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HOLOMORPHIC AND $\mathcal{M}$–HARMONIC FUNCTIONS WITH FINITE DIRICHLET INTEGRAL ON THE UNIT BALL OF $\mathbb{C}^n$

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1. Introduction

For a real or complex-valued C^1 function f defined on the unit disc $\mathbb{D}$ in $\mathbb{C}$ and $\gamma \in \mathbb{R}$, the γ-weighted Dirichlet integral of f is defined by

$$D_\gamma(f) = \frac{1}{\pi} \iint_{\mathbb{D}} (1 - |z|^2)^\gamma |\nabla f(z)|^2 \, dxdy, \tag{1.1}$$

where $\nabla f = (f_x, f_y)$ is the gradient of f and $|\nabla f|^2 = |f_x|^2 + |f_y|^2$. The results of this paper were partially motivated by the following theorem of Yamashita.

Theorem A [20, Theorem 1]. *Let f be a solution of*

$$\Delta f = f_{xx} + f_{yy} = \lambda f, \quad \lambda \geq 0,$$

with $D_\gamma(f) < \infty$ for some γ, $0 < \gamma \leq 1$. Then $|f|^{2/\gamma}$ admits a harmonic majorant in $\mathbb{D}$.

The original goal of the paper was to prove an analogue of Theorem A for eigenfunctions of the Laplace-Beltrami operator $\widetilde{\Delta}$ on B, the unit ball in $\mathbb{C}^n$. For a real or complex-valued C^1 function f defined on B and $\gamma \in \mathbb{R}$, the integral

$$D_\gamma(f) = \int_B (1 - |z|^2)^\gamma |\widetilde{\nabla} f(z)|^2 \, d\tau(z) \tag{1.2}$$

is called the *γ–weighted invariant Dirichlet integral* of f. Here, $\widetilde{\nabla}$ and τ denote, respectively, the gradient and the volume measure corresponding to the Bergman metric on B. We denote by $\mathcal{D}_\gamma$ the weighted Dirichlet space of real or complex-valued C^1 functions f on B satisfying $D_\gamma(f) < \infty$, with the Dirichlet norm

$$\|f\|_{\mathcal{D}_\gamma} = |f(0)| + D_\gamma(f)^{1/2}. \tag{1.3}$$

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When $n = 1$, we have $\widetilde{\Delta} f(z) = (1-|z|^2)^2 \Delta f(z)$, $|\widetilde{\nabla} f(z)|^2 = (1-|z|^2)^2 |\nabla f(z)|^2$ and $d\tau(z) = (1-|z|^2)^{-2} dxdy$. Thus $D_\gamma(f)$ is the ordinary γ–weighted Dirichlet integral of f, defined in (1.1).

When $\gamma = n$, it is known that if f is $\mathcal{M}$–harmonic on B, i.e., if $\widetilde{\Delta} f = 0$, then $|f|^2$ has an $\mathcal{M}$–harmonic majorant if and only if $D_n(f) < \infty$ (see [16]). This suggests that an appropriate generalization of Theorem A is as follows:

If f is a solution of $\widetilde{\Delta} f = \lambda f$, $\lambda \geq 0$, with $D_\gamma(f) < \infty$ for some γ, $0 < \gamma \leq n$, then $|f|^{2n/\gamma}$ admits an $\mathcal{M}$–harmonic majorant on B.

However, we will show in Theorem 3.1 that for $\lambda > 0$, the only function f satisfying $\widetilde{\Delta} f = \lambda f$ with $D_\gamma(f) < \infty$ for some $\gamma \leq n$ is the zero function. Even though Δ and $\widetilde{\Delta}$ annihilate the same class of functions on $\mathbb{D}$, the eigenspaces of Δ and $\widetilde{\Delta}$ corresponding to an eigenvalue λ are significantly different when $\lambda \neq 0$. Thus, a generalization of Theorem A is only possible in the case when $\lambda = 0$, i.e., for the class of $\mathcal{M}$–harmonic functions on B. In addition to proving Theorem 3.1, we will show in Section 3 that, in the case $n \geq 2$, $\mathcal{D}_\gamma$ contains non-constant holomorphic functions if and only if $\gamma > (n-1)$.

In Section 4 we will prove a generalization of Theorem A to $\mathcal{M}$–harmonic functions and holomorphic functions on B, as well as a converse result. Our main result, Theorem 4.2, is as follows.

Theorem B. *(a) Let f be $\mathcal{M}$–harmonic or holomorphic on B. If $f \in \mathcal{D}_\gamma$ for some γ, $0 < \gamma \leq n$, then $|f|^p$ has an $\mathcal{M}$–harmonic majorant for all p, $0 < p \leq 2n/\gamma$.*

(b) Conversely, if f is $\mathcal{M}$–harmonic on B and, for some p with $1 < p \leq 2$, $|f|^p$ has an $\mathcal{M}$–harmonic majorant, then $D_\gamma(f) < \infty$ for all $\gamma \geq 2n/p$. For holomorphic functions the result holds for all p, $0 < p \leq 2$.

In Section 5 we restrict ourselves to holomorphic functions on B. In Theorem 5.1 we compute $D_\gamma(f)$ in terms of the series expansion of f. Specifically, if $f(z) = \sum a_\alpha z^\alpha$ is holomorphic in B, then for all $\gamma > (n-1)$,

$$D_\gamma(f) = 2\gamma n! \Gamma(\gamma - n + 1) \sum_{k=1}^{\infty} \frac{k}{\Gamma(\gamma + k + 1)} \sum_{|\alpha| = k} \alpha! |a_\alpha|^2. \tag{1.4}$$

Combining this with the results of Section 4 gives the following result (Theorem 5.2).

Suppose $f(z) = \sum a_\alpha z^\alpha$ is holomorphic in B. If

$$\sum_{k=1}^{\infty} \frac{k}{\Gamma(\frac{2n}{q} + k + 1)} \sum_{|\alpha| = k} \alpha! |a_\alpha|^2 < \infty \tag{1.5}$$

for some q, $2 \leq q < 2n/(n-1)$, then $|f(z)|^q$ has an $\mathcal{M}$–harmonic majorant on B, i.e., f is in the Hardy space H^q. Conversely, if $f \in H^q$ for some q, $0 < q \leq 2$, then the series in (1.5) converges.

When $n \geq 2$, the integral defining the space $\mathcal{D}_\gamma$ for holomorphic functions only makes sense for $\gamma > (n-1)$. On the other hand, the series in (1.4) is defined for all $\gamma > -1$. In Section 6 we consider the space $\widetilde{\mathcal{D}}_\gamma$ of holomorphic functions f on B for which this series converges, i.e., which satisfy

$$\widetilde{D}_\gamma(f) = \sum_{k=1}^{\infty} \frac{k}{\Gamma(\gamma+k+1)} \sum_{|\alpha|=k} \alpha! |a_\alpha|^2 < \infty \tag{1.6}$$

for some γ, $0 < \gamma \leq n$. We will show in Theorem 6.3 that if $\widetilde{D}_{2n/q}(f) < \infty$ for some q, $2 \leq q < \infty$, then f is in the Hardy space H^q. As a consequence of this result, we show that if f is in the unique Möbius invariant Hilbert space $\mathbb{H}$ on B, then $f \in H^p$ for all p, $0 < p < \infty$.

Of particular interest is the case $n = 1$, which we will consider in greater detail in Section 7. In this case, Theorem 5.2 can be stated as follows:

Suppose that $f(z) = \sum a_k z^k$ is holomorphic in $\mathbb{D}$, and that

$$\sum_{k=1}^{\infty} k^{1-2/q} |a_k|^2 < \infty \tag{1.7}$$

for some $q \geq 2$. Then $f \in H^q$. Conversely, if $f \in H^q$ for some q, $0 < q \leq 2$, then the series in (1.7) converges.

This result is closely related to the following theorem of Hardy and Littlewood:

If

$$\sum_{k=1}^{\infty} k^{q-2} |a_k|^q < \infty \tag{1.8}$$

for some $q \geq 2$, then $f \in H^q$. Conversely, if $f \in H^q$ for some q, $0 < q \leq 2$, then the series in (1.8) converges.

We will give an example showing that, for $q \neq 2$, the convergence of one of the series (1.7) and (1.8) does not imply the convergence of the other series. We will also give examples of holomorphic functions f on $\mathbb{D}$ with $f \in \mathcal{D}_{2/p}$, $0 < p < 2$, which are not in H^p, and of functions $f \in H^p$, $2 < p < \infty$, which are not in $\mathcal{D}_{2/p}$.

