute values of the Vassiliev coefficients:

THEOREM 2.5.

$$\begin{array}{rrr} c(I_{2,2}, A_3) = \pm 4, & c(I_{2,3}, A_4) = \pm 2, & c(I_{2,3}, I_{2,2}) = \pm 1, \\ c(I_{2,3}, II_{2,2}) = \pm 1, & c(I_{2,6}, A_7) = \pm 4, & c(I_{2,6}, I_{2,5}) = \pm 2, \\ c(II_{2,6}, I_{2,5}) = \pm 2, & c(IV_4, A_7) = \pm 8, & c(I_{2,7}, A_8) = \pm 2, \\ c(I_{2,7}, I_{2,6}) = \pm 1, & c(I_{2,7}, II_{2,6}) = \pm 1, & c(I_{3,6}, A_8) = \pm 2, \\ c(I_{3,6}, I_{2,6}) = \pm 1, & c(I_{3,6}, II_{2,6}) = \pm 1. & \square \end{array}$$

There only remain the following five Vassiliev coefficients to be calculated: $c((x^2{+}y^3, xy^2), I_{2,5})$ and $c((x^2{+}y^3, y^4), \eta)$ with $\eta = I_{2,6}, II_{2,6}, IV_4, (x^2{+}y^3, xy^2)$. Here again the main work is to determine the equations for the ξ-points in the source of a prototype of η. This can be done using the Hilbert functions of the local algebras, $h : i \mapsto \dim M^i/M^{i+1}$, where M is the unique maximal ideal. We will sketch the procedure in the particular case of $c((x^2{+}y^3, y^4), IV_4)$. A prototype of $(x^2 + y^3, y^4)$ is

$$\kappa : (x, y, \underline{u}, \underline{v}) \mapsto (x^2{+}y^3{+}u_1y{+}u_2y^2, y^4{+}v_1x{+}v_2y{+}v_3xy{+}v_4y^2{+}v_5xy^2, \underline{u}, \underline{v}),$$

where $\underline{u} = (u_1, u_2)$ and $\underline{v} = (v_1, \ldots, v_5)$. By differentiating we get the following equations for the Σ^2-points in the source:

$$x = 0, \quad u_1 = -3y^2 - 2u_2y, \quad v_1 = -v_3y - v_5y^2, \quad v_2 = -4y^3 - 2v_4y.$$

Thus it is a graph of a map $\mathbb{R}^5(y, u_2, v_3, v_4, v_5) \longrightarrow \mathbb{R}^4(x, u_1, v_1, v_2)$, so it is smooth. Let us choose a point p on this graph. So p is of the form

$$p := (0, \bar{y}, (-3\bar{y}^2 - 2\bar{u}_2\bar{y}), \bar{u}_2, (-\bar{v}_3\bar{y} - \bar{v}_5\bar{y}^2), (-4\bar{y}^3 - 2\bar{v}_4\bar{y}), \bar{v}_3, \bar{v}_4, \bar{v}_5).$$

The germ of κ at p, i.e., the germ of $\kappa((x,y,\underline{u},\underline{v})+p)-\kappa(x,y,\underline{u},\underline{v})$ at 0, is the unfolding of

$$\mu_p:(x,y)\to(x^2+y^3+(\bar{u}_2+3\bar{y})y^2,\quad y^4+\bar{v}_5xy^2 \\ +(2\bar{v}_5\bar{y}+\bar{v}_3)xy+(4\bar{y})y^3+(6\bar{y}^2+\bar{v}_4)y^2).$$

