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COMPACT COMPOSITION OPERATORS ON A HILBERT SPACE OF DIRICHLET SERIES

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ABSTRACT. We study the compactness of composition operators on the Hilbert space of Dirichlet series with square summable coefficients. In particular, we give some necessary and sufficient conditions for compactness. We also describe the spectrum of such operators, and we extend our work to some weighted spaces.

1. Introduction

Let $\mathcal{H}$ be the Hilbert space of Dirichlet series with square summable coefficients:

$$\mathcal{H}=\left\{f(s)=\sum_{1}^{+\infty}a_n n^{-s}:\ \|f\|_2=\left(\sum_{1}^{+\infty}|a_n|^2\right)^{1/2}<+\infty\right\}.$$

By the Cauchy-Schwarz inequality, the functions in $\mathcal{H}$ are all holomorphic on the half-plane $\mathbb{C}_{1/2}$ (where, for θ real, $\mathbb{C}_\theta=\{s\in\mathbb{C}:\ \Re(s)>\theta\}$ and $\mathbb{C}_+=\mathbb{C}_0$). Taking $a_n=1/(n^{1/2}\log n)$ shows that the functions in $\mathcal{H}$ are in general not defined on a larger domain. In [5], J. Gordon and H. Hedenmalm solved the following problem:

> For which analytic mappings $\phi:\mathbb{C}_{1/2}\to\mathbb{C}_{1/2}$ is the composition operator $C_\phi(f)=f\circ\phi$ a bounded linear operator on $\mathcal{H}$?

THEOREM 1. *An analytic function $\phi:\mathbb{C}_{1/2}\to\mathbb{C}_{1/2}$ defines a bounded composition operator $C_\phi:\mathcal{H}\to\mathcal{H}$ if and only if:*

(a) *It is of the form*
$$\phi(s)=c_0 s+\varphi(s),$$
where $c_0\in\mathbb{N}$, and $\varphi(s)=\sum_1^{+\infty}c_n n^{-s}$ admits a representation by a Dirichlet series that is convergent in some half-plane.

(b) *ϕ has an analytic extension to $\mathbb{C}_+$, also denoted by ϕ, such that:*

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(i) $\phi(\mathbb{C}_+) \subset \mathbb{C}_+$ *if* $c_0 \geq 1$.
(ii) $\phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$ *if* $c_0 = 0$.

In this statement, conditions (a) and (b) have two different meanings: Condition (a) is an arithmetic condition ($f \circ \phi$ must be a Dirichlet series), whereas (b) is an analytic condition ($f \circ \phi$ must be in $\mathcal{H}$).

The next step in the study of composition operators on a Banach space of analytic functions is to compare the properties of the operator C_ϕ and of its symbol ϕ. We began this comparison in [1], where, for example, we characterized completely the Fredholm composition operators on $\mathcal{H}$: C_ϕ is Fredholm if and only if $\phi(s) = s + i\tau$, $\tau \in \mathbb{R}$.

Here we consider the compactness question: What conditions should we impose on ϕ for C_ϕ to be a compact operator? In [1], we gave some sufficient conditions: If $\phi(\mathbb{C}_+)$ is strictly smaller than it can be, C_ϕ is compact. More precisely, if $\phi(\mathbb{C}_+) \subset \mathbb{C}_\varepsilon$, $\varepsilon > 0$, for $c_0 \geq 1$, or if $\phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2+\varepsilon}$, $\varepsilon > 0$, for $c_0 = 0$, then C_ϕ is compact. One of our aims is to obtain less trivial sufficient conditions, and to give necessary conditions.

This paper is organized as follows. In Section 2, we give the background material necessary to make this paper as self-contained as possible. In Section 3, we explain the main difficulties which we encounter. Next, we give some partial results on the problem of finding sufficient (Section 4) and necessary (Section 5) conditions for compactness. In Section 6, we describe the spectrum of compact composition operators on $\mathcal{H}$, and in Section 7, we extend our results to some other Hilbert spaces of Dirichlet series recently introduced by J. McCarthy [8].

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2. Background material

Let Θ be the dual group of $\mathbb{Q}_+$, where $\mathbb{Q}_+$ denotes the multiplicative discrete group of strictly positive rational numbers. Θ is the set of all characters $\chi : \mathbb{Q}_+ \to \mathbb{C}$:

(a) $\chi(mn) = \chi(m)\chi(n)$ for all m, n in $\mathbb{Q}_+$.
(b) $|\chi(n)| = 1$.

Θ and $\mathbb{T}^\infty$, the Cartesian product of countably many copies of the unit circle, can be identified in the following way. Given a point $z = (z_1, z_2, \dots) \in \mathbb{T}^\infty$, we define the value of χ at the primes through

$$\chi(2) = z_1, \quad \chi(3) = z_2, \dots, \quad \chi(p_m) = z_m, \dots,$$

and extend the definition multiplicatively. This then yields a character, and clearly all characters are obtained by this procedure. In the sequel, we will drop the notation Θ and write $\chi \in \mathbb{T}^\infty$ (see [6] for details).

Characters are connected with vertical limit functions of $\mathcal{H}$. Indeed, fix any element $f(s)=\sum_1^{+\infty} a_n n^{-s}$ of $\mathcal{H}$. The vertical translations of f are the functions $f_\tau(s)=f(s+i\tau)$. To every sequence (τ_n) of translations there exists a subsequence, say $(\tau_{n(k)})$, such that $f_{\tau_{n(k)}}$ converges uniformly on compact subsets of the domain $\mathbb{C}_{1/2}$ to a limit function, say $\tilde{f}(s)$. We call $\tilde{f}$ a vertical limit function of f. In [6], the following result was proved.

Lemma 1. *The vertical limit functions of the function* $f(s)=\sum_1^{+\infty} a_n n^{-s}$ *coincide with the functions of the form*

$$f_\chi(s)=\sum_1^{+\infty} a_n \chi(n) n^{-s},$$

where χ *is a character.*

In [6], it was also explained that it is illuminating to consider all functions f_χ to obtain properties of f and of $\mathcal{H}$. For example, for almost all (with respect to the Haar measure m of $\mathbb{T}^\infty$) characters χ, the function f_χ can be extended to $\mathbb{C}_+$. Moreover, we can compute the norm of f in terms of the function f_χ (see [6, Theorem 4.1] or [1, Lemma 5]):

Lemma 2. *Let* μ *be a finite Borel measure on* $\mathbb{R}$. *Then*

$$\|f\|_2^2\mu(\mathbb{R})=\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}|f_\chi(it)|^2 d\mu(t)dm(\chi).$$