Dirichlet type spaces of holomorphic or $\mathcal{M}$–harmonic functions defined in terms of the invariant gradient $\widetilde{\nabla}$ have been studied by many other authors. K. T. Hahn and E. H. Youssfi [6] considered the spaces $\mathcal{B}_p(B)$ of holomorphic functions f on B for which $|\widetilde{\nabla} f| \in L^p(\tau)$. These spaces were also considered by Arazy, Fisher, Janson and Peetre [2] who showed that $\mathcal{B}_p(B)$ contains non-constant holomorphic functions if and only if $p > 2n$. The analogous spaces of $\mathcal{M}$–harmonic functions were investigated by Hahn and Youssfi [7, 8]. More general types of Dirichlet or Besov spaces of holomorphic functions have been studied by M. Peloso [12].

2. Notation and preliminary results

Let B denote the unit ball in $\mathbb{C}^n$ with boundary S. We will use the notation B_n or S_n if we wish to emphasize the dimension n. Throughout this paper B_1 will be denoted by $\mathbb{D}$. For $z \in B$, let φ_z denote the Möbius transformation of B satisfying $\varphi_z(0) = z$ and $\varphi_z \circ \varphi_z = I$, where I is the identity map. By [13, p. 26], φ_z satisfies

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z, w\rangle|^2}. \tag{2.1}$$

Let $\mathcal{M}$ denote the group of all biholomorphic automorphisms of B. Then any $\psi \in \mathcal{M}$ has a unique representation $\psi = U \circ \varphi_a$ for some $a \in B$ and some unitary transformation U. The invariant volume measure τ corresponding to the Bergman metric is given by $d\tau(w) = (1-|w|^2)^{-(n+1)}\, dm(w)$, where m is normalized Lebesgue measure on B.

The Laplace-Beltrami operator or the invariant Laplacian $\widetilde{\Delta}$ on B is given by

$$\widetilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0) = 4(1-|z|^2) \sum_{i,j=1}^{n} (\delta_{i,j} - z_j \overline{z}_i) \frac{\partial^2 f(z)}{\partial z_j \partial \overline{z}_i},$$

where Δ is the ordinary Laplacian. Similarly, for a C^1 function f the invariant real gradient $\widetilde{\nabla}$ with respect to the Bergman metric on B is defined by

$$\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0),$$

where ∇ is the real gradient in $\mathbb{R}^{2n}$ given by

$$\nabla f(z) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial y_n} \right), \quad z_k = x_k + i y_k.$$

As in [11] we have

$$|\widetilde{\nabla} f(z)|^2 = 2(1-|z|^2) \left[|\partial f(z)|^2 + |\partial \bar{f}(z)|^2 - |Rf(z)|^2 - |R\bar{f}(z)|^2 \right],$$

where ∂f is the complex gradient of f and Rf is the radial derivative of f given by

$$\partial f(z) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right) \quad \text{and} \quad Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j},$$

respectively. Thus, for a holomorphic function f it follows that

$$|\widetilde{\nabla} f(z)|^2 = 2(1-|z|^2) \left[|\partial f(z)|^2 - |Rf(z)|^2 \right],$$

and for a real-valued C^1 function u we have

$$|\widetilde{\nabla} u|^2 = 4(1-|z|^2) \left[|\partial u|^2 - |Ru|^2 \right].$$

Hence we have

$$4(1-|z|^2)^2 |\partial u(z)|^2 \le |\widetilde{\nabla} u(z)|^2 \le 4(1-|z|^2)|\partial u(z)|^2 \tag{2.2}$$

for all real-valued C^1 functions u. A similar inequality holds for complex-valued functions. The Laplacian $\widetilde{\Delta}$ and the gradient $\widetilde{\nabla}$ are both invariant under $\mathcal{M}$; that is, we have $\widetilde{\Delta}(f \circ \psi) = (\widetilde{\Delta} f) \circ \psi$ and $|\widetilde{\nabla}(f \circ \psi)| = |(\widetilde{\nabla} f) \circ \psi|$ for all $\psi \in \mathcal{M}$.

An upper-semicontinuous function $f : B \to [-\infty, \infty)$ is $\mathcal{M}$-*subharmonic* or *invariant subharmonic* if, for each $a \in B$,

$$f(a) \le \int_S f(\varphi_a(rt))\, d\sigma(t), \quad 0 < r < 1, \tag{2.3}$$

where $d\sigma$ denotes the normalized Lebesgue measure on S. For a C^2 function f this is equivalent to $\widetilde{\Delta} f \ge 0$. A continuous real or complex-valued function f is $\mathcal{M}$–*harmonic* on B if equality holds in (2.3). This is the case if and only if f is C^∞ and satisfies $\widetilde{\Delta} f = 0$. For an $\mathcal{M}$-subharmonic function f on B, the *Riesz measure* of f is the nonnegative Borel measure μ_f on B satisfying

$$\int_B \psi\, d\mu_f = \int_B f\, \widetilde{\Delta}\psi\, d\tau \tag{2.4}$$

for all ψ in $C_c^2(B)$, the class of twice continuously differentiable functions on B with compact support. If f is C^2, then by Green's identity, $d\mu_f = \widetilde{\Delta} f d\tau$.

Throughout this paper we fix δ, $0 < \delta < 1$, and for $a \in B$ we set

$$E(a) = E(a, \delta) = \{z \in B : |\varphi_a(z)| < \delta\}. \tag{2.5}$$

By [17, p. 33] we have $\tau(E(a,\delta)) = \delta^{2n}/(1-\delta^2)^n$. Also, there exists a constant $c > 0$, depending only on δ, such that

$$c^{-1}(1 - |a|^2) \le (1 - |w|^2) \le c(1 - |a|^2) \quad \text{for all} \quad w \in E(a). \tag{2.6}$$

The following result, due to M. Pavlovic [11, Theorem 2.1], will be used several times in this paper.

LEMMA 2.1. *Let f be a solution of $\widetilde{\Delta} f = \lambda f$, $\lambda \in \mathbb{C}$, and let $0 < p < \infty$.*
(a) If $\lambda \ne 0$, then there exists a constant $C = C(|\lambda|, p, \delta)$ such that

$$F_1^p(a) \le C \int_{E(a)} F_2^p(w) d\tau(w), \qquad a \in B, \tag{2.7}$$

whenever F_1, $F_2 \in \{|f|, |\widetilde{\nabla} f|\}$.
(b) If $\lambda = 0$, then inequality (2.7) holds except for the case when $F_1 = |f|$ and $F_2 = |\widetilde{\nabla} f|$.

Throughout this paper we write $f(z) \approx g(z)$ to indicate that there exist positive constants C_1 and C_2 such that

$$C_1 f(z) \le g(z) \le C_2 f(z)$$

for all appropriate z.

3. Eigenfunctions of $\widetilde{\Delta}$ with non-negative eigenvalues

Let $\mathcal{D}_\gamma = \mathcal{D}_\gamma(B_n)$ denote the weighted Dirichlet space of C^1 real or complex-valued functions f on B_n, as defined in the Introduction. Our first result shows that the only eigenfunction f of $\widetilde{\Delta}$ with positive eigenvalue and satisfying $D_\gamma(f) < \infty$ for some $\gamma \le n$ is the zero function.

Theorem 3.1. *If f is a solution of $\widetilde{\Delta} f = \lambda f$, $\lambda > 0$, with $D_\gamma(f) < \infty$ for some $\gamma \le n$, then $f(z) = 0$ for all $z \in B$.*

Proof. Suppose f satisfies $\widetilde{\Delta} f = \lambda f$ for some $\lambda > 0$. Write $f = u + iv$, where u and v are real-valued functions. Since λ is real, u and v are both eigenfunctions of $\widetilde{\Delta}$ with eigenvalue λ. Also, since $|\widetilde{\nabla} f|^2 = |\widetilde{\nabla} u|^2 + |\widetilde{\nabla} v|^2$, $D_\gamma(f)$ is finite if and only if both $D_\gamma(u)$ and $D_\gamma(v)$ are finite. Hence without loss of generality we can assume that f is real-valued.