The local algebra Q_p of κ at p is $\mathbb{R}[[x,y]]/I$, where I is the ideal generated by the two coordinate functions of μ_p. Our task is to obtain the values of p for which the local algebra Q_p is isomorphic to that of IV_4. The algebra of IV_4 has Hilbert function $(h(0),h(1),\ldots)=(1,2,2,2,1,0,\ldots)$. For Q_p we have $h(0)=1$ and $h(1)=2$ for any values of

$$A:=\bar{u}_2+3\bar{y},\quad B:=\bar{v}_5,\quad C:=2\bar{v}_5\bar{y}+\bar{v}_3,\quad D:=4\bar{y},\quad E:=6\bar{y}^2+\bar{v}_4,$$

but

$$h(2)=3-\operatorname{rank}\begin{pmatrix}1&0&A\\0&C&E\end{pmatrix},$$

so the condition for $h(2)$ to be equal to 2 is $C=E=0$. Similarly,

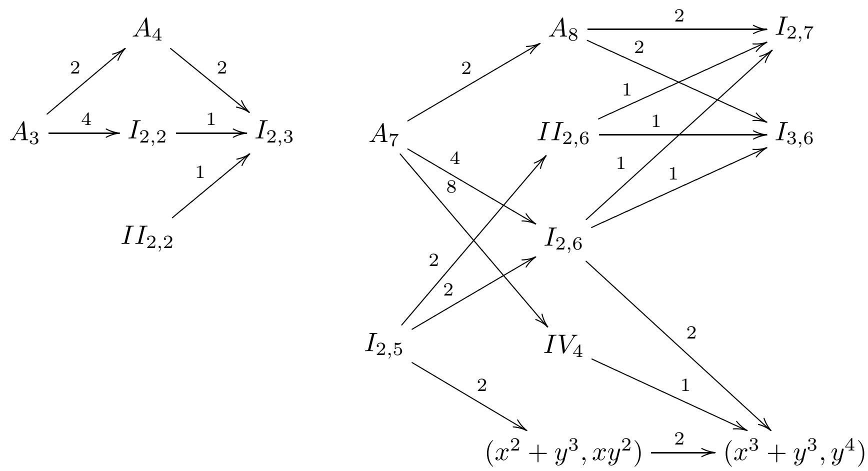
$$h(3)=7-\operatorname{rank}\begin{pmatrix}1&.&A&.&.&.&1\\.&C&E&.&.&B&D\\.&.&.&1&.&A&.\\.&.&.&.&1&.&A\\.&.&.&.&C&E&.\\.&.&.&.&.&C&E\end{pmatrix},$$

so the condition for $h(3)$ to be equal to 2 (using $C=E=0$) is $B=D=0$. Thus there is a 1-dimensional curve (as A varies) in the source of κ where the Hilbert function starts as $(1,2,2,2)$. Studying the Hilbert functions of the singularities of codimension 8 we see that the singularities on this curve are either IV_4 or $I_{4,4}$. (Let us remark that the complexification of these two are the same, so one needs real arguments to distinguish them.) In our case the algebra $(x^2+y^3+Ay^2,y^4)$ is isomorphic to the algebra of IV_4 if $A>0$, and isomorphic to that of $I_{4,4}$ if $A<0$. (In the latter case use the substitution $x=u+v, y=u-v$.) Hence the IV_4-points of κ are on a ray coming out of 0 in the source of κ, so $c((x^2+y^3,y^4),IV_4)=\pm1$. Using similar arguments we can compute the remaining Vassiliev coefficients (up to sign):

THEOREM 2.6.

$$c((x^2+y^3,xy^2),I_{2,5})=\pm2,\qquad c((x^2+y^3,y^4),I_{2,6})=\pm2,$$
$$c((x^2+y^3,y^4),(x^2+y^3,xy^2))=\pm2,$$
$$c((x^2+y^3,y^4),IV_4)=\pm1,\qquad c((x^2+y^3,y^4),II_{2,6})=0. \qquad \square$$

The following graph encodes our results. Here an arrow always connects two singularities in consecutive codimension, and the label of the arrow is the absolute value of the Vassiliev coefficient of these singularities. A missing arrow means the Vassiliev coefficient is 0.