We shall need to extend the notation f_χ to the class of functions of the form $\phi(s)=c_0 s+\varphi(s)$, where $c_0\in\mathbb{N}$ and φ is a Dirichlet series. For such functions, ϕ_χ will be defined by

$$\phi_\chi(s)=c_0 s+\varphi_\chi(s).$$

It should be pointed out that in this case we cannot interpret ϕ_χ as a vertical limit function of ϕ: ϕ_χ is a vertical limit of the functions $\phi_\tau(s)=c_0 s+\varphi(s+i\tau)$. The connection between the composition operator C_ϕ and C_{ϕ_χ} is clarified in [5], where it was shown that for any holomorphic mapping $\phi:\mathbb{C}_{1/2}\to\mathbb{C}_{1/2}$ of the form $\phi(s)=c_0 s+\varphi(s)$, for any $f\in\mathcal{H}$, and for any $\chi\in\mathbb{T}^\infty$, the following relation holds:

$$(f\circ\phi)_\chi(s)=f_{\chi^{c_0}}\circ\phi_\chi(s),\ s\in\mathbb{C}_{1/2}.$$

Moreover, for almost all $\chi\in\mathbb{T}^\infty$, this relation remains true in $\mathbb{C}_+$. Before proceeding further, we mention that this formula, and the fact that almost every f_χ is defined on $\mathbb{C}_+$, explain the strange appearance of the half-plane $\mathbb{C}_+$ in Theorem 1.

Of course, Dirichlet series are connected with arithmetical conditions. We recall a theorem of Kronecker in a form which will be useful for us:

Definition 1. A sequence (q_j) of integers is said multiplicatively independent if, for any $d \geq 1$ and for any $c_1, \dots, c_d$ in $\mathbb{Z}$ the equality

$$c_1 \log q_1 + \cdots + c_d \log q_d = 0$$

implies $c_1 = \cdots = c_d = 0$.

Lemma 3. *Let $q_1, \dots, q_d$ be multiplicatively independent integers. Then the function*

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{T}^d \\ t & \mapsto & (q_1^{it}, \dots, q_d^{it}) \end{array}$$

has dense range.

In particular, if $P(s) = a_1 q_1^{-s} + \cdots + a_d q_d^{-s}$ is a Dirichlet polynomial with spectrum in the q_j's, then

$$\sup \{|P(s)| : \ \Re(s) = 0\} = \sum_1^d |a_j|.$$

The last tool that we will need is the following lemma (Lemma 11 of [1], which is a strengthening of Proposition 4.3 of [5]).

Lemma 4. *Let $\phi(s) = c_0 s + \varphi(s)$, $\phi : \mathbb{C}_+ \to \mathbb{C}_+$. If $\phi(s) \neq s + i\tau$, $\tau \in \mathbb{R}$, then there exist $\eta > 0$ and $\varepsilon > 0$ so that $\phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}$.*

3. Main difficulties

Let $\phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ be an analytic function of the form $\phi(s) = c_0 s + \varphi(s)$, $c_0 \geq 1$. We denote by $\psi_1 : \mathbb{C}_+ \to \mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ the conformal transformation of $\mathbb{C}_+$ onto $\mathbb{D}$ defined by $\psi_1(s) = (s-1)/(s+1)$. Let us define $\psi = \psi_1 \circ \phi \circ \psi_1^{-1}$. Then ψ is a holomorphic mapping from $\mathbb{D}$ to $\mathbb{D}$, and by the Littlewood subordination principle [10], C_ψ is a continuous operator on the classical Hardy space

$$H^2(\mathbb{D}) = \left\{ f = \sum_0^{+\infty} a_n z^n : \ \sum_0^{+\infty} |a_n|^2 < +\infty \right\}.$$

To obtain the continuity of C_ϕ on $\mathcal{H}$, the main idea of Gordon and Hedenmalm was to transfer the continuity of C_ψ through the identity $\psi = \psi_1 \circ \phi \circ \psi_1^{-1}$. One might expect that similar arguments would allow us to obtain the compactness of C_ϕ from that of C_ψ.

Unfortunately, this is hopeless since, because of the behavior of ϕ near $+\infty$, C_ψ is never compact on $H^2(\mathbb{D})$. Let us recall a classical result on the compactness of composition operators on $H^2(\mathbb{D})$ (see [10, Chapter 3]). Let ψ be a holomorphic mapping from $\mathbb{D}$ to $\mathbb{D}$. By using the images of reproducing kernels by C_ψ^*, it can be shown that the compactness of C_ψ implies

$$\lim_{|z|\to 1}\frac{1-|\psi(z)|}{1-|z|}=+\infty. \tag{1}$$

If ψ is written as $\psi=\psi_1\circ\phi\circ\psi_1^{-1}$, where $\phi(s)=c_0s+\varphi(s)$, $c_0\geq 1$, this condition is never satisfied. Indeed, there exists a half-plane in which the Dirichlet series φ is absolutely convergent, and in this half-plane, $|\varphi(s)|\leq A$, where A is a constant. Let s_1 be a sufficiently large real number, and let $s_2=\phi(s_1)$. Let us set $z_1=\psi_1(s_1)$ and $z_2=\psi_1(s_2)$, so that $z_2=\psi(z_1)$. It is clear that z_1 tends to 1 if s_1 tends to $+\infty$. Now,

$$1-|z_1|=1-\left|\frac{s_1-1}{s_1+1}\right|=1-\left|\frac{\frac{s_2-\varphi(s_1)}{c_0}-1}{\frac{s_2-\varphi(s_1)}{c_0}+1}\right|=1-\left|\frac{s_2-\varphi(s_1)-c_0}{s_2-\varphi(s_1)+c_0}\right|.$$

As s_1 is large, $|s_2|$ is large too, whereas $\varphi(s_1)$ and c_0 remain bounded. Hence there exists a constant C' such that

$$\left|\frac{s_2-\varphi(s_1)-c_0}{s_2-\varphi(s_1)+c_0}\right|\leq 1-\frac{C'}{|s_2|}.$$

Moreover,

$$1-|z_2|=1-\left|\frac{s_2-1}{s_2+1}\right|\leq\frac{C''}{|s_2|}.$$

In particular,

$$\frac{1-|\psi(z_1)|}{1-|z_1|}\leq\frac{C''}{C'}.$$

Since z_1 can be chosen arbitrarily close to the circle, this is in contradiction with (1). Hence C_ψ is not compact.

On the other hand, it is not as easy as usual to obtain good necessary conditions for the compactness. Recall that on a Hilbert space H of analytic functions on a domain U, a reproducing kernel at $w\in U$ is a function K_w of H which satisfies

$$\forall f\in H,\ \langle f,K_w\rangle=f(w).$$

For any composition operator C_ψ on H it is almost trivial that $C_\psi^*(K_w)=K_{\psi(w)}$. (The proof given in [10] for $H^2(\mathbb{D})$ can be transferred to this more general setting.) In general, by considering the images of certain sequences of normalized reproducing kernels one obtains conditions like (1) which ψ must satisfy for C_ψ to be compact.