Since $\lambda \neq 0$, we have by (2.6) and (2.7)

$$(1 - |z|^2)^\gamma |f(z)|^2 \le C \int_{E(z)} (1 - |w|^2)^\gamma |\widetilde{\nabla} f(w)|^2 d\tau(w),$$

where $E(z)$ is defined by (2.5). Integrating over B gives

$$\int_B (1 - |z|^2)^\gamma |f(z)|^2 d\tau(z) \le C \int_B \left[\int_{E(z)} (1 - |w|^2)^\gamma |\widetilde{\nabla} f(w)|^2 d\tau(w) \right] d\tau(z),$$

which by Fubini's theorem is equal to

$$C \int_B \tau(E(z)) (1 - |w|^2)^\gamma |\widetilde{\nabla} f(w)|^2 d\tau(w).$$

Since $\tau(E(z)) = \tau(E(z, \delta)) = \delta^{2n} / (1 - \delta^2)^n$, we obtain

$$\int_B (1 - |z|^2)^\gamma |f(z)|^2 d\tau(z) \le C_\delta D_\gamma(f).$$

Since f is real-valued, a straightforward computation gives $\widetilde{\Delta} f^2 = 2|\widetilde{\nabla} f|^2 + 2\lambda f^2$, and thus $\widetilde{\Delta} f^2 \ge 0$, since $\lambda > 0$. Hence f^2 is $\mathcal{M}$–subharmonic on B. But by Theorem 4.1 of [18], the only non-negative $\mathcal{M}$–subharmonic function g on B satisfying

$$\int_B (1 - |z|^2)^\gamma g(z)\, d\tau(z) < \infty$$

for some $\gamma \le n$ is the zero function. Hence $f(z) = 0$ for all $z \in B$. □

Remark 3.2. In [19] the author showed that if f is a solution of $\widetilde{\Delta} f = \lambda f$ with $\lambda \neq 0$ and $\lambda \ge -n^2$, then $f \in \mathcal{D}_\gamma$ if and only if $\gamma > \sqrt{n^2 + \lambda}$.

We now consider the case $\lambda = 0$. In this case the set of eigenfunctions of $\widetilde{\Delta}$ with eigenvalue zero is precisely the class $h(B)$ of $\mathcal{M}$–harmonic functions on B, which contains the class $\mathcal{H}(B)$ of holomorphic functions on B. When $n = 1$, the class of $\mathcal{M}$–harmonic functions coincides with the class of Euclidean harmonic functions on $\mathbb{D}$, as mentioned in the Introduction, and we have

$$D_\gamma(f) = \frac{1}{\pi}\iint_{\mathbb{D}} (1 - |z|^2)^\gamma |\nabla f(z)|^2 \, dxdy.$$

Now, if f is holomorphic or harmonic on $\mathbb{D}$, then $|\nabla f|^2$ is subharmonic on $\mathbb{D}$. From this it easily follows that if f is holomorphic or harmonic on $\mathbb{D}$ with $D_\gamma(f) < \infty$ for some $\gamma \le -1$, then f must be constant. On the other hand, any polynomial $p(z, \overline{z})$ on $\mathbb{D}$ satisfies $D_\gamma(p) < \infty$ for all $\gamma > -1$. Thus when $n = 1$, D_γ contains non-constant harmonic or holomorphic functions if and only if $\gamma > -1$.

For the case $n \ge 2$ the author [17, Proposition 10.9] showed that if f is $\mathcal{M}$–harmonic on B_n with $D_\gamma(f) < \infty$ for some $\gamma \le (n - 2)$, then f must be constant on B_n. However, for holomorphic functions on B_n, $n \ge 2$, we have the following theorem.

Theorem 3.3. *Let $n \ge 2$. Then $\mathcal{D}_\gamma$ contains non-constant holomorphic functions if and only if $\gamma > (n - 1)$.*

This result is an immediate consequence of the following lemma of Arazy, Fischer, Janson and Peetre [2]; it will also follow from the computations in Theorem 5.1.

Lemma 3.4 [2, Lemma 4.1]. *Let V be a linear subspace of the space $\mathcal{H}(B)$ of holomorphic functions on B such that*

(V1) if $f \in V$ and $\phi \in \mathcal{M}$, then $f \circ \phi \in V$,

(V2) if $f \in V$ then $g(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta} f(e^{i\theta} z)\, d\theta \in V$.

Then either V contains only constant functions, or V contains the linear function z_1.

It is easily shown that $\mathcal{D}_\gamma \cap \mathcal{H}(B)$ satisfies (V1) and (V2). But when $n \ge 2$, the linear function z_1 is in D_γ if and only if $\gamma > (n - 1)$.

The above lemma has been used, in some form or other, by several authors in proving similar results.

Remark 3.5. It is not known whether the space $\mathcal{D}_\gamma(B_n)$ $(n \ge 2)$ contains non-constant $\mathcal{M}$–harmonic functions for $(n - 2) < \gamma \le (n - 1)$. The proofs which show that $\mathcal{D}_\gamma \cap \mathcal{H}(B)$ is trivial (i.e., contains only constant functions) for $\gamma \le (n - 1)$ do not seem to translate to the case of $\mathcal{M}$–harmonic functions.

4. Harmonic majorants of Dirichlet finite harmonic functions

In this section we investigate the relationship between the Dirichlet spaces $\mathcal{D}_\gamma$ of $\mathcal{M}$–harmonic and holomorphic functions and the Hardy H^p spaces.

For $0 < p < \infty$, we denote by h^p (respectively H^p) the Hardy space of $\mathcal{M}$–harmonic (respectively holomorphic) functions f on B satisfying

$$\|f\|_p = \sup_{0<r<1} \left[\int_S |f(r\zeta)|^p d\sigma(\zeta) \right]^{1/p} < \infty.$$

Clearly, if f is holomorphic or $\mathcal{M}$-harmonic on B and if $|f|^p$ has an $\mathcal{M}$–harmonic majorant, then $f \in H^p$ or $f \in h^p$. Conversely, if $f \in H^p$ $(0 < p < \infty)$ or $f \in h^p$ $(1 \le p < \infty)$, then $|f|^p$ has an $\mathcal{M}$–harmonic majorant, and the least $\mathcal{M}$–harmonic majorant of $|f(z)|^p$ is given by

$$P[|f^*|^p](z) = \int_S P(z,t)|f^*(t)|^p d\sigma(t),$$

where $P(z,t) = (1-|z|^2)^n / |1 - \langle z,t \rangle|^{2n}$ is the Poisson kernel for $\widetilde{\Delta}$ on B, and f^* is the boundary function defined by $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$ a.e. on S. If $f \in h^1$, then the least $\mathcal{M}$–harmonic majorant of f is given by the Poisson integral of a measure. Since $P(z,\zeta) \le C\,(1-|z|^2)^{-n}$, we have

$$|f(z)|^p \le C\,(1-|z|^2)^{-n} \|f\|_p^p \tag{4.1}$$

for all $f \in H^p$ $(0 < p < \infty)$ or $f \in h^p$ $(1 \le p < \infty)$.

LEMMA 4.1. *Let f be $\mathcal{M}$–harmonic or holomorphic on B. Then $f \in h^p$ $(1 < p < \infty)$ or $f \in H^p$ $(0 < p < \infty)$ if and only if*

$$\int_B (1-|w|^2)^n d\mu_{|f|^p}(w) < \infty, \tag{4.2}$$

where $\mu_{|f|^p}$ is the Riesz measure of $|f|^p$ defined by (2.4). Furthermore, if this condition is satisfied, then

$$\|f\|_p^p = \int_B |f(z)|^p \, dm(z) + \frac{1}{4n^2} \int_B (1-|z|^2)^n \, d\mu_{|f|^p}(z). \tag{4.3}$$

Proof. Since, for the given range of p, $|f|^p$ is $\mathcal{M}$–subharmonic on B, $|f|^p$ has an $\mathcal{M}$–harmonic majorant if and only if (4.2) holds; see [16, Proposition 4; 17, Theorem 6.14]. Furthermore, by the Riesz decomposition theorem,

$$|f(z)|^p = P[|f^*|^p](z) - \int_B G(z,w)\, d\mu_{|f|^p}(w), \tag{4.4}$$

where $G(z,w)$ is the Green function for the Laplace-Beltrami operator $\widetilde{\Delta}$ on B given by $G(z,w) = g(\varphi_z(w))$, where for $z \in B$,

$$g(z) = \frac{1}{2n} \int_{|z|}^1 t^{-2n+1}(1-t^2)^{n-1}\, dt.$$

Integrating the identity (4.4) with respect to the normalized Lebesgue measure m gives

$$\int_B |f(z)|^p\, dm(z) = \int_B P[|f^*|^p](z)\, dm(z) - \int_B \int_B G(z,w)\, d\mu_{|f|^p}(w)\, dm(z),$$

which, by Fubini's theorem and the fact that $P[|f^*|^p](z)$ is $\mathcal{M}$–harmonic on B, equals

$$P[|f^*|^p](0) - \int_B \int_B G(z,w)\, dm(z)\, d\mu_{|f|^p}(w).$$

Let $\psi(z) = (1-|z|^2)^n$. Then $\widetilde{\Delta}\psi(z) = -4n^2(1-|z|^2)^{n+1}$. Therefore,

$$\int_B G(z,w)\, dm(z) = -\frac{1}{4n^2}\int_B G(z,w)\, \widetilde{\Delta}\psi(z)\, d\tau(z) = \frac{1}{4n^2}\psi(w).$$

The last equality follows from the Riesz decomposition theorem. This integral can also be evaluated directly using polar coordinates. The result now follows from the fact that $P[|f^*|^p](0) = \|f\|_p^p$. $\square$

We are now ready to state and prove the main result of the paper.