Since we determined only the absolute values of the coefficients, we apparently do not have enough information to write down the Vassiliev complex exactly. However, we know that the signs are to be distributed so that $d_1 \circ d_1 = 0$ is satisfied, and, in fact, we are (again) lucky, because *essentially* there is only one way to distribute the signs so that this condition is satisfied. Consequently we get the cohomology groups of the Vassiliev complex:

Theorem 2.7.

$$H^1 = H^2 = H^3 = H^5 = H^6 = H^7 = 0, H^0 = \mathbb{Z}\langle A_0\rangle,$$
$$H^4 = \mathbb{Z}\langle I_{2,2} + II_{2,2}\rangle \oplus \mathbb{Z}_2\langle A_4 + 2I_{2,2}\rangle,$$
$$H^8 = \mathbb{Z}\langle (x^2+y^3, xy^2) - 2IV_4\rangle \oplus \mathbb{Z}_2\langle I_{2,6} + II_{2,6} + (x^2+y^3, xy^2)\rangle$$
$$\oplus\, \mathbb{Z}_2\langle A_8 + 2I_{2,6} + 4IV_4\rangle,$$

where in the brackets we indicated possible generators, which mean singularities with some (determinable but undetermined) coorientations. □

3. Calculation of the Thom polynomials

Let us briefly recall the definition of the Thom polynomial. The Vassiliev complex $\mathbf{V}C$ is the 0-th row of the E_2-table of the Kazarian spectral sequence (see [Kaz97]), so we have the following edge homomorphism:

$$\mathrm{Tp} : \mathbf{V}C \to H^*(BG).$$

In the case of real singularities $G = \mathrm{Diff}(\mathbb{R}^\infty, 0) \times \mathrm{Diff}(\mathbb{R}^\infty, 0)$, so the Thom polynomials live in $H^*(BO \times BO; \mathbb{Z})$. In classical terms, given a smooth map $f : N \to P$, the Poincaré dual of the singular points of type ξ can be expressed as a polynomial of characteristic classes of TN and f^*TP.

PROPOSITION 3.1. $\mathrm{Tp}(\xi)$ *depends only on the characteristic classes of* $f^*TP \ominus TN$.

The proof can be found in [FR] for the complex case, but it applies word for word to the real case as well. We adopt the notation $\mathrm{tp}(\xi)$ for the corresponding element in $H^*(BO; \mathbb{Z})$.

We need a convenient notation for the elements of $H^*(BO; \mathbb{Z})$:

THEOREM 3.2 ([MS74, Prop. 15C]).

$$H^*(BO; \mathbb{Z}) \cong \mathbb{Z}[p_1, \dots, p_i, \dots] \oplus \operatorname{Im} \mathrm{Sq}^1 .$$

We will use the notation $\sum v_I$ for the unique second order element in $H^*(BO; \mathbb{Z})$ such that $r(\sum v_I) = \sum w_I$, where r denotes the mod 2 reduction and w_I is a monomial of Stiefel-Whitney classes corresponding to the multiindex I. So we write elements of $H^*(BO; \mathbb{Z})$ in the form $\sum a_I p_I + \sum b_I v_I$, where I runs through multiindices and p_I and v_I are the corresponding monomials. We can assume that $\sum b_I w_I \in \operatorname{Im} \mathrm{Sq}^1$.

Our major tool in the calculations is the following theorem of Borel and Haefliger:

THEOREM 3.3 ([BH61]). *Let* $\eta_{\mathbb{C}}$ *be the complexification of a real singularity* η. *Suppose that* $\mathrm{tp}(\eta_{\mathbb{C}}) = \sum a_I c_I$. *Then* $\mathrm{tp}(\eta; \mathbb{Z}_2) = \sum a_I w_I$.

REMARK 3.4. We say that $\eta_{\mathbb{C}}$ is the *complexification* of the real singularity η if they are defined by the same equation and $\operatorname{codim}_{\mathbb{C}} \eta_{\mathbb{C}} = \operatorname{codim}_{\mathbb{R}} \eta$. The codimension condition is not always satisfied; see [VS91].