In the case of $\mathcal{H}$, the reproducing kernel at w in $\mathbb{C}_{1/2}$ is given by $K_w(s)=\sum_{n=1}^{+\infty}n^{-\overline{w}}n^{-s}$, whose norm equals $\zeta(2\Re w)^{1/2}$. The previous arguments give in this context the following result.

Proposition 1. *Let C_ϕ be a compact composition operator on $\mathcal{H}$. Then*
$$\frac{\zeta\big(2\Re(\phi(w))\big)}{\zeta\big(2\Re(w)\big)} \xrightarrow{\Re(w)\to 1/2} 0.$$

Proof. Let (w_n) be a sequence in $\mathbb{C}_{1/2}$, whose real part tends to $1/2$. The sequence $(K_{w_n}/\|K_{w_n}\|)$ converges weakly to 0. Now, the compactness of C_ϕ implies that of C_ϕ^*, and
$$C_\phi^*\left(\frac{K_{w_n}}{\|K_{w_n}\|}\right) \xrightarrow{n\to+\infty} 0,$$
or
$$\frac{\|K_{\phi(w_n)}\|}{\|K_{w_n}\|} = \frac{\zeta\big(2\Re\phi(w_n)\big)}{\zeta\big(2\Re w_n\big)} \xrightarrow{n\to+\infty} 0. \qquad \square$$

Nevertheless, this proposition is not useful. Indeed, if $\phi(s) \neq s+i\tau$, Lemma 4 asserts that $\phi(\mathbb{C}_{1/2}) \subset \mathbb{C}_{1/2+\varepsilon}$, and in this case the condition is always satisfied. Thus the proposition just says that if $\phi(s) = s + i\tau$, then C_ϕ is not compact. But this is clear since in this case C_ϕ is even invertible!

In the following, we will handle the problem of compactness by different and more efficient ways.

4. Sufficient conditions

Compact composition operators on $H^2(\mathbb{D})$ have been completely characterized by J. Shapiro [9]. Let us recall his method. His starting point is a formula to compute the norm of an element of $H^2(\mathbb{D})$ by an area integral: If $f \in H^2(\mathbb{D})$, then
$$\|f\|_2^2 = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z), \tag{2}$$
where $dA = \frac{1}{\pi}dxdy$. This led Shapiro to introduce, for a holomorphic mapping $\psi : \mathbb{D} \to \mathbb{D}$, its counting function defined by
$$N_\psi(w) = \begin{cases} \displaystyle\sum_{z\in\psi^{-1}(w)} \log\frac{1}{|z|} & \text{if } w \in \psi(\mathbb{D}),\\ 0 & \text{if } w \notin \psi(\mathbb{D}).\end{cases}$$
The condition (satisfied by any holomorphic function $\psi : \mathbb{D} \to \mathbb{D}$) $N_\psi(z) = O(\log(1/|z|))$ as $|z| \to 1^-$ is a way to interpret the continuity of C_ψ on $H^2(\mathbb{D})$. Shapiro showed that the strengthening of this condition to
$$N_\psi(z) = o\left(\log\frac{1}{|z|}\right) \text{ as } |z| \to 1^-$$
characterizes the compactness of C_ψ.

We now apply the same idea to $\mathcal{H}$. We begin by giving a new expression for the norm of an element of $\mathcal{H}$.

PROPOSITION 2. *Let μ be a probability Borel measure on $\mathbb{R}$. Then, for all $f \in \mathcal{H}$,*

$$\|f\|_2^2 = 4\int_{\mathbb{T}^\infty}\int_{\sigma=0}^{+\infty}\int_{t\in\mathbb{R}} \sigma |f'_\chi(\sigma+it)|^2 d\mu(t)d\sigma dm(\chi) + |f(\infty)|^2. \tag{3}$$

Proof. By Lemma 2, if $\sigma > 0$,

$$\int_{\mathbb{T}^\infty}\int_{\mathbb{R}_+} \sigma|f'_\chi(\sigma+it)|^2 d\mu(t)dm(\chi) = \sum_{n\geq 2}\sigma|a_n|^2 n^{-2\sigma}\log^2(n).$$

Now, an integration by parts shows that

$$\int_0^{+\infty} n^{-2\sigma}\sigma d\sigma = \frac{1}{4\log^2 n}.$$

This gives the proposition. □

Inspired by Shapiro's method and by the above proposition, it seems natural to introduce the following definition.

DEFINITION 2. Let $\phi : \mathbb{C}_+ \to \mathbb{C}_+$, $\phi(s) = c_0 s + \varphi(s)$. The counting function of ϕ is defined by

$$\mathcal{N}_\phi(s) = \begin{cases} \displaystyle\sum_{w\in\phi^{-1}(s)} \Re(w) & \text{if } s \in \phi(\mathbb{C}_+), \\ 0 & \text{otherwise.} \end{cases}$$

We begin by proving a Littlewood-like inequality (see [10, Section 10.3]) for this counting function.

PROPOSITION 3. *Let $\phi : \mathbb{C}_+ \to \mathbb{C}_+$, $\phi(s) = c_0 s + \varphi(s)$, $c_0 \geq 1$. Then,*

$$\mathcal{N}_\phi(s) \leq \frac{1}{c_0}\Re(s) \text{ for all } s \in \mathbb{C}_+.$$

Proof. If $s \notin \phi(\mathbb{C}_+)$, the result is trivial. Otherwise, let $w_1, \dots, w_N$ be any distinct pre-images of s under ϕ, where N is any finite number. For $\zeta > 0$ let us set $\psi_\xi(s) = (s-\xi)/(s+\xi)$, which maps $\mathbb{C}_+$ conformally onto $\mathbb{D}$. We define $\psi = \psi_{c_0\xi} \circ \phi \circ \psi_\xi^{-1}$, which is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$, with

$$\psi(0) = \frac{\varphi(\xi)}{2c_0\xi + \varphi(\xi)}.$$

Clearly, $\psi(\psi_\xi(w_k)) = \psi_{c_0\xi}(s)$, and Littlewood's inequality asserts that

$$\sum_1^N \log\left|\frac{1}{\psi_\xi(w_k)}\right| \leq \log\left|\frac{1-\overline{\psi_{c_0\xi}(s)}\psi(0)}{\psi_{c_0\xi}(s)-\psi(0)}\right|. \tag{4}$$

Now, if ω denotes any w_i, observe that

$$\begin{aligned}\log\left|\frac{1}{\psi_\xi(\omega)}\right| &= \frac{1}{2}\log\left|\frac{|\omega|^2+2\xi\Re(\omega)+|\xi|^2}{|\omega|^2-2\xi\Re(\omega)+|\xi|^2}\right|\\ &= \frac{1}{2}\log\left|1+\frac{4\xi\Re(\omega)}{|\omega|^2-2\xi\Re(\omega)+|\xi|^2}\right|.\end{aligned}$$

