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AN ALTERNATIVE TO THE HILBERT FUNCTION FOR THE IDEAL OF A FINITE SET OF POINTS IN $\mathbb{P}^n$

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1. Introduction

Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of s distinct points in the projective space $\mathbb{P}^n(k)$, where $k = \overline{k}$ is an algebraically closed field. Then $P_i \leftrightarrow \wp_i = (L_{i1}, \ldots, L_{in}) \subset R = k[x_0, x_1, \ldots, x_n]$, where the L_{ij}, $j = 1, \ldots, n$, are n linearly independent linear forms and $\wp_i$ is the (homogeneous) prime ideal of R generated by all the forms which vanish at P_i. The ideal

$$I = I_{\mathbb{X}} := \wp_1 \cap \cdots \cap \wp_s$$

is the ideal generated by all the forms which vanish at all the points of $\mathbb{X}$.

Since $R = \oplus_{i=0}^{\infty} R_i$ (R_i being the vector space of dimension $\binom{i+n}{n}$ generated by all the monomials in R having degree i) and $I = \oplus_{i=0}^{\infty} I_i$, we obtain that

$$A = R/I = \oplus_{i=0}^{\infty}(R_i/I_i) = \oplus_{i=0}^{\infty} A_i$$

is a graded ring. The numerical function

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}(A, t) := \dim_k A_t = \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the set $\mathbb{X}$ (or of the ring A).

In this paper, which is the first in a series, we introduce a new "character" (the *n-type vector*), which is an alternative to the Hilbert function for the set of points $\mathbb{X}$. Our main theorem (Theorem 2.6) shows that our new character is equivalent to the Hilbert function as a tool to describe finite sets of points in $\mathbb{P}^n$. The proof of this result occupies all of Section 2.

In Section 3 we connect our character with the *numerical character* introduced in 1978 by Gruson and Peskine [13] in their study of the points in $\mathbb{P}^2$ which are hyperplane sections of a curve in $\mathbb{P}^3$. Gruson-Peskine used the *numerical character* to reveal properties of all sets of points with a given Hilbert function. We translate their results using our new character; these

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translations suggest possible generalizations of the Gruson-Peskine results in $\mathbb{P}^2$ to results in $\mathbb{P}^n$. Indeed, we give some initial applications (Theorem 3.7 and Proposition 3.8) in this direction, which establish an extremal property of the collection of all sets of points in $\mathbb{P}^n$ with a fixed Hilbert function. The study of such extremal subsets is developed further in the third paper of this series [8].

We conclude this paper with a discussion of particular families of sets of points in $\mathbb{P}^n$ whose construction is strongly suggested by our character. We had done something similar in $\mathbb{P}^2$ and $\mathbb{P}^3$ (see [11], [12]), but it is only now, with our definition of the *n-type vector* well understood, that we can give the definition in higher dimensional spaces. A detailed study of these families of point sets is undertaken in [7].

We now define some notation and make some preliminary observations. The collection of functions

$$\mathcal{H}_n := \{\mathbf{H}_{\mathbb{X}} : \mathbb{N} \to \mathbb{N} \mid \mathbb{X} \text{ is a non-degenerate finite set of points in } \mathbb{P}^n\}$$

has been much studied. For example, we know:

(I) (Macaulay) If $\mathbf{H} \in \mathcal{H}_n$, then the values of $\mathbf{H}$, i.e.,
$$\mathbf{H}(0) = 1,\ \mathbf{H}(1) = n+1,\ \mathbf{H}(2),\ \ldots$$
form an O-sequence (see [18] for definition).

(II) If $\mathbf{H} \in \mathcal{H}_n$ and $\mathbf{H} = \mathbf{H}_{\mathbb{X}}$ for some set $\mathbb{X}$ then, for all $t \gg 0$, $\mathbf{H}(t) = |\mathbb{X}|$.

(III) If $\mathbf{H} \in \mathcal{H}_n$ and we define the function $\Delta\mathbf{H}$ by $\Delta\mathbf{H}(0) = 1$ and $\Delta\mathbf{H}(t) = \mathbf{H}(t) - \mathbf{H}(t-1)$ for $t > 0$, then the values of $\Delta\mathbf{H}$, i.e.,
$$\Delta\mathbf{H}(0) = 1,\ \Delta\mathbf{H}(1) = n,\ \Delta\mathbf{H}(2),\ \cdots$$
form an O-sequence which is eventually 0.

One can prove (see, e.g., [6]) that (III) is equivalent to saying that there is a homogeneous ideal $J \subset k[x_1, \ldots, x_n]$ satisfying

(1) $J \cap (x_1, \ldots, x_n)_1 = (0)$;
(2) $\sqrt{J} = (x_1, \ldots, x_n)$;
(3) If $B = k[x_1, \ldots, x_n]/J = \oplus_{i=0}^{\infty} B_i$, then $\Delta\mathbf{H}(t) = \dim_k B_t$.

That is, $\Delta\mathbf{H}$ is the Hilbert function of some Artinian quotient of $k[x_1, \ldots, x_n]$. In fact, in the terminology of [10] one has the following characterization of $\mathcal{H}_n$:

- $\mathbf{H} \in \mathcal{H}_n$ (for some n) if and only if $\mathbf{H}(1) = n+1$,
- $\mathbf{H}$ is a 0-dimensional (condition (II) above), differentiable (III), O-sequence (I).

We use (III) above to define the set of functions

$$\mathcal{H} - Art_n := \{\mathbf{H} : \mathbb{N} \to \mathbb{N} \mid \mathbf{H} \text{ is the Hilbert function of some Artinian graded quotient of } k[x_1, \ldots, x_n] \text{ and } \mathbf{H}(1) = n.\}$$

In light of the above remarks, we can consider Δ as a function from $\mathcal{H}_n$ to $\mathcal{H}-Art_n$. Since "integration" of a function in $\mathcal{H}-Art_n$ is a left inverse to Δ, we obtain that Δ is actually a 1-1 function. It is well-known (see, e.g., [6] or [15]) that Δ is also a surjective function. Thus, we can often reduce questions about $\mathcal{H}_n$ to analogous questions about $\mathcal{H}-Art_n$.

Given $\mathbf{H} \in \mathcal{H}_n$, we define:

$$\begin{aligned} \widetilde{\alpha}(\mathbf{H}) &= \text{least integer } t \text{ such that } \mathbf{H}(t) < \binom{t+n}{n}, \\ \sigma(\mathbf{H}) &= \text{least integer } t \text{ such that } \Delta\mathbf{H}(t+\ell) = 0 \text{ for all } \ell \geq 0. \end{aligned}$$

Notice that if, as above, B is a graded Artinian quotient of $k[x_1, \ldots, x_n]$ and if $B_t = 0$ for some t, then $B_{t+\ell} = 0$ for all $\ell \geq 0$. It follows from this observation that we could have defined $\sigma(\mathbf{H})$ as the least integer t such that $\Delta\mathbf{H}(t) = 0$. Clearly, $\widetilde{\alpha}(\mathbf{H}) \leq \sigma(\mathbf{H})$, and $\mathbf{H} \in \mathcal{H}_n$ is completely known once we know the first $\sigma(\mathbf{H})$ values of $\mathbf{H}$, i.e.,

$$\mathbf{H}(0), \mathbf{H}(1) = n+1, \cdots, \mathbf{H}(\sigma(\mathbf{H})-1).$$

We shall also need to consider degenerate sets of points in $\mathbb{P}^n$ and their Hilbert functions. In order to do that in a systematic manner we define

$$\mathcal{S}_n = \bigcup_{i \leq n} \mathcal{H}_i.$$

Thus, $\mathcal{S}_n$ is the collection of Hilbert functions of all sets of points in $\mathbb{P}^n$.

Unfortunately, in the case $\mathbf{H} \in \mathcal{S}_n$ the above definition of $\widetilde{\alpha}(\mathbf{H})$ is not appropriate. In order to avoid the possibility of confusion we define, for $\mathbf{H} \in \mathcal{S}_n$,

$$\alpha(\mathbf{H}) = \begin{cases} 1 & \text{if } \mathbf{H} \in \mathcal{H}_i, \ i < n, \\ \widetilde{\alpha}(\mathbf{H}) & \text{if } \mathbf{H} \in \mathcal{H}_n. \end{cases}$$

Notice that the definition of $\sigma(\mathbf{H})$ does not depend on where we consider $\mathbf{H}$.

In [13], Gruson and Peskine studied the case of $\mathcal{S}_2$ and observed that $\mathbf{H} \in \mathcal{S}_2$ could, in fact, be completely described by only $\alpha(\mathbf{H})$ numbers, which they called the *numerical character* of $\mathbf{H}$.

To understand the Gruson-Peskine result we use the fact that Δ gives an isomorphism between the sets $\mathcal{H}_n$ and $\mathcal{H}-Art_n$ and consider only $\Delta\mathbf{H} \in \mathcal{H}-Art_2$. Since $\Delta\mathbf{H}$ is the Hilbert function of some graded Artinian quotient of $k[x_1, x_2]$, it is easy to see that

$$\Delta\mathbf{H} := 1 \;\; 2 \;\; 3 \;\; \cdots \;\; \alpha \;\; h_\alpha \;\; h_{\alpha+1} \cdots \;\; h_{\sigma-1} \;\; 0 \qquad (\alpha \geq 2),$$

where $\alpha \geq h_\alpha \geq h_{\alpha+1} \geq \ldots \geq h_{\sigma-1} > 0$ is any non-increasing collection of non-zero integers and $\alpha = \alpha(\mathbf{H}), \ \sigma = \sigma(\mathbf{H})$.

Then the numerical character of $\mathbf{H}$ is defined as the sequence $(b_1, \ldots, b_\alpha)$ with

$$\alpha \leq b_1 \leq b_2 \cdots \leq b_\alpha$$

such that, if there are u_1 occurrences of b_1 in the numerical character then $\Delta\mathbf{H}$ takes on the value $\alpha - u_1$ at b_1 and stays at that value until we arrive at b_{u_1+1}; if there are u_2 occurrences of b_{u_1+1} in the numerical character then $\Delta\mathbf{H}$ takes on the value $\alpha - u_1 - u_2$ at b_{u_2+1} and stays at that value until we arrive at $b_{u_1+u_2+1}$; and so on. (For more details the reader is referred to [9].)

Example 1.1. We will consider the numerical characters of all possible Hilbert functions for sets of 6 nondegenerate points in $\mathbb{P}^2$.

(a) $\mathbb{X}$ consists of 6 points not on a conic in $\mathbb{P}^2$. Then $\mathbf{H} = \mathbf{H}_{\mathbb{X}}$ is given by

$$\mathbf{H} := \ 1 \ 3 \ 6 \ 6 \ \to \quad \text{and so} \quad \Delta\mathbf{H} := \ 1 \ 2 \ 3 \ 0,$$

and the numerical character is $(3,3,3)$.

(b) $\mathbb{X}$ consists of 6 points on an irreducible conic. Then

$$\mathbf{H} := \ 1 \ 3 \ 5 \ 6 \ 6 \ \to \quad \text{and so} \quad \Delta\mathbf{H} := \ 1 \ 2 \ 2 \ 1 \ 0,$$

and the numerical character is $(3,4)$.