THEOREM 4.2. *(a) Let f be $\mathcal{M}$–harmonic (or holomorphic) on B. If $f \in \mathcal{D}_\gamma$ for some γ, $0 < \gamma \le n$, then $f \in h^p$ (or $f \in H^p$) for all p, $0 < p \le 2n/\gamma$, with*

$$\|f\|_p \le C\|f\|_{\mathcal{D}_\gamma}.$$

(b) Conversely, if $f \in H^p$, $0 < p \le 2$ (or $f \in h^p$, $1 < p \le 2$), then $f \in \mathcal{D}_\gamma$ for all $\gamma \ge 2n/p$ with

$$\|f\|_{\mathcal{D}_\gamma} \le C\|f\|_p.$$

Proof. (a) Suppose f is $\mathcal{M}$–harmonic on B. Without loss of generality we can assume that f is real-valued and that $f(0) = 0$.

Let $p = 2n/\gamma$. Since $0 < \gamma \le n$, we have $p \ge 2$ and thus $|f|^p$ is C^2 on B. A straightforward computation shows that $\widetilde{\Delta}|f|^p = p(p-1)|f|^{p-2}|\widetilde{\nabla} f|^2$. Thus the Riesz measure of $|f|^p$ is given by

$$d\mu_{|f|^p}(z) = \widetilde{\Delta}|f(z)|^p\, d\tau(z) = p(p-1)|f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 d\tau(z).$$

If f is holomorphic on B, then the Riesz measure of $|f|^p$ is given by $d\mu_{|f|^p}(z) = \frac{1}{4}p^2|f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 d\tau(z)$. By Lemma 4.1, f is in h^p if and only if

$$\int_B (1-|w|^2)^n\, \widetilde{\Delta}|f(w)|^p\, d\tau(w) < \infty.$$

This, however, is the case if and only if the integral

$$I = \int_B (1-|w|^2)^n |f(w)|^{p-2}|\widetilde{\nabla} f(w)|^2\, d\tau(w)$$

is finite. Since $(1-|z|^2) \approx (1-|w|^2)$ for all $z \in E(w)$, (2.7) gives

$$(1-|z|^2)^\gamma |\widetilde{\nabla} f(z)|^2 \leq C \int_{E(z)} (1-|w|^2)^\gamma |\widetilde{\nabla} f(w)|^2 \, d\tau(w) \leq C D_\gamma(f) < \infty.$$

Therefore

$$|\widetilde{\nabla} f(z)| \leq C D_\gamma(f)^{1/2} (1-|z|^2)^{-\gamma/2} \tag{4.5}$$

for some positive constant C. For $z \in B$ and $t \in [0,1]$ set $g(t) = f(tz)$. Since $f(0) = 0$, we obtain from (2.2) and (4.5)

$$\begin{aligned} |f(z)| &\leq \int_0^1 |g'(t)| dt \leq 2|z| \int_0^1 |\partial f(tz)| \, dt \leq |z| \int_0^1 \frac{|\widetilde{\nabla} f(tz)|}{(1-t^2|z|^2)} \, dt \\ &\leq |z| D_\gamma(f)^{1/2} \int_0^1 (1-t^2|z|^2)^{-(\gamma/2)-1} \, dt. \end{aligned}$$

Thus

$$|f(z)| \leq C D_\gamma(f)^{1/2} (1-|z|^2)^{-\gamma/2} \tag{4.6}$$

for some positive constant C. It follows that

$$\begin{aligned} I &= \int_B (1-|z|^2)^n |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 \, d\tau(z) \\ &\leq C \, D_\gamma(f)^{(1/2)(p-2)} \int_B (1-|z|^2)^{n-(p-2)(\gamma/2)} |\widetilde{\nabla} f(z)|^2 \, d\tau(z). \end{aligned}$$

With $p = 2n/\gamma$ we have $n - (p-2)(\gamma/2) = \gamma$ and therefore

$$I \leq C \, D_\gamma(f)^{(1/2)(p-2)} \int_B (1-|z|^2)^\gamma |\widetilde{\nabla} f(z)|^2 \, d\tau(z) \leq C \, D_\gamma(f)^{p/2},$$

or

$$\int_B (1-|z|^2)^n d\mu_{|f|^p}(z) \leq C \, D_\gamma(f)^{p/2}, \quad p = 2n/\gamma. \tag{4.7}$$

Hence, by Lemma 4.1, we have $f \in h^{2n/\gamma}$, and thus $|f|^p$ has an $\mathcal{M}$–harmonic majorant on B for all p, $0 < p \leq 2n/\gamma$.

It remains to show that $\int_B |f(z)|^p dm(z) \leq C \, D_\gamma(f)^{p/2}$. In the case $p = 2n/\gamma$ we have

$$\int_B |f(z)|^p dm(z) = \int_B (1-|z|^2)^{n+1} |f(z)|^p d\tau(z),$$

which by [17, Theorem 10.10] is

$$\leq C \int_B (1-|z|^2)^{n+1} |\widetilde{\nabla} f(z)|^p d\tau(z).$$

By inequality (4.5) the last integral is

$$\le C\, D_\gamma(f)^{(1/2)(p-2)} \int_B (1-|z|^2)^{\gamma+1}|\widetilde{\nabla} f(z)|^2 d\tau(z) \le C\, D_\gamma(f)^{p/2}.$$

Combining this with (4.7) and Lemma 4.1 gives $\|f\|_{2n/\gamma} \le C D_\gamma(f)^{1/2}$ for $p = 2n/\gamma$, from which the result follows for all p with $0 < p \le 2n/\gamma$.

(b) We first prove the result for a real-valued $\mathcal{M}$–harmonic function f in the class h^p $(1 < p \le 2)$. Note that, for $1 < p < 2$, the function $|f|^p$ is, in general, not C^2 on B. To overcome this difficulty, we set $f_\epsilon(z) = f(z) + i\epsilon$, $\epsilon > 0$. Then f_ϵ is $\mathcal{M}$–harmonic on B with $f_\epsilon \in h^p$ and $f_\epsilon(z) \ne 0$ for all $z \in B$. Thus $|f_\epsilon|^p$ is C^2 on B. A straightforward computation shows

$$\begin{aligned}\widetilde{\Delta}|f_\epsilon|^p &= p|f_\epsilon|^{p-2}\left[\frac{(p-1)|f|^2+\epsilon^2}{|f|^2+\epsilon^2}\right]|\widetilde{\nabla} f|^2\\ &\ge p(p-1)|f_\epsilon|^{p-2}|\widetilde{\nabla} f|^2.\end{aligned}$$

Moreover, by (4.1) we have $(1-|z|^2) \le C|f_\epsilon(z)|^{-p/n}\|f_\epsilon\|_p^{p/n}$ for all $z \in B$. Hence, if $\gamma \ge n$, then

$$\begin{aligned}D_\gamma(f) &= \int_B (1-|z|^2)^\gamma |\widetilde{\nabla} f(z)|^2 d\tau(z)\\ &= \int_B (1-|z|^2)^{n+(\gamma-n)}|\widetilde{\nabla} f(z)|^2 d\tau(z)\\ &\le C\|f_\epsilon\|_p^{(p/n)(\gamma-n)} \int_B (1-|z|^2)^n |f_\epsilon(z)|^{-(p/n)(\gamma-n)}|\widetilde{\nabla} f(z)|^2\, d\tau(z).\end{aligned}$$

In the case $\gamma = 2n/p$ we have $(p/n)(\gamma-n) = 2-p$. Therefore

$$\begin{aligned}D_{2n/p}(f) &\le C\|f_\epsilon\|_p^{2-p}\int_B (1-|z|^2)^n |f_\epsilon(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 d\tau(z)\\ &\le \frac{C}{p(p-1)}\|f_\epsilon\|_p^{2-p}\int_B (1-|z|^2)^n \widetilde{\Delta}|f_\epsilon(z)|^2 d\tau(z),\end{aligned}$$

which by Lemma 4.1 is

$$\le C_{n,p}\|f_\epsilon\|_p^{2-p}\|f_\epsilon\|_p^p = C_{n,p}\|f_\epsilon\|_p^2.$$

Since $\|f_\epsilon\|_p \to \|f\|_p$ as $\epsilon \to 0$, we have $D_{2n/p}(f) \le C_{n,p}\|f\|_p^2$. Finally, since $|f(0)|^p \le P[|f^*|^p](0) = \|f\|_p^p$, we obtain

$$\|f\|_{\mathcal{D}_{2n/p}} \le C_{n,p}\|f\|_p.$$

The conclusion now follows since $D_\gamma(f) \le D_{2n/p}(f)$ for all $\gamma \ge 2n/p$.