Since ω is in a finite set, if $\varepsilon>0$ is fixed, then for ξ large enough and $i=1,\dots,N$ one has

$$\log\left|\frac{1}{\psi_\xi(w_i)}\right|\geq 2(1-\varepsilon)\frac{\Re(w_i)}{\xi}. \tag{5}$$

Likewise, if ξ is large enough, then

$$\log\left|\frac{1-\overline{\psi_{c_0\xi}(s)}\psi(0)}{\psi_{c_0\xi}(s)-\psi(0)}\right|\leq(1+\varepsilon)\log\left|\frac{1}{\psi_{c_0\xi}(s)}\right|\leq 2(1+\varepsilon)^2\frac{\Re(s)}{c_0\xi}. \tag{6}$$

Now, inequalities (4), (5) and (6) give

$$\sum_1^N\Re(w_k)\leq\frac{(1+\varepsilon)^2}{(1-\varepsilon)}\frac{\Re(s)}{c_0}.$$

By letting $\varepsilon\to 0$ and $N\to+\infty$, we obtain the proposition. □

Formula (3) requires an integration on $\mathbb{T}^\infty$. Therefore, we will need estimates for all functions $\mathcal{N}_{\phi_\chi}$, $\chi\in\mathbb{T}^\infty$. Nevertheless, some estimates for $\mathcal{N}_\phi$ transfer to $\mathcal{N}_{\phi_\chi}$, as the following result illustrates.

Proposition 4. *Let $\phi(s):\mathbb{C}_+\to\mathbb{C}_+$, $\phi(s)=c_0s+\varphi(s)$, $c_0\geq 1$. Suppose that there exists $\varepsilon>0$ and $\theta>0$ such that, for any $s\in\mathbb{C}_+$ with $\Re(s)\leq\theta$,*

$$\mathcal{N}_\phi(s)\leq\varepsilon\Re(s).$$

Then, for any $\chi\in\mathbb{T}^\infty$ and any $s\in\mathbb{C}_+$ with $\Re(s)\leq\theta$ we have

$$\mathcal{N}_{\phi_\chi}(s)\leq\varepsilon\Re(s).$$

Proof. Let us recall that, for $\tau\in\mathbb{R}$, $\phi_\tau(w-i\tau)=\phi(w)-ic_0\tau$. Therefore, $\mathcal{N}_{\phi_\tau}(s-ic_0\tau)=\mathcal{N}_\phi(s)$.

Let us assume that the proposition is does not hold for some $\chi\in\mathbb{T}^\infty$, and for a complex number $s\in\mathbb{C}_+$, with $\Re(s)\leq\theta$. In particular, there exist elements $w_1,\dots,w_N$ of $\mathbb{C}_+$ satisfying $\phi_\chi(w_k)=s$ and

$$\Re(w_1)+\cdots+\Re(w_N)>\varepsilon\Re(s).$$

Let us fix $\eta>0$ such that $\Re(w_1)+\cdots+\Re(w_N)-N\eta>\varepsilon\Re(s)$. We set $B_k=B(w_k,\eta)=\{w\in\mathbb{C}_+:\ |w-w_k|<\eta\}$. There exists a sequence (τ_n) such that ϕ_{τ_n} converges uniformly to ϕ_χ on each B_k. Since $s\in\phi_\chi(B_k)$ for each k, Hurwitz's lemma implies that we can find an integer n such that $s \in \phi_{\tau_n}(B_k)$ for $k = 1, \dots, N$. Let us consider $w'_k \in B_k$ with $\phi_{\tau_n}(w'_k) = s$. Then

$$\Re(w'_1) + \cdots + \Re(w'_N) > \varepsilon \Re(s).$$

This is in contradiction with $\mathcal{N}_{\phi_{\tau_n}}(s) \leq \varepsilon \Re(s)$. $\square$

We are now able to prove the main result of this paper.

THEOREM 2. *Let $\phi : \mathbb{C}_+ \to \mathbb{C}_+$, $\phi(s) = c_0 s + \varphi(s)$, $c_0 \geq 1$. Suppose that:*

(a) *$\Im\varphi$ is bounded on $\mathbb{C}_+$.*
(b) *$\mathcal{N}_\phi(s) = o\big(\Re(s)\big)$ if $\Re(s) \to 0$.*

Then C_ϕ is compact on $\mathcal{H}$.

Proof. Let (f_n) be a sequence in $\mathcal{H}$ which converges weakly to 0 and satisfies $\|f_n\| \leq 1$. Let A be a constant such that $\big|\Im\varphi\big| \leq A$. By formula (3) we have

$$\|f_n \circ \phi\|_2^2 = |f_n \circ \phi(\infty)|^2 + 4 \int_{\mathbb{T}^\infty} \int_{\mathbb{R}_+} \int_0^1 \sigma \left| f'_{n,\chi^{c_0}}(\phi_\chi(\sigma + it)) \right|^2 \left| \phi'_\chi(\sigma + it) \right|^2 dt d\sigma dm(\chi).$$

The first term is easy to handle. $|f_n(+\infty)|$ converges to 0 if $n \to +\infty$. To deal with the second term, we begin by making the non-univalent change of variables $w = \phi_\chi(\sigma + it)$. Observe that, since $t \in [0,1]$, $-A \leq \Im w \leq A + c_0$. By applying, for example, Theorem 2.4.18 of [3], we have

$$\int_{\mathbb{R}_+} \int_0^1 \sigma \left| f'_{n,\chi^{c_0}}(\phi_\chi(\sigma + it)) \right|^2 \left| \phi'_\chi(\sigma + it) \right|^2 dt d\sigma \leq \int_{\mathbb{R}_+} \int_{-A}^{A+c_0} \left| f'_{n,\chi^{c_0}}(s) \right|^2 \mathcal{N}_{\phi_\chi}(s) dt d\sigma.$$

We fix $\varepsilon > 0$ and $\theta > 0$ such that for $s = \sigma + it$, $\Re(s) < \theta$ implies $\mathcal{N}_\phi(s) \leq \varepsilon \Re(s)$. We split the integral in two parts:

(1) On the one hand,

$$\int_{\mathbb{T}^\infty} \int_0^\theta \int_{-A}^{A+c_0} \left| f'_{n,\chi^{c_0}}(s) \right|^2 \mathcal{N}_{\phi_\chi}(s) dt d\sigma dm \leq \varepsilon \int_{\mathbb{T}^\infty} \int_{\mathbb{R}_+} \int_{-A}^{A+c_0} \left| f'_{n,\chi^{c_0}}(s) \right|^2 \Re(s) dt d\sigma dm.$$

Now, as in the proof of formula (3), this last quantity is dominated by $(2A + c_0)\varepsilon \|f_n\|_2^2$.