(c) $\mathbb{X}$ consists of 5 points on a line and one point off that line. Then

$$\mathbf{H} := \ 1 \ 3 \ 4 \ 5 \ 6 \ 6 \ \to \ \text{and so} \ \Delta\mathbf{H} := \ 1 \ 2 \ 1 \ 1 \ 1 \ 0,$$

and the numerical character is $(2,5)$.

(d) $\mathbb{X}$ consists of 6 points on a line. Then

$$\mathbf{H} := 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 6 \ \to \ \text{and so} \ \Delta\mathbf{H} := 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ \to \ .$$

Notice that in the last case we have $\mathbf{H} \in \mathcal{H}_1$. It follows that the numerical character of $\mathbf{H}$ is (6).

It is easy to see that the set $\mathcal{S}_2$ is in 1-1 correspondence with the set of numerical characters. Thus, the numerical character is an alternative to the Hilbert function for distinguishing sets of points in $\mathbb{P}^2$. In fact, Gruson-Peskine used the numerical character to characterize the Hilbert functions of points sets in $\mathbb{P}^2$ which are general hyperplane sections of irreducible curves in $\mathbb{P}^3$ (see also [9]).

We are now ready to define our new "character" (called "type vectors"), and we show that there is a 1-1 correspondence between $\mathcal{S}_n$ and "*n-type vectors*". When $n = 2$ and $\mathbf{H} \in \mathcal{S}_2$ then the 2-*type vector* corresponding to $\mathbf{H}$ is an $\alpha(\mathbf{H})$-tuple of non-negative integers (similar to, but not equal to, the numerical character) which characterizes $\mathbf{H}$. We will show in Proposition 3.2 how to pass back and forth between our 2-*type vector* and the numerical character of Gruson and Peskine.

Remark 1.2. For $n \geq 2$ and $\mathbf{H} \in \mathcal{S}_n$ we also explain how the *n-type vector* associated to $\mathbf{H}$ can describe certain features of the point sets that have Hilbert function $\mathbf{H}$.

There is an ambiguity in the above discussion relating to the set $\mathcal{H}_0 = \mathcal{S}_0$. There is only one Hilbert function in this set, namely the constant function 1.

This function is precisely the Hilbert function of the ring $k[x_0]$. In this case we set $\alpha(\mathbf{H}) = -1$ and $\sigma(\mathbf{H}) = 1$.

2. Type vectors

DEFINITION 2.1.

(1) A 0-*type vector* is defined to be $\mathcal{T} = 1$. This vector is the only 0-*type vector*. We define $\alpha(\mathcal{T}) = -1$ and $\sigma(\mathcal{T}) = 1$.

(2) A 1-*type vector* is a vector of the form $\mathcal{T} = (d)$, where $d \geq 1$ is a positive integer. For such a vector we define $\alpha(\mathcal{T}) = d = \sigma(\mathcal{T})$.

(3) A 2-*type vector* is a vector of the form

$$\mathcal{T} = ((d_1), (d_2), \ldots, (d_m)),$$

where $m \geq 1$, the (d_i) are 1-*type vectors*, and $\sigma(d_i) = d_i < \alpha(d_{i+1}) = d_{i+1}$. For such a vector $\mathcal{T}$ we define $\alpha(\mathcal{T}) = m$ and $\sigma(\mathcal{T}) = \sigma((d_m)) = d_m$. Clearly, $\alpha(\mathcal{T}) \leq \sigma(\mathcal{T})$, with equality if and only if $\mathcal{T} = ((1), (2), \ldots, (m))$.

Remark. For simplicity of notation we usually write the 2-*type vector* $((d_1), \ldots, (d_m))$ as $(d_1, \ldots, d_m)$.

(4) A 3-*type vector* is an ordered collection of 2-*type vectors* $\mathcal{T}_1, \ldots, \mathcal{T}_r$,

$$\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_r),$$

where $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, \ldots, r-1$. For such a vector $\mathcal{T}$ we define $\alpha(\mathcal{T}) = r$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_r)$.

(5) Let $n \geq 3$. An n-*type vector* is an ordered collection of $(n-1)$-*type vectors*, $\mathcal{T}_1, \ldots, \mathcal{T}_s$,

$$\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s),$$

such that $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, \ldots, s-1$. For such a vector $\mathcal{T}$ we define $\alpha(\mathcal{T}) = s$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_s)$.

EXAMPLE 2.2. Clearly $\mathcal{T}_1 = (1,2)$, $\mathcal{T}_2 = (1,3,4)$, $\mathcal{T}_3 = (1,2,3)$, and $\mathcal{T}_4 = (2,3,4,5,6)$ are all 2-*type vectors*, but $(\mathcal{T}_3, \mathcal{T}_2) = ((1,2,3),(1,3,4))$ is *not* a 3-*type vector* since $\sigma(\mathcal{T}_3) = 3$ and $\alpha(\mathcal{T}_2) = 3$. However, $(\mathcal{T}_2, \mathcal{T}_4)$ is a 3-*type vector*. Also,

$$(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_4) = ((1,2),(1,3,4),(2,3,4,5,6))$$

is a 3-*type vector* since $\sigma(\mathcal{T}_1) = 2 < \alpha(\mathcal{T}_2) = 3$ and $\sigma(\mathcal{T}_2) = 4 < \alpha(\mathcal{T}_4) = 5$. We will, from time to time, use the simplified notation

$$((1,2),(1,3,4),(2,3,4,5,6)) = (1,2;1,3,4;2,3,4,5,6)$$

for 3-*type vectors* (see [12]).

Note also that $(\ (\mathcal{T}_1)\ ) = (\ (1,2)\ )$ is a 3-*type vector* and that $(\ (\ (1,2)\ )\ )$ is a 4-*type vector*. A simple check shows that

$$\Bigg(\Big((1,2),(1,3,4),(2,3,4,5,6)\Big),\Big((1,2),(1,2,4),(2,3,4,5,6),\\ (1,2,3,4,5,6,7),(2,4,6,7,8,9,10,12),(1,2,3,4,5,6,7,8,9,11,12,14,15),\\ (3,4,6,8,9,10,12,23,24,25,30,31,40,45,50,60)\Big)\Bigg)$$

is also a 4-*type vector*.

Before we begin the proof of our main theorem we recall a construction given in [10], which is crucial to our discussion of *n-type vectors*. Let $\mathbf{H} = \{b_i\} \in \mathcal{H}_n$ (so that $\mathbf{H}(1) = n+1$) and write $\sigma = \sigma(\mathbf{H})$. Let $\mathbf{H}_{\mathbb{P}^{n-1}}(t) = \{d_t\ \}$, where $d_t = \binom{t+n-1}{n-1}$ and define $c_i = b_{i+1} - d_{i+1}$. Then we have:

$$\begin{array}{lcccccccc}
\mathbf{H}: & 1 & \binom{n+1}{1} & \cdot & \binom{\alpha-1+n}{n} & b_\alpha & \cdot & b_{\sigma-2} & < b_{\sigma-1} = & b_\sigma \\
 & (0) & (1) & \cdot & (\alpha-1) & (\alpha) & \cdot & (\sigma-2) & (\sigma-1) & (\sigma) \\
\mathbf{H}_{\mathbb{P}^{n-1}}: & 1 & \binom{n}{1} & \cdot & \binom{\alpha+n-2}{n-1} & \binom{\alpha+n-1}{n-1} & \cdot & \binom{\sigma+n-3}{n-1} & \binom{\sigma+n-2}{n-1} & \binom{\sigma+n-1}{n-1} \\
 & & \underset{c_0}{\overset{1}{\shortparallel}} & \cdot & c_{\alpha-2} & c_{\alpha-1} & \cdot & c_{\sigma-3} & c_{\sigma-2} & c_{\sigma-1}
\end{array}$$

Since the d_i's are strictly increasing and the b_i's are eventually constant, there is a unique integer h such that

$$1 = c_0 \leq c_1 \leq \cdots \leq c_{h-1} > c_h\ .$$

Theorem 2.3 ([10]). *The sequences*

$$\boldsymbol{H}_1 := 1\ c_1\ \cdots\ c_{h-1} \rightarrow \text{ and } \boldsymbol{H}_1' = \{c_i'\},$$

where

$$c_i' = \begin{cases} \dbinom{i+n-1}{n-1} & \textit{for } i \leq h, \\ b_i - c_{h-1} & \textit{for } i \geq h, \end{cases}$$

are 0-dimensional differentiable O-sequences.

We associate to $\mathbf{H}$ the (ordered) pair of Hilbert functions $(\mathbf{H}_1, \mathbf{H}_1')$.

Remark 2.4. (1) Notice that $c_0 = 1$ (since $\mathbf{H}(1) = n+1$) and so $h-1 \geq 0$, i.e., $h \geq 1$. Thus $c_1' = n$, and this means that $\mathbf{H}_1' \in \mathcal{H}_{n-1}$.
(2) By construction, $\sigma(\mathbf{H}_1) \leq h$ and (since $\mathbf{H}_1' \in \mathcal{H}_{n-1}$) we have $\alpha(\mathbf{H}_1') \geq h+1$. Thus, $\sigma(\mathbf{H}_1) < \alpha(\mathbf{H}_1')$.

The following key lemma will be used often in the sequel.

Lemma 2.5. *Let $\boldsymbol{H} \in \mathcal{H}_n$, $\boldsymbol{H}_1$ and $\boldsymbol{H}_1'$ be as above. Then $\sigma(\boldsymbol{H}) = \sigma(\boldsymbol{H}_1')$.*

Proof. Embedded in the proof that $\mathbf{H}_1'$ is an O-sequence is the fact that

$$c_h' = b_h - c_{h-1} \quad \text{and} \quad c_{h+1}' = b_{h+1} - c_{h-1} \ .$$

Thus, if $b_h < b_{h+1}$ then $c_h' < c_{h+1}'$. It is easy to see that, in this case, the numbers b_h become constant exactly when the numbers c_h' become constant; i.e., we have $\sigma(\mathbf{H}) = \sigma(\mathbf{H}_1')$.

Suppose that $b_h = b_{h+1}$. Then $c_h' = c_{h+1}'$ and we obtain $\sigma(\mathbf{H}_1') \leq h+1$. Since we always have $c_{h-1}' < c_h'$, we also have $\sigma(\mathbf{H}_1') \geq h+1$. Thus the hypothesis $b_h = b_{h+1}$ gives $\sigma(\mathbf{H}_1') = h+1$, and it remains to show that this assumption also implies that $\sigma(\mathbf{H}) = h+1$.

Now, $b_h = b_{h+1}$ certainly implies that $\sigma(\mathbf{H}) \leq h+1$, so it suffices to prove that $b_{h-1} < b_h$. But if $b_{h-1} = b_h$, then

$$c_{h-2} = b_{h-1} - \binom{h+n-2}{n-1} > b_h - \binom{h+n-1}{n-1} = c_{h-1},$$

and this contradicts the definition of h. Thus, we have again $\sigma(\mathbf{H}) = h+1$, and the proof of the lemma is complete. □

We are now ready to prove the main theorem of this paper.

THEOREM 2.6. *There is a 1-1 correspondence*

$$\mathcal{S}_n \leftrightarrow \{n\text{-type vectors }\}$$

such that if $\boldsymbol{H} \in \mathcal{S}_n$ *and* $\boldsymbol{H} \leftrightarrow \mathcal{T}$, *then* $\alpha(\boldsymbol{H}) = \alpha(\mathcal{T})$ *and* $\sigma(\boldsymbol{H}) = \sigma(\mathcal{T})$.