Finally, suppose $f \in H^p$, $0 < p < 2$. In [16] we proved that the Riesz measure of $|f|^p$ is given by $d\mu_{|f|^p} = f_p^\sharp d\tau$ where

$$f_p^\sharp(z) = \tfrac{1}{4}p^2|f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 \qquad \text{a.e. on } B.$$

This is valid for all p, $0 < p < \infty$. Since the zero set of a holomorphic function has τ measure zero, (4.1) gives

$$(1-|z|^2)^{\gamma-n} \le C\|f\|_p^{2-p}|f(z)|^{p-2} \qquad \tau\text{–a.e.}$$

In the case $\gamma = 2n/p$ we conclude, as above,

$$\begin{aligned} D_{2n/p}(f) &= \int_B (1-|z|^2)^{\gamma}|\widetilde{\nabla} f(z)|^2 d\tau(z) \\ &\le C_{n,p}\|f\|_p^{2-p}\int_B (1-|z|^2)^n|f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2 d\tau(z) \le C_{n,p}\|f\|_p^2. \end{aligned}$$

This implies that $\|f\|_{\mathcal{D}_\gamma} \le C_{n,p}\|f\|_p$ for all $\gamma \ge 2n/p$. □

Remark 4.3. One can restate part (a) of Theorem 4.2 as follows:

Let f be $\mathcal{M}$–harmonic (or holomorphic) on B. If $f \in \mathcal{D}_{2n/p}$ for some $p \ge 2$, then $f \in h^p$ (respectively H^p).

As we have shown in Section 3, in the case $n \ge 2$, the only holomorphic functions f on B for which $D_\gamma(f)$ is finite for $\gamma \le (n-1)$ are the constant functions. Thus the hypothesis $(n-1) < \gamma \le n$ forces $2 \le p < 2n/(n-1)$.

Examples 4.4.

(a) We first show that the hypothesis $u \in \mathcal{D}_\gamma$ of part (a) of Theorem 4.2 cannot be replaced by $u \in \mathcal{D}_{\gamma+\epsilon}$ for any $\epsilon > 0$. Let u be defined by $u(z) = \operatorname{Re} f(z)$, where

$$f(z) = (1-z_1)^{-\gamma/2}.$$

We will show that for $(n-1) \le \gamma \le n$, $D_{\gamma+\epsilon}(u)$ is finite for all $\epsilon > 0$, but that $|u(z)|^{2n/\gamma}$ does not have an $\mathcal{M}$–harmonic majorant. Since $2n/\gamma \ge 2$, if $|u(z)|^{2n/\gamma}$ had an $\mathcal{M}$–harmonic majorant, then the generalization of the M. Riesz theorem to the unit ball (see [15]) would imply that f is in the Hardy space $H^{2n/\gamma}$. We will show that this is not the case.

Since f is a function of z_1 only, and $u = \operatorname{Re} f$, we have

$$\begin{aligned} |\widetilde{\nabla} u(z)|^2 &= 4(1-|z|^2)\left[|\partial u(z)|^2 - |Ru(z)|^2\right] \\ &= (1-|z|^2)(1-|z_1|^2)|f'(z_1)|^2 \\ &= \frac{\gamma^2(1-|z|^2)(1-|z_1|^2)}{4|1-z_1|^{\gamma+2}}. \end{aligned}$$

By the inequality $(1-|z_1|^2) \le 2|1-z_1|$ it follows that $|\widetilde{\nabla} u(z)|^2 \le C(1-|z|^2)|1-z_1|^{-(\gamma+1)}$ for some constant C. Hence

$$\begin{aligned} D_{\gamma+\epsilon}(u) &= \int_B (1-|z|^2)^{\gamma+\epsilon}|\widetilde{\nabla} u(z)|^2\, d\tau(z) \\ &\le C\int_0^1 r^{2n-1}(1-r^2)^{\gamma+\epsilon-n}\int_S \frac{d\sigma(\zeta)}{|1-r\zeta_1|^{\gamma+1}}\, dr. \end{aligned}$$

By [13, Proposition 1.4.10], we have for all $z \in B$ and c real,

$$\int_S \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+c}} \approx \begin{cases} (1-|z|^2)^{-c}, & c>0, \\ \log \dfrac{1}{(1-|z|^2)}, & c=0, \\ \quad 1, & c<0. \end{cases} \tag{4.8}$$

Thus, with $c = \gamma - (n-1)$ we have

$$D_{\gamma+\epsilon}(u) \le C \int_0^1 r^{2n-1}(1-r^2)^{\epsilon-1} dr$$

for $\gamma > (n-1)$, and

$$D_{\gamma+\epsilon}(u) \le C \int_0^1 r^{2n-1}(1-r^2)^{\epsilon-1} \log \frac{1}{(1-r^2)}\, dr.$$

for $\gamma = (n-1)$. Both of these integrals are finite for all $\epsilon > 0$. On the other hand, we have

$$\int_S |f(r\zeta)|^{2n/\gamma} d\sigma(\zeta) = \int_S \frac{d\sigma(\zeta)}{|1-r\zeta_1|^n} \approx \log \frac{1}{(1-r^2)}.$$

Hence $f \notin H^{2n/\gamma}$. □

(b) We next show that the hypothesis $f \in H^p$ $(0 < p \le 2)$ of part (b) of Theorem 4.2 cannot be replaced by $f \in H^q$ for all $q < p$. As in (a), let

$$f(z) = (1-z_1)^{-n/p}.$$

Then for $q < p$ we have, by (4.8),

$$\int_S |f(r\zeta)|^q d\sigma(\zeta) = \int_S \frac{d\sigma(\zeta)}{|1-r\zeta_1|^{n+n((q/p)-1)}} \le C$$

for all r, $0 \le r < 1$. Thus $f \in H^q$ for all $q < p$. On the other hand, we now show that $f \notin \mathcal{D}_{2n/p}$. As above, we let

$$|\widetilde{\nabla} f(z)|^2 = \frac{n^2(1-|z|^2)(1-|z_1|^2)}{p^2|1-z_1|^{(2n/p)+2}} \ge \frac{n^2(1-|z|^2)^2}{p^2|1-z_1|^{(2n/p)+2}}.$$

By integration in polar coordinates it follows that

$$D_{2n/p}(f) \ge C_{n,p} \int_0^\rho r^{2n-1}(1-r^2)^{(2n/p)-n+1} \int_S \frac{d\sigma(\zeta)}{|1-r\zeta_1|^{(2n/p)+2}}\, dr$$

for any ρ with $0 < \rho < 1$. But by (4.8) we have

$$\int_S \frac{d\sigma(\zeta)}{|1-r\zeta_1|^{(2n/p)+2}} \ge C(1-r^2)^{n-(2n/p)-2}$$

and thus

$$D_{2n/p}(f) \ge C \log \frac{1}{(1-\rho^2)}.$$

for any ρ, $0 < \rho < 1$. Hence $D_{2n/p}(f) = \infty$ and thus $f \notin D_{2n/p}$.

(c) Our final example shows that the conclusion of Theorem 4.2(b) need not hold for $h \in h^1$. Set

$$h(z) = P(z, e_1) = \frac{(1-|z|^2)^n}{|1-z_1|^{2n}}.$$

Then h is a non-negative $\mathcal{M}$–harmonic function on B and thus is an element of the $\mathcal{M}$–harmonic Hardy space h^1. We will show that $h \notin \mathcal{D}_{2n}$. Since h is harmonic, we have $|\widetilde{\nabla} h(z)|^2 = (1/2)\widetilde{\Delta} h^2(z)$. But $P^2(z, e_1)$ is an eigenfunction of $\widetilde{\Delta}$ with eigenvalue $8n^2$ (see [13, Theorem 4.2.2]). Thus

$$|\widetilde{\nabla} h(z)|^2 = 4n^2 h^2(z) = 4n^2 \frac{(1-|z|^2)^{2n}}{|1-z_1|^{4n}}.$$

In the case $\gamma = 2n$, an integration in polar coordinates shows that, for any ρ with $0 < \rho < 1$,

$$\begin{aligned} D_{2n}(h) &\geq 2n \int_0^\rho r^{2n-1}(1-r^2)^{n-1} \int_S |\widetilde{\nabla} h(r\zeta)|^2 \, d\sigma(\zeta)\, dr \\ &= 8n^3 \int_0^\rho r^{2n-1}(1-r^2)^{3n-1} \int_S \frac{d\sigma(\zeta)}{|1-r\zeta_1|^{4n}}\, dr \end{aligned}$$

which by (4.8) is

$$\geq C \int_0^\rho r^{2n-1}(1-r^2)^{-1}\, dr \approx \log \frac{1}{(1-\rho^2)}.$$

Therefore $D_{2n}(h) = \infty$ and hence $h \notin D_{2n}$. □

It is easily seen that in the last example, $h \in \mathcal{D}_\gamma$ for all $\gamma > 2n$. We now show that this is always the case.