(2) On the other hand,

$$\begin{aligned}\int_{\mathbb{T}^\infty}\int_\theta^{+\infty}\int_{-A}^{A+c_0}&\left|f'_{n,\chi^{c_0}}(s)\right|^2\mathcal{N}_{\phi_\chi}(s)dtd\sigma dm\\ &\leq \int_{\mathbb{T}^\infty}\int_\theta^{+\infty}\int_{-A}^{A+c_0}\left|f'_{n,\chi^{c_0}}(s)\right|^2\frac{\Re(s)}{c_0}dtd\sigma dm\\ &\leq \frac{2A+c_0}{c_0}\int_\theta^{+\infty}\sigma\sum_{k\geq 1}|a_{n,k}|^2(\log^2 k)k^{-2\sigma}d\sigma,\end{aligned}$$

where we have written $f_n(s)=\sum_{k\geq 1}a_{n,k}k^{-s}$. We fix K large enough such that, for $k\geq K$,

$$\log^2 k\int_\theta^{+\infty}\sigma k^{-2\sigma}d\sigma\leq\varepsilon.$$

By setting

$$M=\max_k\ \log^2 k\int_\theta^{+\infty}\sigma k^{-2\sigma}d\sigma,$$

we obtain

$$\begin{aligned}\int_{\mathbb{T}^\infty}\int_\theta^{+\infty}\int_{-A}^{A+c_0}&\left|f'_{n,\chi^{c_0}}(s)\right|^2\mathcal{N}_{\phi_\chi}(s)dtd\sigma dm\\ &\leq\frac{2A+c_0}{c_0}\left(M\sum_{k=1}^K|a_{n,k}|^2+\varepsilon\right).\end{aligned}$$

It remains to observe that, for each $k=1,\dots,K$, we have $a_{n,k}\to 0$ as $n\to+\infty$, and the compactness of C_ϕ is proved. □

Corollary 1. *Let* $\phi:\mathbb{C}_+\to\mathbb{C}_+$, $\phi(s)=c_0s+c_1+\sum_{n\geq 2}c_nn^{-s}$. *Suppose that:*

(a) $\displaystyle\sum_{n\geq 2}|c_n|\log n\leq c_0$.

(b) $\Re\phi(s)/\Re s\xrightarrow{\Re s\to 0^+}+\infty$.

Then C_ϕ *is compact.*

Proof. Condition (a) ensures that $\Im\varphi$ is bounded on $\mathbb{C}_+$, and that ϕ is univalent. But in this case, if $w\in\phi(\mathbb{C}_+)$, then $\mathcal{N}_\phi(w)=\Re\big(\phi^{-1}(w)\big)$. Hence, condition (b) of the corollary implies condition (b) of the theorem. □

Remark. For other sufficient conditions for compactness, with $c_0=0$, we refer to [4].

Question. The condition "$\Im\varphi$ bounded on $\mathbb{C}_+$" seems to be just a technical one. Does the theorem remain true without this condition?

5. Necessary conditions

DEFINITION 3. For $w \in \mathbb{C}_+$ and $l \geq 1$ we define the partial reproducing kernel of order l in w by

$$K_{l,w}(s) = \prod_{j=1}^{l} \left(\sum_{n\geq 1} p_j^{-n(\overline{w}+s)} \right) = \sum_{\substack{n\geq 1 \\ P^+(n)\leq p_l}} n^{-\overline{w}} n^{-s},$$

where $P^+(n)$ denotes the greatest prime divisor of n.

These partial reproducing kernels are defined on $\mathbb{C}_+$ and not only on $\mathbb{C}_{1/2}$. Clearly, by Euler's identity we have

$$\|K_{l,w}\|_2 = \prod_{j=1}^{l} \left(\frac{1}{1 - p_j^{-2\Re(w)}} \right)^{1/2}.$$

$K_{l,w}$ reproduces partially $\mathcal{H}$: If $f(s) = \sum_1^{+\infty} a_n n^{-s} \in \mathcal{H}$, then

$$\langle f, K_{l,w} \rangle = \sum_{P^+(n)\leq p_l} a_n n^{-w}.$$

For certain composition operators C_ϕ, it is easy to compute $C_\phi^*(K_{l,w})$.

PROPOSITION 5. *Let $\phi(s) = c_0 s + \sum_{n=1}^{+\infty} c_n n^{-s}$, with $c_n = 0$ if $P^+(n) > l$.*
(a) *If $c_0 \neq 0$, then $C_\phi^*(K_{l,w}) = K_{l,\phi(w)}$.*
(b) *If $c_0 = 0$, then $C_\phi^*(K_{l,w}) = K_{\phi(w)}$.*

Proof. (a) If $c_0 \neq 0$ and $n \geq 1$, we compute $n^{-\phi(s)}$:

$$\begin{aligned}
n^{-\phi(s)} &= (n^{c_0})^{-s}\, n^{-\varphi(s)} \\
&= (n^{c_0})^{-s} \exp\left(- \sum_{\substack{k=1 \\ P^+(k)\leq p_l}}^{+\infty} c_k k^{-s} \log n \right) \\
&= (n^{c_0})^{-s} \prod_{\substack{k=1 \\ P^+(k)\leq p_l}}^{+\infty} \sum_{j=0}^{+\infty} \frac{(-c_k \log n)^j}{j!} \left(k^j\right)^{-s} \\
&= (n^{c_0})^{-s} \left(\sum_{k\geq 1} a_k k^{-s} \right),
\end{aligned}$$

where $a_k = 0$ if $P^+(k) > p_l$. (This formal computation of the Dirichlet series of $n^{-\phi(s)}$ is justified in [5, Section 3].) Therefore, if $P^+(n) > p_l$, the Dirichlet series $n^{-\phi(s)}=\sum_1^{+\infty}b_kk^{-s}$ satisfies $b_k=0$ for $P^+(k)<p_l$, and so

$$\langle n^{-s},C_\phi^*(K_{l,w})\rangle=\langle n^{-\phi(s)},K_{l,w}\rangle=0.$$

On the other hand, if $P^+(n)<p_l$, then the Dirichlet series $n^{-\phi(s)}=\sum_1^{+\infty}b_kk^{-s}$ satisfies $b_k=0$ for $P^+(k)>p_l$, and so

$$\langle n^{-s},C_\phi^*(K_{l,w})\rangle=\langle n^{-\phi(s)},K_{l,w}\rangle=\langle n^{-\phi(s)},K_w\rangle=n^{-\phi(w)}.$$

(b) If $c_0=0$, then for $n\geq 1$,

$$n^{-\phi(s)}=\sum_1^{+\infty}b_kk^{-s},$$

with $b_k=0$ if $P^+(k)>p_l$. This gives directly, for every $n\geq 1$,

$$\langle n^{-\phi(s)},K_{l,w}\rangle=\langle n^{-\phi(s)},K_w\rangle=n^{-\phi(w)},$$

and so $C_\phi^*(K_{l,w})=K_{\phi(w)}$. □

We deduce from these considerations the following result.