Proof. We begin defining an assignment of an *n-type vector* to an element of $\mathcal{S}_n$.

Case $n = 0$*:* When $n = 0$, $\mathcal{H}_0 = \mathcal{S}_0$ and the only element $\mathbf{H} \in \mathcal{H}_0$ is $\mathbf{H} := 1 \ \ \rightarrow$. We associate the only 0-*type vector*, $\mathcal{T} = 1$, to $\mathbf{H}$. By the definition, we then have $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$ and $\sigma(\mathbf{H}) = \sigma(\mathcal{T})$.

Case $n = 1$*:* Let $\mathbf{H} \in \mathcal{S}_1$ and consider $\mathbf{H}(1)$. If $\mathbf{H}(1) = 1$ then $\mathbf{H} \in \mathcal{S}_0$ and, by induction, $\mathbf{H}$ (considered as an element of $\mathcal{S}_0$) corresponds to the 0-*type vector* 1. We let $\mathbf{H}$, now considered as an element of $\mathcal{S}_2$, correspond to $\mathcal{T} = (1)$. Then, by definition, $\alpha(\mathbf{H}) = 1$ and $\alpha(\mathcal{T}) = 1$. Also, $\sigma(\mathbf{H}) = 1$ (this value has not changed) and, by definition, $\sigma(\mathcal{T}) = 1$. Thus we are done in this case.

We may therefore assume that $\mathbf{H} \in \mathcal{H}_1$, i.e., $\mathbf{H}(1) = 2$ and so $\alpha = \alpha(\mathbf{H}) > 1$, i.e.,

$$\mathbf{H} := \underset{(0)}{1} \quad \underset{(1)}{2} \quad \cdots \quad \underset{(\alpha-1)}{\alpha} \quad \underset{(\alpha)}{\alpha} \quad \cdots$$

We associate to $\mathbf{H}$ the 1-*type vector* $(\alpha) = \mathcal{T}$. All conditions are clearly satisfied in this case since $\alpha(\mathbf{H}) = \alpha = \sigma(\mathbf{H})$ and $\alpha(\mathcal{T}) = \alpha = \sigma(\mathcal{T})$.

Case $n = 2$*:* Now suppose that $\mathbf{H} \in \mathcal{S}_2$ and consider $\mathbf{H}(1)$. If $\mathbf{H}(1) < 3$ then $\mathbf{H} \in \mathcal{S}_1$ and by induction, $\mathbf{H}$ (considered as an element of $\mathcal{S}_1$) corresponds to the 1-*type vector* $\mathcal{T} = (e)$ where $\mathbf{H}$ (again considered as an element of $\mathcal{S}_1$) satisfies

$$\alpha(\mathbf{H}) = \alpha(\mathcal{T}) = e = \sigma(\mathbf{H}) = \sigma(\mathcal{T}) \ .$$

Now, considering $\mathbf{H}$ as an element of $\mathcal{S}_2$, we let $\mathbf{H} \leftrightarrow (\ (e)\ ) = (\mathcal{T}) = \mathcal{T}'$. Then, by definition, $\alpha(\mathbf{H}) = \alpha(\mathcal{T}') = 1$ and $\sigma(\mathbf{H}) = e$ with $e = \sigma(\mathcal{T})$. Thus, $\sigma(\mathbf{H}) = \sigma(\mathcal{T}')$, and we are done in this case.

We may therefore assume that $\mathbf{H}(1) = 3$, i.e., $\mathbf{H} \in \mathcal{H}_2$ and $\alpha = \alpha(\mathbf{H}) > 1$. Writing $\mathbf{H}(i) = b_i$, we have

$$\begin{array}{cccccccccccc} \mathbf{H} := & 1 & 3 & \cdots & \binom{\alpha+1}{2} & b_\alpha & \cdots & b_{\sigma-2} & < & b_{\sigma-1} & = & b_\sigma & \cdots \\ & {\scriptstyle (0)} & {\scriptstyle (1)} & \cdots & {\scriptstyle (\alpha-1)} & {\scriptstyle (\alpha)} & \cdots & {\scriptstyle (\sigma-2)} & & {\scriptstyle (\sigma-1)} & & {\scriptstyle (\sigma)} & \end{array}$$

where $\sigma = \sigma(\mathbf{H})$. Thus, $b_\alpha < \binom{\alpha+2}{2}$.

We now apply the above-mentioned construction in [10] to $\mathbf{H}$, this time letting $\{d_i\} = \mathbf{H}_{\mathbb{P}^1}(i)$, to obtain $\mathbf{H}_1$ and $\mathbf{H}_1'$, and we let $\mathbf{H} \to (\mathbf{H}_1, \mathbf{H}_1')$. There are two separate cases to consider: $\alpha(\mathbf{H}) = 2$ and $\alpha(\mathbf{H}) > 2$.

Case 1 ($\alpha(\boldsymbol{H})$=2): In this case we have $b_2 < 6$ and so $c_1 = b_2 - d_2 = b_2 - 3 < 6 - 3 = 3$. Since $c_1 < 3$ we have $\mathbf{H}_1 \in \mathcal{S}_1$, and so by induction $\mathbf{H}_1 \to (e_1)$, and since $\mathbf{H}_1' \in \mathcal{S}_1$ (by Remark 2.4(1) above), we obtain $\mathbf{H}_1' \to (e_2)$. By Remark 2.4(2) we have $e_1 < e_2$. Thus $\mathcal{T} = ((e_1), (e_2))$ is a 2-*type vector*.

In order to associate $\mathcal{T}$ with $\mathbf{H}$ we must ensure that $\alpha(\mathcal{T}) = \alpha(\mathbf{H})$ (this is obvious by construction) and that $\sigma(\mathcal{T}) = \sigma(\mathbf{H})$. To obtain the latter condition note that, by definition, $\sigma(\mathcal{T}) = \sigma((e_2)) = e_2 = \sigma(\mathbf{H}_1')$. Thus, it suffices to show that $\sigma(\mathbf{H}_1') = \sigma(\mathbf{H})$, and this follows from Lemma 2.5.

Case 2 ($\alpha(\boldsymbol{H}) > 2$): As in the previous case we let $\mathbf{H} \to (\mathbf{H}_1, \mathbf{H}_1')$. In this case, $c_{\alpha-2} = \binom{\alpha+1}{2} - \alpha = \binom{\alpha}{2}$ and, since $\alpha = \alpha(\mathbf{H}) > 2$, we have $c_1 = \mathbf{H}_1(1) = 3$. Thus, $\mathbf{H}_1 \in \mathcal{H}_2$. Moreover,

$$c_{\alpha-1} = b_\alpha - (\alpha+1) < \binom{\alpha+2}{2} - (\alpha+1) = \binom{\alpha+1}{2}$$

and we conclude that $\alpha(\mathbf{H}_1) = \alpha(\mathbf{H}) - 1$. Hence, by induction on α, we have

$$\mathbf{H}_1 \to ((e_1), \ldots, (e_{\alpha(\mathbf{H}_1)})),$$

where the (e_i) are 1-*type vectors* and $\sigma(\mathbf{H}_1) = \sigma(\ (e_{\alpha(\mathbf{H}_1)})\ ) = e_{\alpha(\mathbf{H}_1)}$.

We have already remarked that $\mathbf{H}_1' \in \mathcal{H}_1$, so we have $\mathbf{H}_1' \to (e)$. We now define the association

$$\mathbf{H} \to ((e_1), \ldots, (e_{\alpha(\mathbf{H}_1)}), (e)),$$

but to do that we must verify the following:

(1) $\mathcal{T} = ((e_1), \ldots, (e_{\alpha(\mathbf{H}_1)}), (e))$ is a 2-*type vector*;
(2) $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$;
(3) $\sigma(\mathbf{H}) = \sigma(\mathcal{T})$.

To prove (1) it suffices to prove that

$$\sigma((e_1), \ldots, (e_{\alpha(\mathbf{H}_1)})) < \alpha(\ (e)\ ),$$

i.e., $e_{\alpha(\mathbf{H}_1)} = \sigma(\mathbf{H}_1) < \alpha(\mathbf{H}'_1)$. But this is precisely the content of Remark 2.4(2). As for (2) and (3), we have $\alpha(\mathbf{H}) = \alpha(\mathbf{H}_1) + 1$ and so $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$. Since $\sigma(\mathbf{H}'_1) = \sigma(\mathbf{H})$ by Lemma 2.5, we also have $\sigma(\mathbf{H}) = \sigma(\mathcal{T})$. This completes the proof for the case $n = 2$.

Case $n \geq 3$: Let $\mathbf{H} \in \mathcal{S}_n$ $(n \geq 3)$ and consider $\mathbf{H}(1)$. If $\mathbf{H}(1) \leq n$, then, by induction, we have an assignment $\mathbf{H} \to \mathcal{T}$, where $\mathcal{T}$ is an $(n-1)$-*type vector*, with $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$ and $\sigma(\mathbf{H}) = \sigma(\mathcal{T})$. In this case we assign $\mathbf{H} \to (\mathcal{T}) = \mathcal{T}'$. Since $\mathbf{H} \in \mathcal{S}_{n-1}$ also, we have $\alpha(\mathbf{H}) = 1$ and $\alpha(\mathcal{T}') = 1$. By definition, $\sigma(\mathcal{T}') = \sigma(\mathcal{T})$, so using induction we obtain $\sigma(\mathcal{T}') = \sigma(\mathbf{H})$. Thus we are done in this case. Now assume that $\mathbf{H}(1) = n+1$, i.e., $\mathbf{H} \in \mathcal{H}_n$ and $\alpha = \alpha(\mathbf{H}) > 1$. We write $\mathbf{H}(i) = b_i$. We have

$$\begin{array}{cccccccccccc} \mathbf{H} := & 1 & \binom{n+1}{1} & \cdots & \binom{\alpha-1+n}{n} & b_\alpha & \cdots & b_{\sigma-2} & < & b_{\sigma-1} & = & b_\sigma & \cdots \\ & {\scriptstyle (0)} & {\scriptstyle (1)} & \cdots & {\scriptstyle (\alpha-1)} & {\scriptstyle (\alpha)} & \cdots & {\scriptstyle (\sigma-2)} & & {\scriptstyle (\sigma-1)} & & {\scriptstyle (\sigma)} & \cdots \end{array}$$

where $\sigma = \sigma(\mathbf{H})$. So $b_\alpha < \binom{\alpha+n}{n}$.

As in the case $n = 2$, there are two cases to consider: $\alpha(\mathbf{H}) = 2$ and $\alpha(\mathbf{H}) > 2$.

Case 1 ($\alpha(\mathbf{H})=2$): We have

$$c_1 = b_2 - \binom{n+1}{n-1} < \binom{n+2}{n} - \binom{n+1}{n-1} = n+1,$$

and there are three possibilities for c_1, namely $c_1 \leq 0$, $c_1 = 1$, and $c_1 > 1$.