For a cooriented singularity η we have $r\big(\mathrm{tp}(\eta; \mathbb{Z})\big) = \mathrm{tp}(\eta; \mathbb{Z}_2)$ and $\mathrm{Ker}(r) = 2 \cdot \mathbb{Z}[p_1, \dots, p_i, \dots]$, so we need some additional information to determine the coefficients of the Pontryagin classes.

$I_{22} + II_{22}$: This Thom polynomial coincides with the Thom polynomial of the Σ^2 Thom-Boardman singularity, which was calculated by Ronga ([Ron71]):

$$\mathrm{tp}(I_{22} + II_{22}) = p_1 + v_1 v_3.$$

$A_4 + 2I_{22}$: This cocycle has order 2 in the Vassiliev complex, so the Thom polynomial cannot contain Pontryagin classes. So by the Borel-Haefliger theorem,

$$\mathrm{tp}(A_4 + 2I_{22}) = v_1^4 + v_1 v_3.$$

$I_{26}+II_{26}+(x^2+y^3,xy^2)$: This cocycle also has order 2 in the Vassiliev complex, so

$$\mathrm{tp}(I_{26}+II_{26}+(x^2+y^3,xy^2)) = v_1^2v_2v_4+v_1v_2v_5+v_1v_3v_4+v_1v_2^2v_3+v_1^2v_3^2+v_1^3v_5.$$

$A_8+2I_{26}+4IV_4$: This cocycle also has order 2 in the Vassiliev complex, so

$$\mathrm{tp}(A_8+2I_{26}+4IV_4) = v_1^8+v_1^3v_5+v_1^2v_2v_4+v_1v_2v_5+v_1v_3v_4+v_1v_2^2v_3.$$

$\eta=(x^2+y^3,xy^2)-2IV_4$: Again by the Borel-Haefliger theorem it is enough to calculate rationally, i.e., to find the coefficients A and B for $\mathrm{tp}(\eta;\mathbb{Q}) = Ap_1^2+Bp_2$. First we apply the restriction equation method from [Rim01] (see also [FR]) to the 'test germ' $II_{2,2}$. As was explained in [FR], we need the representations λ_0 and λ_1 of $O(2)$ on the source and target space of II_{22}. These were calculated in [Rim02b]: $\lambda_0=\rho_1\oplus\rho_3$ and $\lambda_1=\rho_2\oplus\rho_3$, where ρ_n is the unique 2-dimensional representation of $O(2)$ which restricts to α^n on $U(1)\cong SO(2)$, where α is the standard representation of $U(1)$. Thus, using the notation of [FR], we have

$$p(\lambda_1\ominus\lambda_0)=\frac{1+4p_1}{1+p_1}=1+3p_1-3p_1^2+\cdots$$

and

$$j^*_{II_{22}}\,\mathrm{tp}(\eta;\mathbb{Q}) = A(3p_1)^2+B(-3p_1^2)=0,$$

which implies that $B=3A$. We need one more equation. For this we need another 'test germ' which has at least a $U(1)$ symmetry and also we need to be able to understand the IV_4 and (x^2+y^3,xy^2)-points near the origin. Such singularities turn up in high codimension, and the computation of the IV_4 and (x^2+y^3,xy^2) points near the origin is usually a huge computation. However, the referee provided us with an example where the calculations are simple.

The idea is the following: Any smooth map $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ induces a G_φ-equivariant map $\tilde\varphi:\mathbb{R}^n\to\mathcal{E}$, where $\mathcal{E}$ is the space of germs as before. If $\tilde\varphi$ is transversal to the closure of a stratum η, then $\tilde\varphi^*\,\mathrm{tp}(\eta)\in H^*(BG_\varphi)$ is equal to the G_φ-equivariant Poincaré dual of $\tilde\varphi^{-1}(\overline{\eta})$. It is easy to see that this equation holds even if $\tilde\varphi$ is not transversal along a subset of $\mathbb{R}^n$ having higher codimension than the codimension of η.