THEOREM 3. *Let l be an integer, and let C_ϕ, $\phi(s)=c_0s+\varphi(s)$, be a composition operator on $\mathcal{H}$ such that $c_n=0$ if $P^+(n)>p_l$. Suppose that C_ϕ is compact.*

(a) *If $c_0\geq 1$, then $\lim_{\Re(s)\to 0}\Re\phi(s)/\Re s=+\infty$.*
(b) *If $c_0=0$, then $\lim_{\Re(s)\to 0}\Re(s)^l\zeta(2\Re\phi(s))=0$.*

Proof. (a) Let (s_n) be a sequence in $\mathbb{C}_+$ with $\Re(s_n)\to 0$. We can always assume that $\Re\phi(s_n)\to 0$. As before, $K_{l,s_n}/\|K_{l,s_n}\|$ converges weakly to 0. The compactness of C_ϕ^* implies that

$$C_\phi^*\left(\frac{K_{l,s_n}}{\|K_{l,s_n}\|}\right)=\frac{K_{l,\phi(s_n)}}{\|K_{l,s_n}\|}\text{ converges to 0,}$$

or equivalently

$$\prod_{j=1}^{l}\left(\frac{1-p_j^{-2\Re s_n}}{1-p_j^{-2\Re\phi(s_n)}}\right)\xrightarrow{n\to+\infty}0.$$

Now,

$$1-p_j^{-2\Re\phi(s_n)}\sim_{+\infty}2\Re\phi(s_n)\log p_j,$$

where $u_n\sim_{+\infty}v_n$ means that $u_n/v_n\to 1$ if n tends to $+\infty$. Similarly,

$$1-p_j^{-2\Re(s_n)}\sim_{+\infty}2\Re(s_n)\log p_j.$$

Finally, we obtain

$$\frac{\Re\phi(s_n)}{\Re(s_n)}\xrightarrow{n\to+\infty}+\infty,$$

which is the result.

(b) In this case, since $C_\phi^*(K_{l,s_n}) = K_{\phi(s_n)}$, the same reasoning shows that

$$\prod_{j=1}^{l}\left(1 - p_j^{-2\Re(s_n)}\right)\zeta\left(2\Re\phi(s_n)\right) \xrightarrow{n\to+\infty} 0.$$

Using $1 - p_j^{-2\Re(s_n)} \sim_{+\infty} 2\Re(s_n)\log p_j$ gives the result. $\square$

Corollary 2. *Let $\phi(s) = c_0 s + c_1 + \sum_{j=1}^{d} c_{q_j} q_j^{-s}$, $c_0 \neq 0$, where (q_j) are multiplicatively independent integers, and $c_{q_j} \neq 0$. Then the following are equivalent:*

(i) $\Re(c_1) > |c_{q_1}| + \cdots + |c_{q_d}|$.
(ii) $\phi(\mathbb{C}_+) \subset \mathbb{C}_\varepsilon$, *where* $\varepsilon > 0$.
(iii) C_ϕ *is compact.*

Proof. Observe that, by Kronecker's theorem, if we want C_ϕ to be bounded on $\mathcal{H}$ (equivalently, $\phi(\mathbb{C}_+) \subset \mathbb{C}_+$), we have to assume $\Re(c_1) \geq |c_{q_1}|+\cdots+|c_{q_d}|$. By the same theorem, assertions (i) and (ii) are equivalent, and as mentioned in the introduction (or by an application of Theorem 2), (ii) implies (iii). Therefore it remains to prove that (iii) implies (i).

If $\Re(c_1) = |c_{q_1}|+\cdots+|c_{q_d}|$, there exists a sequence (s_n) in $\mathbb{C}_+$ with $\Re(s_n) = 1/n$ and

$$\Re\left(\sum_{1}^{d} c_{q_j} q_j^{-s_n}\right) \leq -\sum_{1}^{d}\left|c_{q_j}\right| q_j^{-1/n} + \frac{1}{n^2}.$$

Then,

$$\begin{aligned}\Re\left(\phi(s_n)\right) &\leq \frac{c_0}{n} + \Re(c_1) - \sum_{1}^{q}\left|c_{q_j}\right| q_j^{-1/n} + \frac{1}{n^2}\\ &= \frac{c_0}{n} + \Re(c_1) - \sum_{1}^{q}\left|c_{q_j}\right| + \frac{\sum_1^d \left|c_{q_j}\right|\log q_j}{n} + o\left(\frac{1}{n}\right)\\ &= \frac{c_0 + \sum_1^d \left|c_{q_j}\right|\log q_j}{n} + o\left(\frac{1}{n}\right).\end{aligned}$$

In particular, $\Re\phi(s_n)/\Re(s_n)$ cannot converge to $+\infty$, so C_ϕ is not compact. $\square$

Corollary 3. *Let $\phi(s) = c_1 + c_2 2^{-s}$, with $\Re(c_1) \geq |c_2| + 1/2$. Then the following are equivalent:*

(i) $\Re(c_1) > |c_2| + 1/2$.
(ii) $\phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2+\varepsilon}$, *where* $\varepsilon > 0$.
(iii) C_ϕ *is compact.*

Proof. Here, too, it suffices to prove that (iii) implies (i). Without loss of generality, we can assume that $c_1 \in \mathbb{R}$. If $c_1 = |c_2| + 1/2$, there exists a sequence (s_n) in $\mathbb{C}_+$ such that $\Re(s_n) = 1/n$, and $c_2 2^{-s_n} = -|c_2| 2^{-1/n}$. Now,

$$\begin{aligned}\zeta\left(2\Re\phi(s_n)\right) &= \zeta\left(1 + |c_2|(1 - 2^{-1/n})\right)\\ &\sim_{+\infty} \frac{1}{|c_2|(1 - 2^{-1/n})}\\ &\sim_{+\infty} Kn.\end{aligned}$$

In particular, $\Re(s_n)\zeta(2\Re\phi(s_n))$ does not converge to 0. □

Remark. This result was also proved in [4], using different methods.

Remark. For composition operators C_ϕ with $\phi(s) = c_0 s + c_1 + \sum_{j=1}^{d} c_{q_j} q_j^{-s}$, where (q_j) are multiplicatively independent integers, the situation is quite different according to whether $c_0 = 0$ or $c_0 \neq 0$:

If $c_0 \neq 0$, then for C_ϕ to be bounded it is necessary and sufficient that $\Re(c_1) \geq |c_{q_1}| + \cdots + |c_{q_d}|$. C_ϕ is compact if and only if this inequality is strict.

If $c_0 = 0$, then the boundedness of C_ϕ is characterized by the condition $\Re(c_1) \geq \frac{1}{2} + |c_{q_1}| + \cdots + |c_{q_d}|$. The strict inequality is still necessary and sufficient for C_ϕ to be compact if $d = 1$. On the other hand, for $d \geq 2$ it was proved in [4] that C_ϕ is always compact (and even Hilbert-Schmidt if $d \geq 3$).