Case $c_1 \leq 0$: Then $h = 1$ and

$$\mathbf{H}_1 := 1 \quad \to \qquad \text{and} \qquad \mathbf{H}'_1 := 1 \;\; n \;\; c'_2 \;\; \cdots$$

By induction, we have $\mathbf{H}_1 \to \mathcal{T}_1$, where $\mathcal{T}_1$ is an $(n-1)$-*type vector* with $\sigma(\mathbf{H}_1) = 1 = \sigma(\mathcal{T}_1)$ and $\mathbf{H}'_1 \to \mathcal{T}_2$, where $\mathcal{T}_2$ is an $(n-1)$-*type vector* with $\alpha(\mathbf{H}'_1) = \alpha(\mathcal{T}_2)$. But $\mathbf{H}'_1(1) = n$ and so $\alpha(\mathbf{H}'_1) \geq 2$. Thus, $\sigma(\mathcal{T}_1) < \alpha(\mathcal{T}_2)$ and we associate

$$\mathbf{H} \to (\mathcal{T}_1, \mathcal{T}_2) = \mathcal{T}\ .$$

Since $\alpha(\mathcal{T}) = 2$ we have $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$. It remains to show that $\sigma(\mathbf{H}) = \sigma(\mathcal{T}) = \sigma(\mathcal{T}_2)$. This will follow if we can show that $\sigma(\mathbf{H}) = \sigma(\mathbf{H}'_1)$. But the latter relation follows from Lemma 2.5, and we thus have obtained the required result.

Case $c_1 = 1$: In this case we have $h \geq 2$ and

$$\mathbf{H}_1 := \;\; 1 \quad \to \qquad \text{and} \qquad \mathbf{H}'_1 := 1 \;\; n \;\; c'_2 \;\; \cdots \;\; c'_h \;\; c'_{h+1} \;\; \cdots\ .$$

By induction, we have $\mathbf{H}_1 \to \mathcal{T}_1$ with $\sigma(\mathcal{T}_1) = 1$ and $\mathbf{H}'_1 \to \mathcal{T}_2$ with $\alpha(\mathcal{T}_2) \geq h+1$. Thus, $\sigma(\mathcal{T}_1) < \alpha(\mathcal{T}_2)$ and so

$$\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$$

is an *n-type vector*, which we associate to $\mathbf{H}$.

By construction, $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$, so it remains to show that $\sigma(\mathbf{H}) = \sigma(\mathcal{T})$. But $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_2)$ (by definition) and $\sigma(\mathcal{T}_2) = \sigma(\mathbf{H}_1')$ (by induction). Lemma 2.5 now completes the proof in this case.

Case $n \geq c_1 > 1$: As above, we have $\mathbf{H} \to (\mathbf{H}_1, \mathbf{H}_1')$ with $\mathbf{H}_1(1) = c_1$. In this case we have $\mathbf{H}_1 \to \mathcal{T}_1$ and $\mathbf{H}_1' \to \mathcal{T}_2$ and (by Remark 2.4(2)) $\sigma(\mathbf{H}_1) < \alpha(\mathbf{H}_1')$, so $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ is an *n-type vector* with $\alpha(\mathbf{H}) = \alpha(\mathcal{T}) = 2$.

Hence from Lemma 2.5 we obtain $\sigma(\mathbf{H}_1') = \sigma(\mathbf{H})$, which completes the proof for the case $\alpha(\mathbf{H}) = 2$).

Case 2 ($\alpha(\boldsymbol{H}) > 2$): We form $\mathbf{H}_1$ and $\mathbf{H}_1'$ in the usual way from $\mathbf{H}$. But now observe that

$$c_{\alpha-2} = b_{\alpha-1} - d_{\alpha-1} = \binom{\alpha-1+n}{\alpha-1} - \binom{\alpha-2+n}{\alpha-1} = \binom{\alpha-2+n}{\alpha-2} .$$

Since $\alpha > 2$, we have $\alpha - 2 \geq 1$ and $c_1 = n+1$ and so $\mathbf{H}_1 \in \mathcal{H}_n$. Also,

$$c_{\alpha-1} = b_\alpha - d_\alpha < \binom{\alpha+n}{\alpha} - \binom{\alpha-1+n}{\alpha} = \binom{\alpha-1+n}{\alpha-1} .$$

Thus, $\alpha(\mathbf{H}_1) = \alpha(\mathbf{H}) - 1$ Hence by induction on α we obtain

$$\mathbf{H}_1 \to (\mathcal{T}_1, \dots, \mathcal{T}_{\alpha(\mathbf{H}_1)}),$$

where the $\mathcal{T}_i$ are $(n-1)$*-type vectors* and $\sigma(\mathbf{H}_1) = \sigma(\mathcal{T}_{\alpha(\mathbf{H}_1)})$.

Since $\mathbf{H}_1' \in \mathcal{H}_{n-1}$, we have, by induction, $\mathbf{H}_1' \to \mathcal{T}'$, where $\mathcal{T}'$ is an $(n-1)$-*type vector* with $\alpha(\mathbf{H}_1') = \alpha(\mathcal{T}')$ and $\sigma(\mathbf{H}_1') = \sigma(\mathcal{T}')$.

Consider

$$\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_{\alpha(\mathbf{H}_1)}, \mathcal{T}') .$$

By Remark 2.4(2), this is an *n-type vector*. By construction, $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}') = \sigma(\mathbf{H}_1')$. But, by Lemma 2.5, $\sigma(\mathbf{H}_1') = \sigma(\mathbf{H})$ and so $\mathbf{H} \to \mathcal{T}$ is an appropriate correspondence.

Now that we have defined how to associate to a Hilbert function in $\mathcal{S}_n$ an *n-type vector*, we next show that this correspondence is a 1-1 correspondence. We begin by first defining an assignment in the opposite direction. In order to simplify our discussion, let us denote the assignments defined above by the letters χ_n, i.e.,

$$\chi_n : \ \mathcal{S}_n \longrightarrow \{ \ n\text{-}type\ vectors \ \}$$

We now define (inductively) assignments

$$\rho_n : \ \{ \ n\text{-}type\ vectors \ \} \ \longrightarrow \ \mathcal{S}_n,$$

such that $\alpha(\mathcal{T}) = \alpha(\rho_n(\mathcal{T}))$ and $\sigma(\mathcal{T}) = \sigma(\rho_n(\mathcal{T}))$.

Case $n = 0$*:* Since there is only one element in either of the sets involved, the assignment is obvious.

Case $n = 1$: Let $\mathcal{T} = (a)$ be a 1-*type vector* with $a \geq 1$. We define $\rho_1(\mathcal{T}) = \mathbf{H}$ by setting

$$\mathbf{H} := \begin{array}{cccccc} 1 & 2 & \cdots & a & a & \rightarrow \\ {\scriptstyle (0)} & {\scriptstyle (1)} & \cdots & {\scriptstyle (a-1)} & {\scriptstyle (a)} & \end{array}$$

Clearly ρ_1 and χ_1 are inverses of each other, thus proving the 1-1 correspondence of the theorem for $n = 1$. It is also obvious that $\alpha(\mathcal{T}) = \alpha(\rho_1(\mathcal{T}))$ and $\sigma(\mathcal{T}) = \sigma(\rho_1(\mathcal{T}))$.

Case $n \geq 2$: Let $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ be an *n-type vector*. Then the vectors $\mathcal{T}_i$ are $(n-1)$-*type vectors* and, by induction, we have $\rho_{n-1}(\mathcal{T}_i) = \widetilde{\mathbf{H}}_i \in \mathcal{S}_{n-1}$ and ρ_{n-1} is a 1-1 correspondence between the set of $(n-1)$-type vectors and $\mathcal{S}_{n-1}$, which respects both α and σ. We define $\rho_n(\mathcal{T}) = \mathbf{H}$, where

$$\mathbf{H}(t) = \widetilde{\mathbf{H}}_r(t) + \widetilde{\mathbf{H}}_{r-1}(t-1) + \cdots + \widetilde{\mathbf{H}}_1(t-(r-1))$$

(with $\widetilde{\mathbf{H}}_i(j) = 0$ if $j < 0$). We need to verify that this definition actually gives an element of $\mathcal{S}_n$, which respects α and σ.

Let $\mathcal{T}$ be an *n-type vector* and suppose first that $\alpha(\mathcal{T}) = 1$. Then $\mathcal{T} = (\mathcal{T}_1)$ where $\mathcal{T}_1$ is an $(n-1)$-*type vector*. By induction, we have $\rho_{n-1}(\mathcal{T}_1) = \widetilde{\mathbf{H}}_1 \in \mathcal{S}_{n-1}$. Then we also have $\rho_n(\mathcal{T}) = \widetilde{\mathbf{H}}_1$, and obviously $\widetilde{\mathbf{H}}_1$ is a 0-dimensional differentiable O-sequence with $\widetilde{\mathbf{H}}_1(1) \leq n$ (and hence $\widetilde{\mathbf{H}}_1(1) \leq n+1$). Thus, $\widetilde{\mathbf{H}}_1$, considered as an element of $\mathcal{S}_n$, satisfies $\alpha(\widetilde{\mathbf{H}}_1) = \alpha(\mathcal{T}) = 1$. We have $\sigma(\widetilde{\mathbf{H}}_1) = \sigma(\mathcal{T}_1)$, by induction, and since $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_1)$ by definition, we obtain $\sigma(\mathcal{T}) = \sigma(\widetilde{\mathbf{H}}_1)$, and we are done.

Now assume that $\alpha(\mathcal{T}) = u > 1$, i.e., $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_u)$. As above, we consider two cases, $u = 2$ and $u > 2$. We will leave the simple argument in case $u = 2$ to the reader and concentrate on the case $u > 2$.

Let $\mathbf{H}_1(t) = \widetilde{\mathbf{H}}_1(t-(u-2)) + \cdots + \widetilde{\mathbf{H}}_{u-1}(t) = [\rho_n(\mathcal{T}_1, \ldots, \mathcal{T}_{u-1})](t)$ and let $\mathbf{H}'_1(t) = \widetilde{\mathbf{H}}_u(t) = [\rho_{n-1}(\mathcal{T}_u)](t)$. Then $\mathbf{H}_1$ and $\mathbf{H}'_1$ are both 0-dimensional differentiable O-sequences in $\mathcal{S}_n$, as can be seen by induction on u in the case of $\mathbf{H}_1$ and by induction on n in the case of $\mathbf{H}'_1$. We want to prove that the same is true for

$$[\rho_n(\mathcal{T})](t) = \mathbf{H}(t) = \mathbf{H}_1(t-1) + \mathbf{H}'_1(t)\ .$$

We have, by induction, $\alpha(\mathbf{H}_1) = u-1$, $\sigma(\mathbf{H}_1) = \sigma(\rho_{n-1}(\mathcal{T}_{u-1}))$, $\alpha(\mathbf{H}'_1) = \alpha(\rho_{n-1}(\mathcal{T}_u))$ and $\sigma(\mathbf{H}'_1) = \sigma(\rho_{n-1}(\mathcal{T}_u))$.

Let $\alpha = \alpha(\mathbf{H}_1)$. Then $\mathbf{H}_1(t-1)$ is generic for $t-1 < \alpha$ (i.e., for every $t \leq \alpha$). Since $\alpha \leq \sigma(\mathbf{H}_1) < \alpha(\mathbf{H}'_1)$, it follows that $\mathbf{H}'_1(t)$ is generic for $t \leq \alpha$, and hence $\mathbf{H}(t)$ is generic for $t \leq \alpha$. Thus, $\mathbf{H}$ is a differentiable O-sequence for $t \leq \alpha$. Since $[\rho_n(\mathcal{T})](t) = \mathbf{H}(t)$ is generic for $t \leq \alpha$, we have $\alpha(\mathbf{H}) \geq \alpha + 1$. If $\alpha(\mathbf{H}) > \alpha + 1$, then $\mathbf{H}$ is also generic for $t = \alpha + 1$. It follows that $\mathbf{H}'_1(t)$ and $\mathbf{H}_1(t-1)$ are generic for $t \leq \alpha + 1$, which implies that

$\alpha(\mathbf{H}_1) \geq \alpha + 1$, a contradiction. Hence $\alpha(\mathbf{H}) = \alpha(\mathbf{H}_1) + 1 = \alpha + 1$ and so $\alpha(\mathbf{H}) - 1 = \alpha(\mathbf{H}_1) \leq \sigma(\mathbf{H}_1)$. In particular, $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$.