Let the 'test germ' be

$$(z,a_1,\dots,a_4)\to\left(|z|^2,\mathrm{Re}\left(\sum_{i=1}^4 a_iz^i\right),a_1,\dots,a_4\right)$$

(the number 4 can be replaced by larger integers), where z and the a_i's are from $\mathbb{C}\cong\mathbb{R}^2$. This germ clearly has a $U(1)$ symmetry, which acts by $\alpha\oplus\overline{\alpha}\oplus\overline{\alpha}^2\oplus\overline{\alpha}^3\oplus\overline{\alpha}^4$ on the source and by $1_{\mathbb{R}^2}\oplus\overline{\alpha}\oplus\overline{\alpha}^2\oplus\overline{\alpha}^3\oplus\overline{\alpha}^4$ on the target, so its relative Pontryagin class is

$$p(\lambda_1 \ominus \lambda_0) = \frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots ,$$

where t is the first Chern class of α. An easy computation shows that the locus of IV_4-singularity points in the source is $z = a_1 = a_2 = a_3 = 0$, $a_4 \neq 0$, whose closure is smooth. Hence the class dual to this closure is easily computed: $(-t)(-2t)(-3t) = -6t^4$ (i.e., the Euler class of the representation normal to the $a_1 = a_2 = a_3 = 0$ space).

The Thom-Boardman symbol of the singularity (x^2+y^3, xy^2) is $\Sigma^{2,1}$. Differentiation shows that no such singularities are in the source space of this map. Since $\tilde{\varphi}$ is automatically transversal at stable singularities of φ, it is transversal to the closure of the union of the orbit of (x^2+y^3, xy^2) and IV_4, except at 0. Applying the restriction equation we get

$$A(-t^2)^2 + B(t^4) = -2 \cdot (-6t^4).$$

Comparing this with $B = 3A$ we already knew, we obtain the unique solution $A = 3, B = 9$.

So for the proper coorientations we have

$$\mathrm{tp}((x^2{+}y^3, xy^2) - 2IV_4) = 3(p_1^2 + 3p_2) + v_1^2 v_2 v_4 + v_1 v_2 v_5 + v_1 v_3 v_4 + v_3 v_5.$$

4. Complex versus real

In this section we discuss a complexification technique (and its relation to Thom polynomials) different from that of Borel-Haefliger [BH61], as suggested to us by A. Szűcs and R. Szőke. Given a smooth map $f : N^n \to P^p$ between real manifolds consider its complexification $f_{\mathbb{C}} : N_{\mathbb{C}} \to P_{\mathbb{C}}$ in the Bruhat-Whitney sense [BW59] as follows.

First choose real analytic atlases for N and P and perturb f to be real analytic [Hir76, Thm. 5.1]. Then change the real coordinate charts to complex ones and glue them with the original gluing maps now considered as complex analytic maps. In fact, these gluing maps can be defined only in a neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$, so consider only these tubes. Also f, now considered as a complex analytic map on each coordinate chart, is defined only in a (possibly smaller) tube. Choosing these appropriately small tubes we get a map $f_{\mathbb{C}} : N_{\mathbb{C}} \to P_{\mathbb{C}}$. Since $N_{\mathbb{C}} \cong TN$ we identify the cohomology rings of N and $N_{\mathbb{C}}$.

Now let η be a real singularity of codimension c, which is the complete real form of its complexification $\eta_{\mathbb{C}}$ (e.g., $\eta = A_i$, $\eta_{\mathbb{C}} = A_i^{\mathbb{C}}$, or $\eta = I_{2,2} \cup II_{2,2}$, $\eta_{\mathbb{C}} = I_{2,2}^{\mathbb{C}}$). Also suppose that η defines a cocycle in the Vassiliev complex, i.e., the set of η-points $\eta(f) \subset N$ defines a cohomology class $[\eta(f)]$ in N.

In the next lemma η is a coorientable singularity.