6. Spectrum

If T is an operator on a Hilbert space H, we denote by $\mathrm{Sp}(T)$ its *spectrum*:

$$\mathrm{Sp}(T) = \{\lambda \in \mathbb{C} :\ T - \lambda Id_H \text{ is not invertible}\}.$$

Even on $H^2(\mathbb{D})$, our understanding of the spectra of composition operators is far from being complete. If a power of the operator is compact, the situation is much easier, since determining the spectrum becomes equivalent to finding the eigenvalues. In [2], J. Caughran and H. Schwartz gave a complete description of the spectra of compact composition operators on $H^2(\mathbb{D})$. In this section, we will do the same for $\mathcal{H}$.

We recall the following lemma (see [7, p. 270]), which allows us to reduce the eigenvalue problem to a finite dimensional problem:

Lemma 5. *Suppose H is a Hilbert space with $H = K \oplus L$, where K is finite dimensional and C is a bounded operator on H that leaves K or L invariant. If the operator C has the matrix representation*

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \text{ or } \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

with respect to this decomposition, then $\mathrm{Sp}(C) = \mathrm{Sp}(X) \cup \mathrm{Sp}(Z)$.

Here, too, we will distinguish between the two cases $c_0 = 0$ and $c_0 \neq 0$. Observe that, if $c_0 = 0$, then $\phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$ and $\phi(+\infty) \neq +\infty$. In particular, ϕ admits a fixed point in $\mathbb{C}_{1/2}$.

THEOREM 4. *Let C_ϕ be a composition operator on $\mathcal{H}$, $\phi(s) = c_0 s + \varphi(s)$. Suppose that there exists $N \geq 1$ such that C_ϕ^N is compact.*

(a) *If $c_0 = 0$, then $\mathrm{Sp}(C_\phi) = \{0, 1\} \cup \{[\phi'(\alpha)]^k : k \geq 1\}$, where α is the fixed point of ϕ in $\mathbb{C}_{1/2}$.*
(b) *If $c_0 = 1$, then $\mathrm{Sp}(C_\phi) = \{0, 1\} \cup \{k^{-c_1} : \ k \geq 2\}$.*
(c) *If $c_0 > 1$, then $\mathrm{Sp}(C_\phi) = \{0, 1\}$.*

REMARK. If $c_0 = 0$, Lemma 4 implies that $\phi \circ \phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2+\varepsilon}$ ($\varepsilon > 0$). Therefore $(C_\phi)^2$ is compact and Theorem 4 gives the spectrum of all composition operators in this setting.

Proof. The proof uses ideas from the corresponding theorem in Section 7.4 of [7]. We begin by proving (a). We denote by K_α the reproducing kernel at $\alpha \in \mathbb{C}_{1/2}$ and by $K_\alpha^{(m)}$ its m-th derivative. If $f(s) = \sum_1^{+\infty} a_n n^{-s}$, then

$$\langle f, K_\alpha^{(m)} \rangle = \sum_{n \geq 1} (-1)^m (\log n)^m a_n n^{-\alpha} = f^{(m)}(\alpha).$$

Let us set $\mathcal{K}_m = \mathrm{span}(K_\alpha, \dots, K_\alpha^{(m)})$. $\mathcal{K}_m$ is invariant under C_ϕ^*. Indeed,

$$\langle f, C_\phi^*(K_\alpha^{(m)}) \rangle = (f \circ \phi)^{(m)}(\alpha).$$

Now,

$$(f \circ \phi)^{(m)}(\alpha) = [\phi'(\alpha)]^m f^{(m)} \circ \phi(\alpha) + \lambda_1 f^{(m-1)} \circ \phi(\alpha) + \cdots + \lambda_{m-1} f' \circ \phi(\alpha),$$

and so

$$C_\phi^*\left(K_\alpha^{(m)}\right) = [\phi'(\alpha)]^m K_\alpha^{(m)} + \lambda_1 K_\alpha^{(m-1)} + \cdots + \lambda_{m-1} K'_\alpha.$$

Let X_m be the restriction of C_ϕ^* to $\mathcal{K}_m$. The matrix of X_m in the basis $(K_\alpha, \dots, K_\alpha^{(m)})$ is upper-triangular, and the coefficients on the diagonal are 1, $\left[\phi'(\alpha)\right]^k$, $1 \leq k \leq m$. These numbers are in the spectrum of X_m, and therefore also in the spectrum of C_ϕ^*.

Now, for each m, let $\mathcal{L}_m$be the orthogonal complement of $\mathcal{K}_m$ in $\mathcal{H}$. The block matrix for C_ϕ^* is then

$$C_\phi^* = \begin{pmatrix} X_m & Y_m \\ 0 & Z_m \end{pmatrix}.$$

By the lemma, $\mathrm{Sp}(C_\phi^*) = \mathrm{Sp}(X_m) \cup \mathrm{Sp}(Z_m)$, and it is sufficient to prove that the spectral radius of Z_m tends to 0. Suppose that this is not the case. Like C_ϕ, Z_m has compact square, and its spectrum, except for the value 0, reduces to eigenvalues. By passing to subsequences, we obtain a sequence of scalar numbers (λ_m), with $|\lambda_m|\geq\varepsilon>0$, and a norm 1 sequence $(z_m)\in\mathcal{L}_m$, such that $Z_m z_m=\lambda_m z_m$. Since $\overline{\bigcup_m\mathcal{K}_m}=\mathcal{H}^2$ and $z_m\perp\mathcal{K}_m$, (z_m) converges weakly to 0. Now, $C_\phi^*(z_m)=Y_m z_m+Z_m z_m=Y_m z_m+\lambda_m z_m$, and

$$\left(C_\phi^*\right)^2(z_m)=\underbrace{X_m Y_m z_m+\lambda_m Y_m z_m}_{\in\mathcal{K}_m}+\underbrace{\lambda_m^2 z_m}_{\in\mathcal{L}_m}.$$

In particular, $\|(C_\phi^*)^2(z_m)\|$ does not converge to 0, which contradicts the compactness of $(C_\phi^*)^2$. □

Assertions (b) and (c) of Theorem 4 are direct consequences of the following propositions, where $\mathrm{Sp}_p(C_\phi)$ denotes the point spectrum of C_ϕ, i.e.,

$$\mathrm{Sp}_p(C_\phi)=\{\lambda\in\mathbb{C}:\ C_\phi-\lambda Id_{\mathcal{H}}\text{ is not one-to-one}\}.$$

Proposition 6. *Let C_ϕ be a composition operator on $\mathcal{H}$, with $c_0\geq 1$. Then:*

$$\mathrm{Sp}_p(C_\phi)=\{1\}\ \textit{if}\ c_0>1,$$
$$\mathrm{Sp}_p(C_\phi)\subset\{1\}\cup\left\{k^{-c_1}:\ k\geq 2\right\}\ \textit{if}\ c_0=1.$$