By the definition of $\mathbf{H}$ and by induction on u we also have $\sigma(\mathbf{H}_1) < \alpha(\mathbf{H}_1')$ (and, in general, $\alpha(\mathbf{H}_1') \leq \sigma(\mathbf{H}_1')$. Thus, $\Delta\mathbf{H}_1'(t) = 0$ implies that $t \geq \sigma(\mathbf{H}_1')$ and so $t > \sigma(\mathbf{H}_1)$, i.e., $t - 1 \geq \sigma(\mathbf{H}_1)$. Since $\Delta\mathbf{H}(t) = \Delta\mathbf{H}_1(t-1) + \Delta\mathbf{H}_1'(t)$, this shows that $\Delta\mathbf{H}_1'(t) = 0$ and thus $\Delta\mathbf{H}(t) = 0$. Since the reverse implication is obvious, we find that $\sigma(\mathbf{H}) = \sigma(\mathbf{H}_1')$. Thus it only remains to show that $\rho_n(\mathcal{T})$ behaves like an O-sequence in degrees $\geq \alpha$.

We first consider the case when $\alpha(\mathbf{H}) - 1 = \sigma(\mathbf{H}_1)$. Then the Hilbert functions $\mathbf{H}_1$ and $\mathbf{H}_1'$ are, respectively,

$$\begin{array}{cccccccc}
\mathbf{H}_1 & : & 1 & \binom{n+1}{1} & \binom{n+2}{2} & \cdots & \binom{n+\alpha-1}{\alpha-1} & \rightarrow & \\
\mathbf{H}_1' & : & 1 & \binom{n}{1} & \binom{n+1}{2} & \cdots & \binom{n+\alpha-2}{\alpha-1} & \binom{n+\alpha-1}{\alpha} & \cdots \\
 & & {\scriptstyle (0)} & {\scriptstyle (1)} & {\scriptstyle (2)} & \cdots & {\scriptstyle (\alpha-1)} & {\scriptstyle (\alpha)} &
\end{array}$$

Now $\Delta\mathbf{H}(t) = \Delta\mathbf{H}_1(t-1) + \Delta\mathbf{H}_1'(t)$, so if $t - 1 \geq \alpha$ then $\Delta\mathbf{H}_1(t-1) = 0$ and so $\Delta\mathbf{H}(t) = \Delta\mathbf{H}_1'(t)$. Thus, for $t \geq \alpha + 1$, $\mathbf{H}$ behaves like a differentiable O-sequence. Hence, it only remains to verify that

$$\Delta\mathbf{H}(\alpha+1) \leq (\Delta\mathbf{H}(\alpha))^{<\alpha>} .$$

But $\Delta\mathbf{H}(\alpha+1) = \Delta\mathbf{H}_1'(\alpha+1)$, and this is always

$$\leq \binom{(\alpha+1)+(n-2)}{\alpha+1} = \binom{\alpha+n-1}{\alpha+1},$$

since $\Delta\mathbf{H}_1'(1) = n - 1$. Now,

$$\begin{aligned}
\Delta\mathbf{H}(\alpha) &= \Delta\mathbf{H}_1(\alpha-1) + \Delta\mathbf{H}_1'(\alpha) \\
&= \binom{(\alpha-1)+(n-1)}{\alpha-1} + \binom{\alpha+n-2}{\alpha} \\
&= \binom{\alpha+n-2}{\alpha-1} + \binom{\alpha+n-2}{\alpha} \\
&= \binom{\alpha+n-1}{\alpha},
\end{aligned}$$

and thus $(\Delta\mathbf{H}(\alpha))^{<\alpha>} = \binom{\alpha+n}{\alpha+1}$. Since $\binom{\alpha+n-1}{\alpha+1} < \binom{\alpha+n}{\alpha+1}$, this completes the proof of the claim that, in the case $\alpha(\mathbf{H}) - 1 = \sigma(\mathbf{H}_1)$, $\Delta\mathbf{H}$ is a differentiable O-sequence.

Now assume $\alpha(\mathbf{H}) \leq \sigma(\mathbf{H}_1)$ and consider those t for which $\alpha + 1 = \alpha(\mathbf{H}) \leq t \leq \sigma(\mathbf{H}_1)$. We first consider the passage from α to $\alpha + 1$. We have

$$\Delta\mathbf{H}(\alpha) = \Delta\mathbf{H}_1(\alpha-1) + \Delta\mathbf{H}_1'(\alpha).$$

Since $\alpha < \sigma(\mathbf{H}_1) < \alpha(\mathbf{H}_1')$, $\Delta\mathbf{H}_1'(1) = n-1$ and $\Delta\mathbf{H}_1(1) = n$, we have

$$\Delta\mathbf{H}(\alpha) = \binom{(\alpha-1)+(n-1)}{\alpha-1} + \binom{\alpha+n-2}{\alpha} = \binom{\alpha+n-1}{\alpha} .$$

Therefore

$$(\Delta\mathbf{H}(\alpha))^{<\alpha>} = \binom{\alpha+n}{\alpha+1} .$$

Since $\Delta\mathbf{H}(\alpha+1) = \Delta\mathbf{H}_1(\alpha) + \Delta\mathbf{H}_1'(\alpha+1)$, which is

$$\leq \binom{\alpha+n-1}{\alpha} + \binom{(\alpha+1)+(n-2)}{\alpha+1} = \binom{\alpha+n}{\alpha+1} = (\Delta\mathbf{H}(\alpha))^{<\alpha>},$$

we obtain that $\Delta\mathbf{H}$ behaves like an O-sequence when passing from α to $\alpha+1$.

Now consider any t in the range $\alpha+1 \leq t \leq \sigma(\mathbf{H}_1) < \alpha(\mathbf{H}_1')$ and the passage from $\Delta\mathbf{H}(t)$ to $\Delta\mathbf{H}(t+1)$. Since in this range, $\Delta\mathbf{H}_1'(t) = \binom{t+n-2}{t}$, we have

$$\Delta\mathbf{H}(t) = \Delta\mathbf{H}_1(t-1) + \binom{t+n-2}{t} .$$

Since $\Delta\mathbf{H}_1(t-1) < \binom{t+n-2}{t-1}$, the $(t-1)$-*binomial expansion* of $\Delta\mathbf{H}_1(t-1)$ is

$$(\Delta\mathbf{H}_1(t-1))_{(t-1)} = \binom{m_{t-1}}{t-1} + \cdots + \binom{m_j}{j},$$

where $t+n-2 > m_{t-1} > \cdots > m_j \geq j \geq 1$. Thus,

$$\Delta\mathbf{H}(t) = \binom{t+n-2}{t} + \binom{m_{t-1}}{t-1} + \cdots + \binom{m_j}{j},$$

and since $t+n-2 > m_{t-1}$, this is the t-*binomial expansion* of $\Delta\mathbf{H}(t)$. Hence,

$$\begin{aligned}(\Delta\mathbf{H}(t))^{<t>} &= \binom{t+n-1}{t+1} + \binom{m_{t-1}+1}{t} + \cdots + \binom{m_j+1}{j+1} \\ &= (\Delta\mathbf{H}_1'(t))^{<t>} + (\Delta\mathbf{H}_1(t-1))^{<t-1>} .\end{aligned}$$

Since, by induction, $\Delta\mathbf{H}_1'(t+1) \leq (\Delta\mathbf{H}_1'(t))^{<t>}$ and $\Delta\mathbf{H}_1(t) \leq (\Delta\mathbf{H}_1(t-1))^{<t-1>}$, we are done in this case as well.

It only remains to consider the case when $t \geq \sigma(\mathbf{H}_1)+1$. But in this case, $\Delta\mathbf{H}(t) = \Delta\mathbf{H}_1'(t)$, and the result easily follows.

This completes the proof of the existence of assignments ρ_n that respect both α and σ. We now show that ρ_n is injective for each n. We have already seen that this is true for $n=0$ and $n=1$. For the general case, we need the following lemma.

Lemma 2.7. *Let $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_u)$ be an n-*type vector*, where $u \geq 2$. Let $\sigma = \sigma(\mathcal{T}_1)$ and $\rho_{n-1}(\mathcal{T}_i) = \widetilde{\boldsymbol{H}}_i$. Then*

$$\widetilde{\boldsymbol{H}}_i(\sigma+(i-2)) = \binom{n+(\sigma+(i-2))-1}{n-1} \text{ for } i = 2, \ldots, u .$$

In other words, $\widetilde{\mathbf{H}}_i(t)$ is maximal (i.e., generic) in $k[x_1,\dots,x_n]$ for $t \le \sigma + (i-2)$ and $i = 2,\dots,u$.

Proof. Since $\sigma = \sigma(\mathcal{T}_1) \le \alpha(\mathcal{T}_i) - (i-1)$ for $i = 2,\dots,u$, we have $\sigma + (i-2) < \alpha(\mathcal{T}_i)$, for i in this range. The conclusion is immediate from this observation. □

We now return to the proof of Theorem 2.6. Let $n \ge 2$ and let $\mathcal{T} = (\mathcal{T}_1,\dots,\mathcal{T}_u)$ and $\mathcal{T}' = (\mathcal{T}'_1,\dots,\mathcal{T}'_v)$ be two *n-type vectors* such that $\rho_n(\mathcal{T}) = \rho_n(\mathcal{T}')$. Since, by construction, $\rho_n(\mathcal{T})$ is generic up to $u-1$ and $\rho_n(\mathcal{T}')$ is generic up to $v-1$, we obtain $u = v$.

Suppose first that $u = 1$, i.e., $\mathcal{T} = (\mathcal{T}_1)$ and $\mathcal{T}' = (\mathcal{T}'_1)$, where $\mathcal{T}_1$ and $\mathcal{T}'_1$ are both $(n-1)$*-type vectors*. By construction, $\rho_n(\mathcal{T}) = \rho_{n-1}(\mathcal{T}_1)$ and $\rho_n(\mathcal{T}') = \rho_{n-1}(\mathcal{T}'_1)$. So, by induction on n we get $\mathcal{T}_1 = \mathcal{T}'_1$ and so $\mathcal{T} = \mathcal{T}'$.

Now suppose that $u > 1$. If $\mathcal{T}_1 = \mathcal{T}'_1$ then, by construction, $\rho_n(\mathcal{T}_2,\dots,\mathcal{T}_u) = \rho_n(\mathcal{T}'_2,\dots,\mathcal{T}'_u)$. By induction on u we get $\mathcal{T}_i = \mathcal{T}'_i$ for $i = 2,\dots,u$ and so $\mathcal{T} = \mathcal{T}'$ in this case. If $\mathcal{T}_1 \ne \mathcal{T}'_1$ then, by induction on u, $\rho_n(\mathcal{T}_1)(t) \ne \rho_n(\mathcal{T}'_1)(t)$ for some t. Let s be the least such integer t. We can assume, without loss of generality, that $\sigma(\mathcal{T}_1) \le \sigma(\mathcal{T}'_1)$. Then clearly $s \le \sigma = \sigma(\mathcal{T}_1)$.