Proposition 4.5. *If $f \in h^1$, then $f \in \mathcal{D}_\gamma$ for all $\gamma > 2n$.*

Proof. If $h \in h^1$, then $h(z) = P[\nu](z)$, where ν is a signed measure on S with total variation $|\nu|(S) = \|h\|_1$. By [17, Proposition 10.3] we have $|\widetilde{\nabla} h(z)| \leq 2nP[|\nu|](z)$. Thus, by Hölder's inequality,

$$|\widetilde{\nabla} h(z)|^2 \leq 2n|\nu|(S) \int_S P^2(z,t)\, d|\nu|(t),$$

and as above

$$\int_S |\widetilde{\nabla} h(r\zeta)|^2 d\sigma(\zeta) \leq C(1-r^2)^{-n}.$$

From this it follows that $D_\gamma(h) < \infty$ for all $\gamma > 2n$. □

5. Holomorphic functions in $\mathcal{D}_\gamma$

In this section we consider $\mathcal{D}_\gamma \cap \mathcal{H}(B)$, where $\mathcal{H}(B)$ is the set of holomorphic functions on B. We begin by computing $D_\gamma(f)$ for $f \in \mathcal{H}(B)$. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, where each α_i is a non-negative integer, we use the standard notations

$$a_\alpha = a_{\alpha_1,\dots,\alpha_n}, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Also, for $z = (z_1, \dots, z_n)$ we set $Z_\alpha(z) = z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. It is well known that the monomials $\{Z_\alpha\}_\alpha$ are orthogonal on S.

Theorem 5.1. *Suppose $f(z) = \sum_\alpha a_\alpha z^\alpha = \sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha z^\alpha$ is holomorphic on B. Then we have, for all $\gamma > (n-1)$,*

$$D_\gamma(f) = 2\gamma\, n!\, \Gamma(\gamma - n + 1) \sum_{k=1}^\infty \frac{k}{\Gamma(\gamma + k + 1)} \sum_{|\alpha|=k} \alpha!\, |a_\alpha|^2, \tag{5.1}$$

and

$$D_\gamma(f) = \frac{2\gamma\, \Gamma(\gamma - n + 1)}{\Gamma(\gamma - n + 2)} \int_B (1 - |z|^2)^{\gamma+2} |\partial f(z)|^2 d\tau(z). \tag{5.2}$$

Furthermore, if $n \geq 2$, then $\mathcal{D}_\gamma \cap \mathcal{H}(B)$ is nontrivial if and only if $\gamma > (n-1)$.

Proof. Since f is holomorphic, we have

$$|\widetilde{\nabla} f(z)|^2 = 2(1 - |z|^2)[|\partial f(z)|^2 - |Rf(z)|^2].$$

Thus, by integration in polar coordinates,

$$\begin{aligned} D_\gamma(f) &= \int_B (1 - |z|^2)^\gamma |\widetilde{\nabla} f(z)|^2 d\tau(z) \\ &= 2 \int_B (1 - |z|^2)^{\gamma - n} \left[|\partial f(z)|^2 - |Rf(z)|^2 \right] \, dm(z) \\ &= 4n \int_0^1 r^{2n-1} (1 - r^2)^{\gamma - n} \int_S \left[|\partial f(r\zeta)|^2 - |Rf(r\zeta)|^2 \right] \, d\sigma(\zeta) \, dr. \end{aligned}$$

We now compute

$$I = \int_S \left[|\partial f(r\zeta)|^2 - |Rf(r\zeta)|^2 \right] \, d\sigma(\zeta).$$

Consider first

$$I_1 = \int_S |\partial f(r\zeta)|^2 d\sigma(\zeta) = \sum_{j=1}^n \int_S |\partial_j f(r\zeta)|^2 d\sigma(\zeta).$$

For $j = 1, ..., n$, set $\hat{\alpha}(j) = (\alpha_1, ..., \alpha_j - 1, ..., \alpha_n)$. Then

$$\partial_j f(z) = \sum_{k=1}^\infty \sum_{|\alpha|=k} a_\alpha \alpha_j Z_{\hat{\alpha}(j)}(z).$$

Since the monomials $Z_\beta(z)$ are orthogonal on S,

$$\int_S |\partial_j f(r\zeta)|^2 d\sigma(\zeta) = \sum_{k=1}^{\infty}\sum_{|\alpha|=k} |a_\alpha|^2 \alpha_j^2 \int_S |Z_{\hat{\alpha}(j)}(r\zeta)|^2 d\sigma(\zeta)$$
$$= \sum_{k=1}^{\infty}\sum_{|\alpha|=k} |a_\alpha|^2 \alpha_j^2\, r^{2|\alpha|-2} \int_S |Z_{\hat{\alpha}(j)}(\zeta)|^2 d\sigma(\zeta)\, dr.$$

By [13, Proposition 1.4.9] we have

$$\int_S |Z_{\hat{\alpha}(j)}(\zeta)|^2 = \frac{(n-1)!\hat{\alpha}(j)!}{(n-1+|\hat{\alpha}(j)|)!}.$$

Since $\alpha_j\hat{\alpha}(j)! = \alpha!$ and $|\hat{\alpha}(j)| = |\alpha|-1$, we have

$$\int_S |\partial_j f(r\zeta)|^2 d\sigma(\zeta) = (n-1)!\sum_{k=1}^{\infty}\sum_{|\alpha|=k} \frac{|a_\alpha|^2\alpha_j \alpha!\, r^{2|\alpha|-2}}{(n+|\alpha|-2)!}.$$

Finally, summing over $j = 1, ..., n$ gives

$$I_1 = \int_S |\partial f(r\zeta)|^2 d\sigma(\zeta) = (n-1)!\sum_{k=1}^{\infty} \frac{k\, r^{2k-2}}{(n+k-2)!}\sum_{|\alpha|=k} \alpha!|a_\alpha|^2. \tag{5.3}$$

We next evaluate the integral $I_2 = \int_S |Rf(r\zeta)|^2 d\sigma(\zeta)$. Since

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j} = \sum_{k=1}^{\infty}\sum_{|\alpha|=k} a_\alpha \sum_{j=1}^{n} z_j \frac{\partial Z_\alpha}{\partial z_j} = \sum_{k=1}^{\infty}\sum_{|\alpha|=k} a_\alpha |\alpha| Z_\alpha(z)$$

by the orthogonality of $\{Z_\alpha\}$, we obtain

$$I_2 = (n-1)!\sum_{k=1}^{\infty} \frac{k^2 r^{2k}}{(n+k-1)!}\sum_{|\alpha|=k} \alpha!|a_\alpha|^2.$$

Combining this with the above identity for I_1 gives

$$I = I_1 - I_2 = (n-1)!\sum_{k=1}^{\infty} \frac{k\, r^{2k-2}[(n-1)+k(1-r^2)]}{(n+k-1)!}\sum_{|\alpha|=k} \alpha!|a_\alpha|^2.$$

Hence

$$D_\gamma(f) = 2n!\sum_{k=1}^{\infty} \frac{k\, I(k)}{(n+k-1)!}\sum_{|\alpha|=k} \alpha!|a_\alpha|^2, \tag{5.4}$$

where

$$I(k) = 2\int_0^1 r^{2n+2k-3}\left[(n-1)+k(1-r^2)\right](1-r^2)^{\gamma-n}\, dr. \tag{5.5}$$

Since

$$2\int_0^1 r^{2m-3}(1-r^2)^\alpha dr = \int_0^1 s^{m-2}(1-s)^\alpha ds = \frac{\Gamma(m-1)\Gamma(\alpha+1)}{\Gamma(m+\alpha)}$$

for all $m > 1$ and $\alpha > -1$ (where Γ is the Gamma function), we have

$$\begin{aligned} I(k) &= (n-1)\frac{\Gamma(n+k-1)\Gamma(\gamma-n+1)}{\Gamma(\gamma+k)} + k\,\frac{\Gamma(n+k-1)\Gamma(\gamma-n+2)}{\Gamma(\gamma+k+1)} \\ &= \frac{\gamma\,\Gamma(\gamma-n+1)\Gamma(n+k)}{\Gamma(\gamma+k+1)}. \end{aligned}$$

Substituting this into (5.4) gives

$$D_\gamma(f) = 2\gamma\, n!\,\Gamma(\gamma-n+1)\sum_{k=1}^\infty \frac{k}{\Gamma(\gamma+k+1)}\sum_{|\alpha|=k}\alpha!\,|a_\alpha|^2.$$