Proof. Let f be an eigenvector of C_ϕ for λ, so that $f\circ\phi(s)=\lambda f(s)$. We first take $s=+\infty$. Then we have either $\lambda=1$, which is in $\mathrm{Sp}_p(C_\phi)$ since any constant function is an eigenvector, or $\lambda\neq 1$, in which case $f(+\infty)=0$. Next, write $f(s)=\sum_{l>k}a_k k^{-s}$, with $l\geq 2$ and $a_l\neq 0$, and consider the coefficient of l^{-s} in $f\circ\phi(s)$. By [5], the Dirichlet series of $f\circ\phi$ can be obtained by expanding the product in the representation

$$f\circ\phi(s)=\sum_{k\geq l}a_k k^{-c_0 s}k^{-c_1}\prod_{n=2}^{+\infty}\left(1+\sum_{j=1}^{+\infty}\frac{(-c_n\log k)^j}{j!}n^{-js}\right).$$

In particular, if $c_0>1$, there is no term involving l^{-s}, and $\mathrm{Sp}_p(C_\phi)=\{1\}$. If $c_0=1$, the coefficient of l^{-s} is $a_l l^{-c_1}$. Hence, $\lambda a_l=a_l l^{-c_1}$, and $\lambda=l^{-c_1}$. □

Conversely, we have:

Proposition 7. *Let C_ϕ be a composition operator on $\mathcal{H}$, $c_0=1$. Then*

$$\{1\}\cup\left\{k^{-c_1}:\ k\geq 2\right\}\subset\mathrm{Sp}(C_\phi).$$

Proof. We set $\mathcal{K}_m=\{1,2^{-s},\dots,m^{-s}\}$ and $\mathcal{L}_m=\mathcal{K}_m^\perp$. $\mathcal{L}_m$ is invariant under C_ϕ, and we have the block decomposition

$$C_\phi=\begin{pmatrix}X&0\\Y&Z\end{pmatrix},$$

which ensures that $\mathrm{Sp}(X) \subset \mathrm{Sp}(C_\phi)$. Now,

$$C_\phi\left(k^{-s}\right) = k^{-s}k^{-c_1} + \sum_{j>k} a_j j^{-s}.$$

In particular, the matrix of X is lower-triangular and therefore $\mathrm{Sp}(X) = \{1, 2^{-c_1}, \dots, m^{-c_1}\}$. □

7. Other spaces

In [8], J. McCarthy introduced new weighted Hilbert spaces of Dirichlet series

$$\mathcal{H}_\alpha = \left\{ f(s) = \sum_1^{+\infty} a_n n^{-s} : \ \|f\|_{\alpha,2}^2 = |a_1|^2 + \sum_{n\geq 2} |a_n|^2 (\log n)^\alpha < +\infty \right\},$$

where $\alpha \in \mathbb{R}$. For $\alpha = 0$, this is again $\mathcal{H}$, whereas $\mathcal{H}_{-1}$ and $\mathcal{H}_1$ correspond, respectively, to the Bergman space and to the Dirichlet space in the setting of the disk. The methods used in the previous section can be generalized to those spaces. More precisely, we have the following result.

THEOREM 5. *Fix $\alpha < 0$ and $\phi : \mathbb{C}_+ \to \mathbb{C}_+$, $\phi(s) = c_0 s + \varphi(s)$, $c_0 \geq 1$. Suppose that:*

(a) *$\Im\varphi$ is bounded on $\mathbb{C}_+$.*

(b) $\Re\phi(s)/\Re(s) \xrightarrow{\Re(s)\to 0} +\infty$.

Then C_ϕ is a compact composition operator on $\mathcal{H}_\alpha$.

REMARK. It must be pointed out that in this theorem we do not mention any counting functions. The same phenomenon occurs in the disk for Bergman spaces (see [9]).

Proof. First, we give an area integral formula like (3) for the norm of an element of $\mathcal{H}_\alpha$. Recalling that

$$\int_0^{+\infty} n^{-2\sigma}\sigma^{\beta-1} d\sigma = \frac{\Gamma(\beta)}{(\log n)^\beta 2^\beta},$$

we then obtain

$$\|f\|_{\alpha,2}^2 = |a_1|^2 + \frac{2^{-\alpha+2}}{\Gamma(-\alpha+2)} \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} \sigma^{-\alpha+1}|f_\chi'(\sigma+it)|^2 d\mu(t) d\sigma dm(\chi),$$

where μ still denotes a probability measure on $\mathbb{R}$. We introduce a new counting function $\mathcal{N}_{\phi,\alpha}$ by letting

$$\mathcal{N}_{\phi,\alpha}(s) = \begin{cases} \sum_{w\in\phi^{-1}(s)} \Re^{1-\alpha}(s) & \text{if } w \in \phi(\mathbb{C}_+), \\ 0 & \text{otherwise.} \end{cases}$$

By copying word for word the proofs of Propositions 3 and 4 and Theorem 2, and by using the classical inequalities for counting functions on the disk (see [9] or [10, Exercises 12–15]), one easily sees that the conditions (a) and

(b′) $\mathcal{N}_{\phi,\alpha}(s) = o\left(\Re^{1-\alpha}(s)\right)$ if $\Re(s) \to 0$

imply the compactness of C_ϕ on $\mathcal{H}_\alpha$. Thus, it remains to prove that, if $\alpha < 0$, (b′) is a consequence of condition (b) of the theorem. To see this, fix $\varepsilon > 0$ and $\theta > 0$ such that

$$\Re(w) < \theta \implies \Re(w) \leq \varepsilon \Re\phi(w).$$

Let $s \in \mathbb{C}_+$ with $\Re(s) < \theta$. If $s \notin \phi(\mathbb{C}_+)$, then $\mathcal{N}_{\phi,\alpha}(s) = 0$. Otherwise,

$$\begin{aligned} \sum_{w \in \phi^{-1}(\{s\})} \Re^{1-\alpha}(w) &\leq \varepsilon^{-\alpha} \Re^{-\alpha}(s) \sum_{w \in \phi^{-1}(\{s\})} \Re(w) \\ &\leq \varepsilon^{-\alpha} \Re^{-\alpha}(s) \mathcal{N}_\phi(s) \\ &\leq \frac{\varepsilon^{-\alpha}}{c_0} \Re^{1-\alpha}(s). \end{aligned}$$

□

The necessary conditions given in Section 5 remain valid. The partial reproducing kernels are now given by

$$K_{l,w}(s) = 1 + \sum_{\substack{n \geq 2 \\ P^+(n) \leq p_l}} \frac{1}{(\log n)^\alpha} n^{-\bar{w}-s}.$$

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