Write $\widetilde{\mathbf{H}}_i = \rho_{n-1}(\mathcal{T}_i)$ and $\widetilde{\mathbf{H}}'_i = \rho_{n-1}(\mathcal{T}'_i)$. If $s < \sigma$, we have, by Lemma 2.7,

$$\widetilde{\mathbf{H}}_i(s+(i-1)) = \widetilde{\mathbf{H}}'_i(s+(i-1)) = \binom{n+(s+(i-1))-1}{n-1}$$

for $i = 2,\dots,u$. But then

$$\begin{array}{rcl} \mathbf{H}(s+(u-1)) & = & \widetilde{\mathbf{H}}_1(s) + [\widetilde{\mathbf{H}}_2(s+1) + \cdots + \widetilde{\mathbf{H}}_u(s+(u-1))] \\ & \ne & \widetilde{\mathbf{H}}'_1(s) + [\widetilde{\mathbf{H}}'_2(s+1) + \cdots + \widetilde{\mathbf{H}}'_u(s+(u-1))] \\ & = & \rho_n(\mathcal{T}')(s+(u-1)), \end{array}$$

which contradicts the relation $\rho_n(\mathcal{T}) = \rho_n(\mathcal{T}')$.

Now suppose that $s = \sigma(\mathcal{T}_1)$. This forces $\sigma(\mathcal{T}_1) < \sigma(\mathcal{T}_2)$ and hence $\widetilde{\mathbf{H}}_1(s) < \widetilde{\mathbf{H}}'_1(s)$. Since $s < \sigma(\mathcal{T}'_1)$ we have, by Lemma 2.7,

$$\widetilde{\mathbf{H}}'_i(s+(i-1)) = \binom{n+(s+(i-1))-1}{n-1}$$

and clearly

$$\widetilde{\mathbf{H}}_i(s+(i-1)) \le \binom{n+(s+(i-1))-1}{n-1}.$$

Since $\rho_n(\mathcal{T})(s+(u-1)) = \rho_n(\mathcal{T}')(s+(u-1))$ we must have $\widetilde{\mathbf{H}}_1(s) \ge \widetilde{\mathbf{H}}'_1(s)$, which is a contradiction. Therefore $\mathcal{T}_1 = \mathcal{T}'_1$, and so $\mathcal{T} = \mathcal{T}'$ as we wanted to show.

The proof will be complete if we can show that, for each n, the composition $\rho_n\chi_n$ is the identity map. We have already shown this for the cases $n = 0$ and $n = 1$. Now suppose that $n \ge 2$, let $\mathbf{H} \in \mathcal{S}_n$, and consider $\mathbf{H}(1)$. If

$\mathbf{H}(1) < n+1$ then $\mathbf{H} \in \mathcal{S}_{n-1}$ and by induction $\rho_{n-1}\chi_{n-1}(\mathbf{H}) = \mathbf{H}$. If $\chi_{n-1}(\mathbf{H}) = \mathcal{T}$, where $\mathcal{T}$ is an $(n-1)$-*type vector*, then $\chi_n(\mathbf{H}) = (\mathcal{T})$ and $\rho_n((\mathcal{T})) = \rho_{n-1}(\mathcal{T}) = \mathbf{H}$, and we are done.

Suppose now that $\mathbf{H}(1) = n+1$ and, as above, let $\mathbf{H} \to (\mathbf{H}_1, \mathbf{H}'_1)$. If $\alpha(\mathbf{H}) = 2$ then, as we have shown above, $\mathbf{H}_1$ and $\mathbf{H}'_1$ are both in $\mathcal{S}_{n-1}$ and

$$\chi_n(\mathbf{H}) = (\chi_{n-1}(\mathbf{H}_1), \chi_{n-1}(\mathbf{H}'_1)) = (\mathcal{T}_1, \mathcal{T}_2),$$

where the $\mathcal{T}_i$ are $(n-1)$-*type vectors*. By definition,

$$\begin{aligned} \rho_n(\mathcal{T}_1, \mathcal{T}_2)(t) &= \rho_{n-1}(\mathcal{T}_2)(t) + \rho_{n-1}(\mathcal{T}_1)(t-1) \\ &= \mathbf{H}'_1(t) + \mathbf{H}_1(t-1). \end{aligned}$$

by induction on n. Now, it is immediate from the definitions of $\mathbf{H}_1$ and $\mathbf{H}'_1$ that this is the description of $\mathbf{H}(t)$. Thus, we are done in this case as well.

The case $\alpha > 2$ is handled similarly, where now $\mathbf{H} \to (\mathbf{H}_1, \mathbf{H}'_1)$ with $\mathbf{H}_1 \in \mathcal{S}_n$ and $\mathbf{H}'_1 \in \mathcal{S}_{n-1}$. This time, however, $\alpha(\mathbf{H}_1) < \alpha(\mathbf{H})$ and we must also use induction on α. This completes the proof of the main theorem. □

3. Some applications

In this section we give a few applications to illustrate the idea of the "*type vector*" of a Hilbert function $\mathbf{H} \in \mathcal{S}_n$.

The numerical character. As mentioned in the introduction, Gruson and Peskine [13] introduced, for $\mathbf{H} \in \mathcal{S}_2$, an $\alpha(\mathbf{H})$-tuple of non-negative integers called the *numerical character* of $\mathbf{H}$. (See [9] for a thorough discussion.)

Recall that a set of points $\mathbb{X} \in \mathbb{P}^n$ is said to have the *uniform position property* (*UPP* for short) if, whenever $\mathbb{X}_1$ and $\mathbb{X}_2$ are subsets of $\mathbb{X}$ with the same cardinality, then $\mathbf{H}_{\mathbb{X}_1} = \mathbf{H}_{\mathbb{X}_2}$. There has been a great deal of work done in an attempt to characterize the Hilbert functions of points in $\mathbb{P}^n$ with UPP - we will not go into the reasons as to why this is an interesting question, but refer the reader instead to some of the works which consider this problem ([1], [2], [3], [5], and [16]). Combining the work of [13] and [16] we now state the solution to this problem for points in $\mathbb{P}^2$ given in these papers.

THEOREM 3.1. *Let $\boldsymbol{H} \in \mathcal{S}_2$ and let $(p_1, \ldots, p_{\alpha(\boldsymbol{H})})$ be the numerical character of $\boldsymbol{H}$. Then $\boldsymbol{H}$ is the Hilbert function of a set of points in $\mathbb{P}^2$ with* UPP *if and only if*

$$p_{i+1} \le p_i + 1 \quad \text{for } i = 1, \ldots, \alpha(\boldsymbol{H}) - 1 \ .$$

We now exhibit the relationship between the numerical character and the 2-*type vector* for a Hilbert function $\mathbf{H} \in \mathcal{S}_2$. Consider $\mathbf{H}(1)$. If $\mathbf{H}(1) = 2$ then $\alpha(\mathbf{H}) = 1$ and the numerical character is (p) and the 2-*type vector* of $\mathbf{H}$ is $((e)) = (e)$, where $e \ge 1$. In this case $p = \sigma(\mathbf{H}) = e$ and both the numerical character and the 2-*type vector* of $\mathbf{H}$ agree.

Now suppose that $\mathbf{H}(1) = 3$, i.e., that $\alpha(\mathbf{H}) > 1$.

Proposition 3.2. *If $(p_1, p_2, \ldots, p_{\alpha-1}, p_\alpha)$ is the numerical character of $\boldsymbol{H} \in \mathcal{S}_2$, then*

$$(e_1, \ldots, e_\alpha) = (p_1 - (\alpha - 1), p_2 - (\alpha - 2), \ldots, p_{\alpha-1} - 1, p_\alpha)$$

is the 2-type vector *associated to* $\boldsymbol{H}$.

Proof. We leave this simple exercise to the reader. □

It follows from this result that

$$p_{i+1} \le p_i + 1 \Leftrightarrow e_{i+1} \le e_i + 2\ .$$

Thus, the result of Gruson-Peskine and Maggioni-Ragusa can be stated very simply in terms of 2-*type vectors*:

Corollary 3.3. *Let $\boldsymbol{H} \in \mathcal{H}_2$ and let $\mathcal{T} = (e_1, \ldots, e_{\alpha(\boldsymbol{H})})$ be the* 2-type vector *associated to $\boldsymbol{H}$. The following are equivalent:*

(1) *$\boldsymbol{H}$ is the Hilbert function of a set of points in $\mathbb{P}^2$ with* UPP.
(2) $e_{i+1} - e_i \le 2$ *for* $i = 1, \ldots, \alpha(\boldsymbol{H}) - 1$.

There exists a somewhat more precise result which, in the case of $\mathcal{H}_2$, is due to E.D. Davis [4] (see also [1] for a generalization). The result of Davis can be rephrased in terms of 2-*type vectors* as follows. Let $\mathbf{H} \in \mathcal{S}_2$ and let $\mathcal{T} = (e_1, \ldots, e_r)$ be the 2-*type vector* associated to $\mathbf{H}$. Choose i so that $1 < i < r$ and let $\mathcal{T}_1 = (e_1, \ldots, e_i)$ and $\mathcal{T}_2 = (e_{i+1}, \ldots, e_r)$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are also 2-*type vectors*, and so we let $\mathcal{T}_1 \leftrightarrow \mathbf{H}_1$ and $\mathcal{T}_2 \leftrightarrow \mathbf{H}_2$.

Theorem 3.4 ([4]). *Suppose that $e_{i+1} - e_i > 2$ and let $\mathbb{X}$ be* any *set of points in $\mathbb{P}^2$ with Hilbert function $\boldsymbol{H}$. Then $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$ (where the union is disjoint) and $\boldsymbol{H}_{\mathbb{X}_1} = \boldsymbol{H}_1$ and $\boldsymbol{H}_{\mathbb{X}_2} = \boldsymbol{H}_2$.*

In particular, in the above notation we have:

Corollary 3.5. *Suppose that $e_{i+1} - e_i > 2$ for $i = 1, \ldots, r-1$. Then, if $\mathbb{X}$ is* any *set of points in $\mathbb{P}^2$ with Hilbert function $\boldsymbol{H}$, we can find a set of lines $\mathbb{L}_1, \ldots, \mathbb{L}_r$ in $\mathbb{P}^2$ and subsets $\mathbb{X}_i$ of $\mathbb{X}$ with the property that*

(i) *$\mathbb{X}_i \subset \mathbb{L}_i$ and $\mathbb{X}_i \cap \mathbb{X}_j = \varnothing$ if $i \ne j$;*
(ii) *$|\mathbb{X}_i| = e_i$;*
(iii) *$\cup_{i=1}^r \mathbb{X}_i = \mathbb{X}$.*

Thus the 2-*type vectors* of Corollary 3.5 correspond to Hilbert functions of very special point sets in $\mathbb{P}^2$.

Another special class of Hilbert functions in $\mathcal{S}_2$ are the Hilbert functions of complete intersections. A Hilbert function $\mathbf{H} \in \mathcal{S}_2$ is a *complete intersection Hilbert function* if $\Delta\mathbf{H}$ satisfies

$$\Delta\mathbf{H}(\sigma - (i+1)) = \Delta\mathbf{H}(i) \quad \text{for} \quad 0 \le i \le \sigma = \sigma(\mathbf{H})$$

(i.e., if $\Delta\mathbf{H}$ is symmetric). It is a simple matter to verify that, if $\mathbf{H}$ has numerical character $(p_1, \dots, p_r)$ and associated 2-*type vector* $(e_1, \dots, e_r)$, then the following result holds.