We next derive equation (5.2). Using again integration in polar coordinates, we obtain

$$\begin{aligned} &\int_B (1-|z|^2)^{\gamma+2}|\partial f(z)|^2 d\tau(z) \\ &\qquad = 2n\int_0^1 r^{2n-1}(1-r^2)^{\gamma-n+1}\int_S |\partial f(r\zeta)|^2 d\sigma(\zeta)\,dr. \end{aligned}$$

By (5.3), this is equal to

$$\begin{aligned} &n!\sum_{k=1}^\infty\sum_{|\alpha|=k}\frac{k\,\alpha!|a_\alpha|^2}{\Gamma(n+k-1)}2\int_0^1 r^{2n+2k-3}(1-r^2)^{\gamma-n+1}dr \\ &\qquad = n!\Gamma(\gamma-n+2)\sum_{k=1}^\infty\frac{k}{\Gamma(\gamma+k+1)}\sum_{|\alpha|=k}\alpha!|a_\alpha|^2, \end{aligned}$$

and (5.2) now follows. Finally, suppose that $n > 1$. By (5.5) we have

$$I(k) \geq (n-1)\int_0^1 r^{2n+2k-3}(1-r^2)^{\gamma-n}dr,$$

which is finite if and only if $\gamma > (n-1)$. Thus, for $\gamma \leq n-1$, the only holomorphic functions in $\mathcal{D}_\gamma \cap \mathcal{H}(B)$ are the constant functions. □

Combining Theorem 4.2 with Theorem 5.1 gives the following result.

THEOREM 5.2. *Suppose* $f(z) = \sum_\alpha a_\alpha z^\alpha$ *is holomorphic in* B. *If the sequence* $\{a_\alpha\}$ *satisfies*

$$\sum_{k=1}^\infty \frac{k}{\Gamma(\frac{2n}{q}+k+1)}\sum_{|\alpha|=k}\alpha!\,|a_\alpha|^2 < \infty \tag{5.6}$$

for some q, $2 \le q < 2n/(n-1)$, then $f \in H^q$. Conversely, if $f \in H^q$, $0 < q \le 2$, then (5.6) holds.

6. Fractional derivatives and holomorphic functions in $\widetilde{\mathcal{D}}_\gamma$

The restriction $q < 2n/(n-1)$ in Theorem 5.2 is due to the fact that, in the case $n \ge 2$, the integral defining $\mathcal{D}_\gamma(f)$ is only defined for $\gamma > (n-1)$. However, the series in (5.1) is defined for all $\gamma > 0$, and in fact for $\gamma > -1$. Thus, for $\gamma > -1$ it makes sense to consider the space $\widetilde{\mathcal{D}}_\gamma$ of holomorphic functions $f(z) = \sum a_\alpha z^\alpha$ on B for which

$$\widetilde{D}_\gamma(f) = \sum_{k=1}^{\infty} \frac{k}{\Gamma(\gamma+k+1)} \sum_{|\alpha|=k} \alpha!|a_\alpha|^2 \tag{6.1}$$

is finite, with norm

$$\|f\|_{\widetilde{\mathcal{D}}_\gamma} = |f(0)| + (\widetilde{D}_\gamma(f))^{1/2}.$$

It is natural to ask the following question:

Suppose that $f \in \mathcal{H}(B)$ and $\widetilde{D}_\gamma(f)$ is finite for some γ, $0 < \gamma \le (n-1)$. Does it follow that $f \in H^{2n/\gamma}$? Alternately, if $n > 1$ and $f = \sum a_\alpha z^\alpha$ satisfies

$$\sum_{k=1}^{\infty} \frac{k}{\Gamma(\frac{2n}{p}+k+1)} \sum_{|\alpha|=k} \alpha!\,|a_\alpha|^2 < \infty$$

for some p, $2n/(n-1) \le p < \infty$, is f in H^p?

As we will show in Theorem 6.3 below, the answer is yes.

To this end, we introduce the radial fractional derivative of f. As in [1, 3,5], if f is holomorphic in B with homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \quad \text{where} \quad f_k(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha,$$

for $\beta > 0$, the *radial fractional derivative of f of order β*, denoted by $R^\beta f$, is defined by

$$R^\beta f(z) = \sum_{k=0}^{\infty} (k+1)^\beta f_k(z). \tag{6.2}$$

The function $R^\beta f$ is clearly holomorphic on B. When $\beta = 1$, $R^1 f = f + Rf$ where R is the radial derivative introduced in Section 2. For $0 < \beta < n$, let k_β denote the kernel

$$k_\beta(\zeta,\eta) = \frac{1}{|1 - \langle \zeta, \eta \rangle|^{n-\beta}}, \qquad \zeta,\, \eta \in S,$$

and for $g \in L^p(S)$, $p \ge 1$, set

$$(k_\beta * g)(\zeta) = \int_S k_\beta(\zeta,\eta) g(\eta) d\sigma(\eta).$$

The following lemma will be the key step in the proof of Theorem 6.3 below.

Lemma 6.1. *Let $1 < p < \infty$ and $0 < \beta < n$. If $R^\beta f \in H^p$, then $f \in H^q$, where $1/q = 1/p - \beta/n$, with $\|f\|_q \le C\|R^\beta f\|_p$.*

Proof. By [1, Lemma 1.7], if $R^\beta \in H^p$, then there exists $g \in L^p(S)$ with $\|R^\beta f\|_p = \|g\|_p$ such that

$$|f(z)| \le P[k_\beta * g](z).$$

But by Theorem 3 of [5], the mapping $g \to k_\beta * g$ is a bounded mapping of $L^p(S)$ to $L^q(S)$, where $1/q = 1/p - \beta/n$. Thus we have $f \in H^q$ with

$$\|f\|_q \le \|k_\beta * g\|_q \le C\|g\|_p = C\|R^\beta f\|_p. \qquad \square$$

Remark 6.2. Lemma 6.1 has also been proved, using different methods, by Wu Young Chen [3]

Theorem 6.3. *Suppose $f(z) = \sum a_\alpha z^\alpha$ is holomorphic on B. If $\widetilde{D}_\gamma(f) < \infty$ for some γ, $0 < \gamma \le n$, then $f \in H^{2n/\gamma}$ with*

$$\|f\|_{2n/\gamma} \le C\|f\|_{\widetilde{\mathcal{D}}_\gamma}.$$

Proof. Given $0 < \gamma \le (n-1)$, choose $\beta > 0$ such that $(n-1) < \gamma + 2\beta \le n$. Then by Theorem 5.1,

$$D_{\gamma+2\beta}(R^\beta f) = C_{\gamma,\beta} \sum_{k=1}^{\infty} \frac{k(k+1)^{2\beta}}{\Gamma(\gamma+2\beta+k+1)} \sum_{|\alpha|=k} \alpha! |a_\alpha|^2.$$

Since $\lim_{k\to\infty} k^{b-a}\Gamma(k+a)/\Gamma(k+b) = 1$, we have

$$\frac{(k+1)^{2\beta}}{\Gamma(\gamma+2\beta+k+1)} \approx \frac{1}{\Gamma(\gamma+k+1)}. \tag{6.3}$$

Thus $D_{\gamma+2\beta}(R^\beta f) \approx \widetilde{D}_\gamma(f)$. Hence by Theorem 4.2, $R^\beta f \in H^p$ with

$$\|R^\beta f\|_p \le C\|R^\beta f\|_{\mathcal{D}_{\gamma+2\beta}} \le C\|f\|_{\widetilde{\mathcal{D}}_\gamma},$$

where $p = 2n/(\gamma + 2\beta)$. But then by Lemma 6.1, we have $f \in H^q$ with $\|f\|_q \le C\|R^\beta f\|_p$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n} = \frac{\gamma}{2n}.$$

This proves the result. $\square$

From Theorem 5.2 and Theorem 6.3 we deduce the following result.

Corollary 6.4. *Let $f(z) = \sum a_\alpha z^\alpha$ be holomorphic in B. If*

$$\sum_{k=1}^{\infty} \frac{k}{\Gamma(\frac{2n}{p} + k + 1)} \sum_{|\alpha|=k} \alpha! \, |a_\alpha|^2 < \infty$$

for some p, $2 \le p < \infty$, then $f \in H^p$ with $\|f\|_p \le C \|f\|_{\mathcal{D}_{2n/p}}$.

K. Zhu [21] proved the existence of a unique Hilbert space $\mathbb{H}$ of holomorphic functions in B_n which is Möbius invariant. Zhu also showed that a function $f(z) = \sum_\alpha a_\alpha z^\alpha$ is in $\mathbb{H}$ if and only if

$$\sum_\alpha |a_\alpha|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < \infty.$$

This, however, is just the special case $\gamma = 0$ of (6.1). Thus

$$\mathbb{H} = \{ f \in \mathcal{H}(B) : \widetilde{D}_0(f) < \infty \}.$$

When $n = 1$, $\mathbb{H}$ is simply the Dirichlet space $\mathcal{D}_0$ of holomorphic functions f on U satisfying

$$\int_U |f'(z)|^2 \, dA(z) < \infty.$$

The space $\mathbb{H}$ has also been identified by M. Peloso [12] as the 2–Besov space $\mathcal{B}_2$ of holomorphic functions f on B for which $(1 - |z|^2)^m |R^m f(z)| \in L^2(\tau)$, where m is any integer with $m > n/2$.