Lemma 4.1. *If $\eta(f)$ is closed then $[\eta(f)]^2 = [\eta_{\mathbb{C}}(f_{\mathbb{C}})] \in H^{2c}(N)$.*

Proof. A tubular neighborhood of $S := \eta(f)$ in $N_{\mathbb{C}}$ is diffeomorphic to its normal bundle. However

$$\nu(S \subset N_{\mathbb{C}}) = \nu(S \subset N) \oplus TN|_S = \nu(S \subset N) \oplus (TS \oplus \nu(S \subset N)),$$

and the diffeomorphism can be chosen so that $\eta_{\mathbb{C}}(f_{\mathbb{C}})$ is the total space of the middle term. The cohomology class $\eta(f)$ is represented by the total space of the second and third terms. Since the intersection of $\eta(f)$ and its perturbation $(0, t, n) \mapsto (n, t, 0)$ is exactly $\eta_{\mathbb{C}}(f_{\mathbb{C}})$, the lemma is proved. □

Remark 4.2. We used an analytic method to prove this lemma. It is possible to give a more homotopy theoretic proof, valid in the more general case of Thom polynomials for group actions; however, this is beyond the scope of this paper.

Using this lemma we can construct an obstruction for avoiding singularities more complicated than η:

Definition 4.3. Let η be a real singularity of codimension d admitting a complexification $\eta_{\mathbb{C}}$. We define

$$s(\eta) := i^* \operatorname{tp}(\eta_{\mathbb{C}}) - \operatorname{tp}^2(\eta) \in H^{2d}(BO; \mathbb{Z}),$$

where $i : BO \to BU$ is the map induced by the embedding $O \to U$.

Lemma 4.1 implies the following corollary.

Corollary 4.4. *If $s(\eta)(f) \neq 0$, then there exists a singularity ξ more complicated than η such that $\xi(f)$ is not empty.*

We can see that Theorem 3.3 is equivalent to the following result.

Theorem 4.5. *$s(\eta)$ is even, i.e., its mod 2 reduction is 0.* □

It would be interesting to find a direct proof of this theorem.

Example 4.6. To see that $s(\eta)$ is not always 0, consider maps of codimension 1, i.e., maps $N^* \to P^{*+1}$, and let $\eta = A_2 (= \Sigma^{1,1})$. The Thom polynomial of its complexification is (see [Ron72], [Rim01])

$$\operatorname{tp}(A_2^{\mathbb{C}}) = c_2^2 + c_1 c_3 + 2c_4.$$

By Lemma 4.1 we have

$$\operatorname{tp}(A_2; \mathbb{Z}) = Ap_1 + v_1 v_3,$$

where A is an odd integer. (In fact, $A = 1$ as Toru Ohmoto and András Szűcs explained to us.) So

$$s(A_2) = p_1^2 + v_1^2 v_3^2 + 2p_2 - (Ap_1 + v_1 v_3)^2 = (1 - A^2) p_1^2 + 2p_2,$$

which is nonzero.

REMARK 4.7. In some sense $s(\eta)$ is not a new obstruction. In the notation of [FR] it is an element of the avoiding ideal $\mathcal{A}_{\partial\eta}$. Kazarian [Kaz97] calls such classes higher Thom polynomials. On the other hand, these avoiding ideals are not known except in special cases, e.g., Σ^i singularities; see [FP98, Ch. IV].

EXAMPLE 4.8. The case of Thom-Boardman singularities $\Sigma^i(k)$—where k refers to maps $\mathbb{R}^n \to \mathbb{R}^{n+k}$—has some interesting properties. They are coorientable if i and k are even. Their Thom polynomials were calculated in [Ron71] and [And82]. One can also calculate them by the method of restriction (which is somewhat surprising in light of the previous calculations in this paper); the calculation is completely analogous to the complex case in [FR]. The other—probably related—phenomenon is that $s(\Sigma^i(k)) = 0$. This is not a consequence of Remark 4.7, since $\mathcal{A}_{\partial\eta} \cap H^{2d}(BO;\mathbb{Z}) \neq 0$ for $d = \operatorname{codim}\Sigma^i(k) = i(i+k)$.

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