PROPOSITION 3.6. *The following are equivalent:*

(1) $\boldsymbol{H}$ *is a complete intersection Hilbert function;*
(2) $p_{i+1} = p_i + 1$ *for all* $i = 1, \dots, r-1$*;*
(3) $e_{i+1} - e_i = 2$ *for all* $i = 1, \dots, r-1$.

Since, for a set $\mathbb{X}$ of points in $\mathbb{P}^2$, $A = k[x_0, x_1, x_2]/I_{\mathbb{X}}$ is a Gorenstein ring if and only if $I_{\mathbb{X}}$ is a complete intersection ideal in $R = k[x_0, x_1, x_2]$, we obtain that $\mathbf{H}(A, -)$ is a complete intersection Hilbert function. Thus, using Proposition 3.6 we see that the 2-*type vectors* can be used to describe all possible Hilbert functions of Gorenstein sets of points in $\mathbb{P}^2$.

Extremal subsets. Let $\mathbf{H} \in S_n$ and let $\mathbb{X}$ be a set of points in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$. We consider all the subsets of $\mathbb{X}$ which lie on a hyperplane of $\mathbb{P}^n$. (To avoid trivialities, we will assume that not all of $\mathbb{X}$ is in a hyperplane of $\mathbb{P}^n$, i.e., $\mathbf{H}(1) = n + 1$).

We can then partially order the Hilbert functions of the subsets of $\mathbb{X}$ that arise in this way as follows. Suppose that $\mathbb{X}_1$ and $\mathbb{X}_2$ are two subsets of $\mathbb{X}$ which lie in hyperplanes of $\mathbb{P}^n$. Then we define

$$\mathbf{H}_{\mathbb{X}_1} \leq \mathbf{H}_{\mathbb{X}_2} \ := \ \mathbf{H}_{\mathbb{X}_1}(i) \leq \mathbf{H}_{\mathbb{X}_2}(i) \quad \text{for every} \quad i\ .$$

Clearly, if $\mathbb{X}_1 \subseteq \mathbb{X}_2$ then $\mathbf{H}_{\mathbb{X}_1} \leq \mathbf{H}_{\mathbb{X}_2}$. We do this for *every* set $\mathbb{X}$ in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$ and thus obtain a finite, partially ordered set of Hilbert functions in $\mathcal{H}_{n-1}$, which we will call $LinSub(\mathbf{H})$.

Now suppose that $\chi_n(\mathbf{H}) = \mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$. Then we have the following interesting result.

THEOREM 3.7. *$LinSub(\boldsymbol{H})$ contains a maximal element, namely $\rho_{n-1}(\mathcal{T}_r)$.*

Proof. We have stated more than what we will prove in this section. The proof given below will show that $\rho_{n-1}(\mathcal{T}_r)$ is an upper bound for the elements of $LinSub(\mathbf{H})$. The proof will be completed in the next section (more precisely, in Remark 4.3(1)) when we construct, for any Hilbert function $\mathbf{H} \in \mathcal{S}_n$, a set of points with Hilbert function $\mathbf{H}$ having a subset on a hyperplane with Hilbert function $\rho_{n-1}(\mathcal{T}_r)$.

Let $\rho_{n-1}(\mathcal{T}_r) = \mathbf{H}_r$, let $\mathbb{Z}$ be any set of points in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$, and consider $\mathbb{L}$ a hyperplane of $\mathbb{P}^n$. We will show that

$$\Delta\mathbf{H}(\mathbb{Z} \cap \mathbb{L}, j) \leq \Delta\mathbf{H}_r(j) \quad \text{for every} \quad j.$$

This will be enough to prove that $\mathbf{H}_r$ is an upper bound for the elements of $LinSub(\mathbf{H})$.

Now $\mathbf{H}_r(j)$ is generic in $\mathbb{P}^{n-1}$ for $0 \le j < \alpha(\mathbf{H}_r)$, so we obviously have

$$\Delta\mathbf{H}(\mathbb{Z}\cap\mathbb{L}, j) \le \Delta\mathbf{H}_r(j) \quad \text{for} \quad 0 \le j < \alpha(\mathbf{H}_r)\ .$$

The result for $j \ge \alpha(\mathbf{H}_r)$ will follow easily from the following claim:

$$\Delta\mathbf{H}_r(j) = \Delta\mathbf{H}(j) \quad \text{for all } j \ge \alpha(\mathbf{H}_r).$$

To prove this claim, let $\widetilde{\mathcal{T}} = (\mathcal{T}_1, \dots, \mathcal{T}_{r-1})$ and $\rho_n(\widetilde{\mathcal{T}}) = \mathbf{H}_1$. Then, as we have seen,

$$\mathbf{H}(j) = \mathbf{H}_r(j) + \mathbf{H}_1(j-1) \quad \text{for all} \quad j.$$

By definition, $\sigma(\mathbf{H}_1) < \alpha(\mathbf{H}_r)$. Let s be the (eventually) constant value of $\mathbf{H}_1$, i.e., $\mathbf{H}_1(t) = s$ for all $t \ge \sigma(\mathbf{H}_1) - 1$. Then, for all $j \ge \alpha(\mathbf{H}_r) - 1$ we have

$$\mathbf{H}(j) = \mathbf{H}_r(j) + s$$

and so

$$\Delta\mathbf{H}(j) = \Delta\mathbf{H}_r(j)$$

for all $j \ge \alpha(\mathbf{H}_r)$, as we wanted to prove.

Since $\mathbb{Z}\cap\mathbb{L} \subseteq \mathbb{Z}$, we have $\Delta\mathbf{H}(\mathbb{Z}\cap\mathbb{L}, j) \le \Delta\mathbf{H}(j)$ for all j. Combining this with the observations made above completes the proof. $\square$

There is one final observation we would like to make about sets of points $\mathbb{X} \subset \mathbb{P}^n$ which have Hilbert function $\mathbf{H}$, where $\mathbf{H} = \rho_n(\mathcal{T})$, with $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$, an *n-type vector*. Theorem 3.7 tells us that any subset of such a set $\mathbb{X}$, which lies on a hyperplane, must have a Hilbert function which is $\le \rho_{n-1}(\mathcal{T}_r)$. The following proposition deals with the situation in which a set $\mathbb{X}$ with Hilbert function $\mathbf{H}$ actually has a (hyperplane) subset $\mathbb{U}$ for which $\mathbf{H}_{\mathbb{U}} = \rho_{n-1}(\mathcal{T}_r)$.

Proposition 3.8. *Let $\mathbb{X}$, $\boldsymbol{H}$ and $\mathcal{T}$ be as above and let $\mathbb{U} \subset \mathbb{X}$ be such that the Hilbert function of $\mathbb{U}$, $\boldsymbol{H}_{\mathbb{U}}$, satisfies $\boldsymbol{H}_{\mathbb{U}} = \rho_{n-1}(\mathcal{T}_r)$. Then, setting $\mathcal{T}' = (\mathcal{T}_1, \dots, \mathcal{T}_{r-1})$ and $\mathbb{X}' = \mathbb{X} - \mathbb{U}$, we have $\boldsymbol{H}_{\mathbb{X}'} = \rho_n(\mathcal{T}')$.*

Proof. Let L be the linear form in $R = k[x_0, \dots, x_n]$ which describes the hyperplane containing the points of $\mathbb{U}$. We have the exact sequence

$$0 \to I_{\mathbb{X}'}(-1) \xrightarrow{\times L} I_{\mathbb{X}} \to (I_{\mathbb{X}} + (L))/(L) \to 0, \tag{3.1}$$

since $\mathbb{X}'$ is precisely the set of points of $\mathbb{X}$ that do not lie on the hyperplane defined by L. Let $I_{\mathbb{U}}$ be the ideal (in R) of the set of points $\mathbb{U}$. Then $J = I_{\mathbb{X}} + (L) \subseteq I_{\mathbb{U}}$. Thus,

$$\mathbf{H}_{R/J}(t) = \mathbf{H}(R/(I_{\mathbb{X}} + (L)), t) \ge \mathbf{H}_{R/I_{\mathbb{U}}}(t) = \mathbf{H}_{\mathbb{U}}(t)\ . \tag{3.2}$$

From (3.1) we obtain

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}_{\mathbb{X}'}(t-1) + \mathbf{H}_{R/J}(t)\ . \tag{3.3}$$

From our earlier discussion of *n-type vectors* we also have

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}_{\mathcal{T}'}(t-1) + \mathbf{H}_{\mathbb{U}}(t)\ . \tag{3.4}$$

Let β be the smallest integer such that

$$\mathbf{H}_{\mathbb{X}}(\beta) - \binom{n+\beta-1}{\beta} > \mathbf{H}_{\mathbb{X}}(\beta+1) - \binom{n+\beta}{\beta+1}$$

and let $c_{\beta-1} = \mathbf{H}_{\mathbb{X}}(\beta) - \binom{n+\beta-1}{\beta}$. Then the Hilbert function of $\mathbb{U}$ is

$$\mathbf{H}_{\mathbb{U}}(t) = \mathbf{H}_{\mathcal{T}_r}(t) = \begin{cases} \binom{n+t-1}{t} & \text{for } t \le \beta, \\ \mathbf{H}_{\mathbb{X}}(t) - c_{\beta-1} & \text{for } t \ge \beta. \end{cases}$$

Hence

$$\mathbf{H}_{\mathbb{U}}(t) = \mathbf{H}_{\mathcal{T}_r}(t) = \mathbf{H}_{R/J}(t) = \binom{n+t-1}{t} \tag{3.5}$$

for $t \le \beta$. Moreover,

$$\Delta\mathbf{H}_{\mathbb{U}}(t) = \Delta\mathbf{H}_{\mathcal{T}_r}(t) = \Delta\mathbf{H}_{R/J}(t) \tag{3.6}$$

for such t. Since $\sigma(\mathbf{H}_{\mathcal{T}'}) \le \beta$, we see that

$$\Delta\mathbf{H}_{\mathcal{T}'}(t) = 0 \tag{3.7}$$

for every $t \ge \beta$. From (3.3) and (3.4) we have

$$\begin{aligned} \Delta\mathbf{H}_{\mathbb{X}}(t) &= \Delta\mathbf{H}_{\mathcal{T}'}(t-1) + \Delta\mathbf{H}_{\mathbb{U}}(t) && (3.8) \\ &= \Delta\mathbf{H}_{\mathbb{X}'}(t-1) + \Delta\mathbf{H}_{R/J}(t). && (3.9) \end{aligned}$$

Since $\Delta\mathbf{H}_{\mathcal{T}'}(t-1) = 0$ and $\Delta\mathbf{H}_{\mathbb{X}'}(t-1) \ge 0$ for $t-1 \ge \beta$, we have

$$\Delta\mathbf{H}_{\mathbb{U}}(t) \ge \Delta\mathbf{H}_{R/J}(t) \tag{3.10}$$

for every $t \ge \beta+1$. From (3.6) and (3.10), we obtain

$$\Delta\mathbf{H}_{\mathbb{U}}(t) \ge \Delta\mathbf{H}_{R/J}(t) \tag{3.11}$$

for every $t \ge 0$. Hence we have

$$\mathbf{H}_{\mathbb{U}}(t) \ge \mathbf{H}_{R/J}(t) \tag{3.12}$$

for such t. It follows from (3.2) and (3.12) that

$$\mathbf{H}_{\mathbb{U}}(t) = \mathbf{H}_{R/J}(t) \tag{3.13}$$

for every $t \ge 0$. Therefore, we obtain from (3.3), (3.4), and (3.13) that

$$\mathbf{H}_{\mathbb{X}'}(t) = \mathbf{H}_{\mathcal{T}'}(t)$$

for every $t \ge 0$ and we are done. □

Notice that, as a bonus, we obtain that $I_{\mathbb{X}} + (L) = I_{\mathbb{U}}$ in this case.