Since $\Gamma(\gamma + k + 1) \ge \Gamma(k+1)$ for all $\gamma \ge 0$, $\widetilde{D}_0(f) < \infty$ implies $\widetilde{D}_\gamma(f) < \infty$ for all $\gamma > 0$. We thus have the following result, which represents a generalization of a result that is known in the case $n = 1$ (see [4, Exercise 7, p. 106]).

Proposition 6.5. *If f is in the unique Möbius invariant Hilbert space $\mathbb{H}$ on B, then $f \in H^p$ for all p, $0 < p < \infty$.*

7. The special case $n = 1$

In this section we consider the results of the previous sections for the special case of the unit disc $\mathbb{D}$ in $\mathbb{C}$. When $n = 1$, (6.3) gives

$$D_\gamma(f) \approx \sum_{k=1}^{\infty} k^{1-\gamma} |a_k|^2,$$

and Theorem 5.2 therefore reduces to the following result.

Theorem 7.1. *Suppose $f(z) = \sum a_k z^k$ is holomorphic in the unit disc $\mathbb{D}$.*

(a) If the sequence $\{a_k\}$ satisfies

$$\sum_{k=1}^{\infty} k^{1-2/q} |a_k|^2 < \infty \tag{7.1}$$

for some $q \geq 2$, then $f \in H^q(\mathbb{D})$.

(b) Conversely, if $f \in H^q(\mathbb{D})$ for some q, $0 < q \leq 2$, then (7.1) holds.[1]

Part (a) of Theorem 7.1 is Theorem 2 of [20]. This result is closely related to the following results of Hardy and Littlewood (see [4, Theorems 6.2 and 6.3]):

If $f(z) = \sum a_k z^k$ is holomorphic in $|z| < 1$, and if the sequence $\{a_k\}$ satisfies

$$\sum_{k=1}^{\infty} k^{q-2}|a_k|^q < \infty \tag{7.2}$$

for some $q \geq 2$, then $f \in H^q$ Conversely, if $f \in H^q$ for some q, $0 < q \leq 2$, then (7.2) holds.

For a generalization of these theorems see J. H. Shi [14].

Although the convergence of either of the series in (7.1) or (7.2) for some $q \geq 2$ implies that $f(z) = \sum a_k z^k$ is in the Hardy space H^q, we remark that for $q \neq 2$ the convergence of one series does not imply the convergence of the other series. For example, if $a_k = k^{(1/q)-1}(\log k)^{-1/2}$, $k \geq 2$, then

$$\sum_{k=2}^{\infty} k^{1-2/q} a_k^2 = \sum_{k=2}^{\infty} \frac{\log k}{k} = \infty, \text{ but } \sum_{k=2}^{\infty} k^{q-2} a_k^q = \sum_{k=2}^{\infty} \frac{1}{k\,(\log k)^{q/2}} < \infty$$

for all $q > 2$. In the other direction, the following example was suggested by my colleague Stephen Dilworth. Let

$$a_k = \begin{cases} 2^{n((1/q)-(1/2))}\dfrac{1}{n}, & \text{when } k = 2^n, \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$\sum_{k=1}^{\infty} k^{1-2/q} a_k^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \text{ but } \sum_{k=1}^{\infty} k^{q-2} a_k^q = \sum_{n=1}^{\infty} 2^{n((q/2)-1)} \frac{1}{n^q} = \infty$$

for all $q > 2$. Similar examples can be constructed for the case $q < 2$.

Example 7.2. We consider the holomorphic function $f_{p,q}$ on $\mathbb{D}$ defined by

$$f_{p,q}(z) = (1-z)^{-1/p} \left\{ \frac{1}{z} \log \frac{1}{(1-z)} \right\}^{-1/q}.$$

We will prove the following result.

(a) If $0 < p < 2$, then for any q with $p \leq q < 2$, we have $f_{p,q} \in \mathcal{D}_{2/p}$, but $f_{p,q} \notin H^p$.

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[1]Added in proof: Theorem 7.1(b) for the case $1 \leq q \leq 2$ was known to G.H. Hardy and J.E. Littlewood [Math. Ann. **97** (1926), 159–209]. For related results, see the article by P.L. Duren and G.D. Taylor [Illinois J. Math. **14** (1970), 419–423].

(b) If $2 < p < \infty$, then for any q with $2 \le q < p$, we have $f_{p,q} \in H^p$, but $f_{p,q} \notin \mathcal{D}_{2/p}$.

J. E. Littlewood ([10, pp. 93–96]) has shown that the Taylor coefficients $\{a_n\}$ of $f_{p,q}$ satisfy

$$a_n \approx n^{\frac{1}{p}-1} (\log n)^{-1/q}\,.$$

Thus

$$D_{2/p}(f_{p,q}) \approx \sum_{n=2}^{\infty} n^{1-\frac{2}{p}} |a_n|^2 = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{2/q}}.$$

The above series converges for all $q < 2$ and diverges for all $q \ge 2$. Thus

$$f_{p,q} \in \mathcal{D}_{2/p}(\mathbb{D}) \quad \Longleftrightarrow \quad q < 2.$$

On the other hand, we also have that

$$f_{p,q} \in H^p(\mathbb{D}) \quad \Longleftrightarrow \quad q < p.$$

For $q < p$, straightforward estimates give

$$\int_{-\pi}^{\pi} |f_{p,q}(re^{i\theta})|^p \, d\theta \le C_1 + C_2 \left[\log \frac{1}{(1-r)}\right]^{1-p/q}.$$

Thus $f_{p,q} \in H^p$ for all q, $q < p$. On the other hand, if $f \in H^p$, then its boundary function $f^*(e^{i\theta})$ is in $L^p[0, 2\pi]$. But for $0 < |\theta| < \pi$ we have

$$|f^*_{p,q}(e^{i\theta})|^p \approx \frac{1}{\theta |\log \theta|^{p/q}},$$

which is not integrable for any $q \ge p$. Thus $f_{p,q} \notin H^p$ for any $q \ge p$. □

Remark 7.3. The existence of functions having the desired properties (a) and (b) above can also be ascertained from a theorem of Littlewood [9] (see also [4, Theorem A.5]).

8. Comparison with a theorem of Yamashita

For $N \ge 3$, let B_N denote the unit ball in $\mathbb{R}^N$. Yamashita [20, Theorem 3] proved the following generalization of Theorem A for Euclidean harmonic functions on B_N.

THEOREM. *Let u be a harmonic function in B_N $(N \ge 3)$ satisfying*

$$\int_{B_N} (1 - |x|)^{\alpha} |\nabla u(x)|^2 dx < \infty \tag{8.1}$$

for some α, $0 \le \alpha \le 1$. Then for $p = 2(N-1)/(N+\alpha-2)$ the function $|u|^p$ admits a harmonic majorant in B_N.

Let $n > 1$ and suppose that f is holomorphic on $B \subset \mathbb{C}^n$ with $D_\gamma(f) < \infty$ for some γ, $(n-1) < \gamma \le n$. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and B with $B_{2n} \subset \mathbb{R}^{2n}$. If $u = \operatorname{Re} f$, then u is pluriharmonic on B and thus also (Euclidean) harmonic on B_{2n}. Also, since $D_\gamma(f) < \infty$, (5.2) gives

$$\int_B (1-|z|^2)^{\gamma+2}|\partial u(z)|^2\,d\tau(z) = \int_B (1-|z|^2)^{\gamma-n+1}|\partial u(z)|^2\,dm(z) < \infty.$$

But $|\partial u|^2 = (1/4)|\nabla u|^2$. Hence u satisfies (8.1) with $\alpha = \gamma - n + 1$. Thus for $p = 2(2n-1)\big/(n+\gamma-1)$, the function $|u|^p$ admits a (Euclidean) harmonic majorant F on B_{2n}. But since F satisfies the mean-value property,

$$\sup_{0<r<1}\int_S |u(r\zeta)|^p\,d\sigma(\zeta) \le \int_{\partial B_{2n}} F(r\zeta)\,d\sigma(\zeta) = F(0).$$

Since $p > 1$, by [15, Theorem 1] the function f is in H^p with $p = 2(2n-1)\big/(n+\gamma-1)$. However, when $n > 1$ and $(n-1) < \gamma < n$, this value of p is strictly smaller than the value $2n/\gamma$ given by Theorem 4.2.

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