4. k-configurations in $\mathbb{P}^n$

Let $\mathbf{H} \in \mathcal{S}_n$. Then $\mathbf{H}$ can, in general, be the Hilbert function of many different sets of points in $\mathbb{P}^n$. For example, if

$$\mathbf{H} := 1 \;\; 3 \;\; 5 \;\; 6 \quad \rightarrow \quad \in \mathcal{S}_2,$$

then $\mathbf{H}$ is the Hilbert function of the complete intersection of a conic and a cubic. However, $\mathbf{H}$ is also the Hilbert function of the set

$$\begin{array}{llll} \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

which (by Bezout) cannot be the complete intersection of a conic and a cubic.

We will show how to associate, to any Hilbert function $\mathbf{H} \in \mathcal{S}_n$, a special point set in $\mathbb{P}^n$ which, naturally, has Hilbert function $\mathbf{H}$ and is "extremal" with respect to Theorem 3.7.

These types of point sets have been studied in $\mathbb{P}^2$ and $\mathbb{P}^3$ by Geramita, Harima, and Shin [7], Geramita, Pucci, and Shin [11], Geramita and Shin [12], Harima [14], and Shin [17]. In this section we will define the point sets in question and give a few of their elementary properties. A deeper study will be carried out in a subsequent paper [7].

Our assignment of a point set to a Hilbert function $\mathbf{H} \in \mathcal{S}_n$ will be done inductively.

Definition 4.1 (**k-configuration in $\mathbb{P}^n$**).

S_0: The only element in S_0 is $\mathbf{H} := 1 \;\rightarrow$, which is the Hilbert function of $\mathbb{P}^0$, which is a single point. This is the only k-configuration in $\mathbb{P}^0$.

S_1: Let $\mathbf{H} \in S_1$. Then $\chi_1(\mathbf{H}) = T = (e)$, where $e \geq 1$. We associate to $\mathbf{H}$ any set of e distinct points in $\mathbb{P}^1$. Clearly, any set of e distinct points in $\mathbb{P}^1$ has Hilbert function $\mathbf{H}$. A set of e distinct points in $\mathbb{P}^1$ will be called a *k-configuration in $\mathbb{P}^1$ of type* $T = (e)$.

S_2: Let $\mathbf{H} \in S_2$ and let $T = ((e_1), \ldots, (e_r)) = \chi_2(\mathbf{H})$, where $T_i = (e_i)$ is a 1-*type vector*. Choose r distinct sets $\mathbb{P}^1$' in $\mathbb{P}^2$, i.e., lines in $\mathbb{P}^2$, and label these $\mathbb{L}_1, \ldots, \mathbb{L}_r$. By induction we choose, on $\mathbb{L}_i$, a *k-configuration* $\mathbb{X}_i$ in $\mathbb{P}^1$ of type $T_i = (e_i)$ such that no point of $\mathbb{L}_i$ contains a point of $\mathbb{X}_j$ for $j < i$. The set $\mathbb{X} = \bigcup \mathbb{X}_i$ is called a *k-configuration in $\mathbb{P}^2$* of type T.

S_n, $n > 2$: Now suppose that we have defined a *k-configuration* of type $\widetilde{T} \in \mathbb{P}^{n-1}$, where $\widetilde{T}$ is an $(n-1)$-*type vector* associated to $G \in S_{n-1}$. Let $\mathbf{H} \in S_n$ and suppose that $\chi_n(\mathbf{H}) = T = (T_1, \ldots, T_r)$, where the T_i are $(n-1)$-*type vectors*. Then $\rho_{n-1}(T_i) = \mathbf{H}_i$ and $\mathbf{H}_i \in S_{n-1}$. Consider distinct hyperplanes $\mathbb{H}_1, \ldots, \mathbb{H}_r$ in $\mathbb{P}^n$, and let $\mathbb{X}_i$ be a *k-configuration* in $\mathbb{H}_i$ of type T_i such that $\mathbb{H}_i$ does not contain any point of $\mathbb{X}_j$ for any $j < i$. The set $\mathbb{X} = \bigcup \mathbb{X}_i$ is called a *k-configuration in $\mathbb{P}^n$* of type T.

We claim that the set of points so chosen has Hilbert function $\mathbf{H}$. To prove this claim, we proceed by induction on r.

The case $r = 1$ is obvious. Suppose $r \geq 2$. We will have shown, by induction, that $\mathbf{H}_i = \rho_1(\mathcal{T}_i)$ is the Hilbert function of $\mathbb{X}_i$ and that $\widetilde{\mathbf{H}}(t) := \mathbf{H}_1(t - (r-2)) + \cdots + \mathbf{H}_{r-2}(t-1) + \mathbf{H}_{r-1}(t)$ is the Hilbert function of $\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_{r-1}$. By Corollary 2.8 of [10] (which is applicable here since $\sigma(\widetilde{\mathbf{H}}) < \alpha(\mathbf{H}_r)$ and the line containing $\mathbb{X}_r$ contains no point of $\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_{r-1}$) we obtain

$$\mathbf{H}_{\mathbb{X}}(t) = \widetilde{\mathbf{H}}(t-1) + \mathbf{H}_r(t) \ .$$

As we have seen, this is the description of the Hilbert function associated to $\mathcal{T}$, i.e. $\mathbf{H}$. This completes the proof of the claim.

Notation and Terminology: If $\mathbf{H} \in \mathcal{S}_n$ and $\chi_n(\mathbf{H}) = \mathcal{T}$, where $\mathcal{T}$ is an *n-type vector*, and $\mathbb{X}$ is a *k-configuration* associated to $\mathbf{H}$ (or $\mathcal{T}$), then we say that $\mathbb{X}$ *is a k-configuration in $\mathbb{P}^n$ of type $\mathcal{T}$*.

If we write $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ and let $\mathbb{X}$ be a *k-configuration* in $\mathbb{P}^n$ of type $\mathcal{T}$ then, by definition,

$$\mathbb{X} = \mathbb{X}_1 \cup \ldots \cup \mathbb{X}_r \quad \text{with a disjoint union,}$$

where $\mathbb{X}_i$ is a *k-configuration* of type $\mathcal{T}_i$ and $\mathbb{X}_i \subseteq \mathbb{L}_i$, where $\mathbb{L}_i \simeq \mathbb{P}^{n-1}$ is a linear subspace of $\mathbb{P}^n$. We will call the $\mathbb{X}_i$ the *(first) sub-k-configurations* of $\mathbb{X}$.

Now $\mathcal{T}_i = (\mathcal{T}_{i1}, \ldots, \mathcal{T}_{ir_i})$ where the $\mathcal{T}_{ij}$ are *$(n-2)$-type vectors*. Thus

$$\mathbb{X}_i = \mathbb{X}_{i,1} \cup \ldots \cup \mathbb{X}_{i,r_i},$$

where the $\mathbb{X}_{i,j}$ are in linear subspaces $\mathbb{L}_{i,j}$ of $\mathbb{L}_i$ and $\mathbb{X}_{i,j}$ is a *k-configuration* of type $\mathcal{T}_{i,j}$ in $\mathbb{P}^{n-2} \simeq \mathbb{L}_{i,j}$. The spaces $\mathbb{X}_{i,j}$, $1 \leq i \leq r$, $1 \leq j \leq r_i$ are called the *(second) sub-k-configurations* of $\mathbb{X}$. The description of the remainder of this hierarchical decomposition of $\mathbb{X}$ should now be clear.

Example 4.2. Let $\mathbf{H}$ be the Hilbert function

$$\mathbf{H} := 1 \ \ 4 \ \ 9 \ \ 12 \ \ 15 \ \ 17 \ \ 19 \ \ 21 \ \ 22 \ \ \rightarrow$$

Then $\mathbf{H} \leftrightarrow \mathcal{T} = ((1,2),(3,7,9))$.

A *k-configuration* in $\mathbb{P}^3$ of type $\mathcal{T}$ is a set of points $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$ where $\mathbb{X}_1 \subseteq \mathbb{L}_1$ and $\mathbb{X}_2 \subseteq \mathbb{L}_2$ (where $\mathbb{L}_1$ and $\mathbb{L}_2$ are two distinct linear subspaces of $\mathbb{P}^3$) and no point of $\mathbb{X}_1 \cup \mathbb{X}_2$ is in $\mathbb{L}_1 \cap \mathbb{L}_2$. Moreover, $\mathbb{X}_1$ is a *k-configuration* in $\mathbb{L}_1 \simeq \mathbb{P}^2$ of type $(1,2)$, $\mathbb{X}_2$ a *k-configuration* in $\mathbb{L}_2 \simeq \mathbb{P}^2$ of type $(3,7,9)$, and $\mathbb{X}_1$ and $\mathbb{X}_2$ are the *first sub-k-configurations* of $\mathbb{X}$. Now $\mathbb{X}_1$ consists of 3 points on two distinct lines in $\mathbb{L}_1 \simeq \mathbb{P}^2$, $\mathbb{L}_{1,1}$ and $\mathbb{L}_{1,2}$, with one point in $\mathbb{L}_{1,1}$ (say, $\mathbb{X}_{1,1}$) and 2 points on $\mathbb{L}_{1,2}$ (say, $\mathbb{X}_{1,2}$) of $\mathbb{X}$. Similarly $\mathbb{X}_2 = \mathbb{X}_{2,1} \cup \mathbb{X}_{2,2} \cup \mathbb{X}_{2,3}$ where $\mathbb{X}_{2,1}$ contains 3 points, $\mathbb{X}_{2,2}$ contains 7 points and $\mathbb{X}_{2,3}$ contains 9 points, on three separate lines $\mathbb{L}_{2,1}$, $\mathbb{L}_{2,2}$, and $\mathbb{L}_{2,3}$ in $\mathbb{L}_2 \simeq \mathbb{P}^2$.

The sets $\mathbb{X}_{1,1},\ \mathbb{X}_{1,2},\ \mathbb{X}_{2,1},\ \mathbb{X}_{2,2},\ \mathbb{X}_{2,3}$ are the (second) *sub-k-configurations* of $\mathbb{X}$.

Remark 4.3.

(1) Notice that if $\mathbf{H} \leftrightarrow \mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$ and if $\mathbb{X}$ is a *k-configuration* of type $\mathcal{T}$, then the first sub-k-configuration $\mathbb{X}_r$ has Hilbert function $\rho_{n-1}(\mathcal{T}_r)$. This remark, then, completes the proof of Theorem 3.7.

(2) Corollary 3.5 shows that, for some Hilbert functions $\mathbf{H} \in \mathcal{S}_2$, the *only* possible point sets with Hilbert function $\mathbf{H}$ are *k-configurations* in $\mathbb{P}^2$.

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