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ON THE DYNAMICS OF $e^{2\pi i\theta}\sin(z)$

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ABSTRACT. We prove that for any bounded type irrational number $0 < \theta < 1$ the boundary of the Siegel disk of $e^{2\pi i\theta}\sin(z)$ is a quasi-circle which passes through exactly two critical points $\pi/2$ and $-\pi/2$.

1. Introduction

In this paper, we consider the Siegel entire function $f_\theta(z) = e^{2\pi i\theta}\sin(z)$, where $0 < \theta < 1$ is a *bounded type* irrational number. Here an irrational number $0 < \theta < 1$ is said to be of *bounded type* if $\sup\{a_n\} < \infty$, where $[a_1, \dots, a_n \dots]$ is its continued fraction. Clearly, f_θ has a Siegel disk centered at the origin which has rotation number θ. This function was studied in [10], where it was shown that the boundary of the Siegel disk of f_θ must contain a critical point for every *diophantine* number θ. For *bounded type* rotation numbers θ, the existence of the critical points on the boundary of the Siegel disk also follows from a recent result of Graczyk and Swiatek [5]. There are still two unresolved questions:

(1) Is the boundary of the Siegel disk of f_θ a Jordan curve?
(2) Which critical point lies on the boundary of the Siegel disk?

In this paper, we will answer these two questions under the assumption that θ is of *bounded type*. We prove the following result:

MAIN THEOREM. *Let $0 < \theta < 1$ be an irrational number of bounded type. Then the boundary of the Siegel disk of the entire function $f_\theta(z) = e^{2\pi i\theta}\sin(z)$ is a quasi-circle which passes through exactly two critical points $\pi/2$ and $-\pi/2$.*

The main idea of our proof can be sketched as follows. We consider the map $g(z) = (\sin z)/2$. The map $g(z)$ has an attracting fixed point at the origin. Let Ω be the maximal linearizable domain of $g(z)$ which is centered at the origin. It will be proved that Ω is a bounded and simply connected domain, and, moreover, that the boundary $\partial\Omega$ is a quasi-circle which passes through

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exactly two critical points $\pi/2$ and $-\pi/2$. Let $\xi = g(-\pi/2)$ and Ω' be the unbounded component of $\widehat{\mathbb{C}} - g(\partial\Omega)$. Then for each $\eta \in \partial\Omega$, by the Riemann mapping theorem, there exists a unique conformal map $\mu_\eta : \Omega' \to \widehat{\mathbb{C}} - \overline{\Omega}$ such that $\mu_\eta(\xi) = \eta$ and $\mu_\eta(\infty) = \infty$ (see Figure 1). Let $F_\eta = \mu_\eta \circ g$. By Proposition 11.1.9 [6], it follows that there exists a unique $\eta \in \partial\Omega$ such that the map $F_\eta|_{\partial\Omega} : \partial\Omega \to \partial\Omega$ is a topological circle homeomorphism of rotation number θ (Lemma 5). Using the map F_η, we will construct a model map $\tilde{f}_\theta$ that, when restricted to Ω, is quasiconformally conjugate to the rigid rotation R_θ on the unit disk, and, moreover, satisfies $\tilde{f}_\theta(z+\pi) = -\tilde{f}_\theta(z)$. The proof is then completed by showing that the map f_θ is quasiconformlly conjugate to $\tilde{f}_\theta$.

We would like to mention that A. Cheritat had provided a similar construction in his Ph.D. thesis, by which one can construct Blascke fractions that serve as models for a certain class of maps with Siegel disks. The reader may refer to [4] for the details of his construction. The new feature of our construction in this case is that the model map $\tilde{f}_\theta$ preserves the periodicity, which plays a crucial role in the whole proof, but which does not hold for the Blaschke model constructed in [4].

2. Proof of the Main Theorem

Notations and definitions. We use Δ, $\mathbb{C}$, $\widehat{\mathbb{C}}$ to denote the unit disk, complex plane, and the Riemann sphere, respectively. An irrational number $0 < \theta < 1$ is said to be of *bounded type* if $\sup\{a_n\} < \infty$, where $\theta = [a_1, a_2, \dots,]$ is its continued fraction. For any entire function $f(z)$ we say that β is an *asymptotic value* of f if there exists a continuous curve $\gamma(t) \subset \mathbb{C}$, $0 \le t < \infty$, such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to \beta$ as $t \to \infty$.

Let $g(z) = (\sin z)/2$. Then $g(0) = 0$ and $g'(0) = 1/2$. It follows that g has an attracting fixed point at the origin. Let Ω be the maximal linearizable domain of g at the origin and $\phi : \Omega \to \Delta$ be a holomorphic homeomorphism which conjugates g to the linear map $L_{1/2} : z \to z/2$ on Δ. We may assume that $\phi'(0) > 0$. It follows that ϕ must be unique.

Lemma 1. $\sin z$ *does not have any finite asymptotic value.*

Proof. Assume β is a finite *asymptotic value* of $\sin z$. Then, by definition, there exists a continuous curve $\gamma(t) \subset \mathbb{C}$, $0 \le t < \infty$, such that $\gamma(t) \to \infty$ as $t \to \infty$ and $\sin(\gamma(t)) \to \beta$ as $t \to \infty$. Let $\gamma(t) = x(t) + iy(t)$. Since $|\sin(z)| \to \infty$ as $|\Im(z)| \to \infty$, it follows that $|y(t)| \le M$ for some constant $M > 0$. This implies that $x(t) \to \infty$. By a simple calculation, we have

$$\sin(\gamma(t)) = \sin x(t) \left[\frac{e^{y(t)} + e^{-y(t)}}{2}\right] + i \cos x(t) \left[\frac{e^{y(t)} - e^{-y(t)}}{2}\right].$$

Since $x(t) \to \infty$, there is a sequence $t_k \to \infty$ such that $x(t_k) = k\pi$, and it follows that $\Re(\beta) = 0$. On the other hand, there is a sequence $t_{k'} \to \infty$ such that $x(t_{k'}) = 2k'\pi + \pi/2$. Since

$$\Re \sin(\gamma(t_{k'})) = \frac{e^{y(t_{k'})} + e^{-y(t_{k'})}}{2} > 1,$$

it follows that $\Re\beta \geq 1$. This is a contradiction. □

LEMMA 2. *The domain Ω is bounded and symmetric about the origin. Moreover, $\partial\Omega$ contains exactly two critical points of g, $\pi/2$ and $-\pi/2$.*

Proof. The fact that Ω is bounded follows from Lemma 1, since otherwise $g(\partial\Omega)$ would contain a finite *asymptotic value* of g and this would imply that $\sin z$ has a finite *asymptotic value*, a contradiction. By the uniqueness of the linearization map of g near the origin and the fact that $g(-z) = -g(z)$, it follows that Ω is symmetric about the origin.

Now let us prove the second assertion of the lemma. Let $t(z) = \overline{\phi(\overline{z})}$ be defined in a small neighborhood of the origin. By the assumption that $\phi'(0) > 0$ we have $t'(0) = \phi'(0)$. Since in a small neighborhood of the origin,

$$t^{-1} \circ L_{1/2} \circ t(z) = g(z),$$

it follows that $t(z) = \phi(z)$. This implies that the restriction of ϕ to the real line is a real function, and so is ϕ^{-1}. Since Ω is bounded and g has no finite *asymptotic value*, there must be at least one critical point of g on $\partial\Omega$ (Theorem 2.4.1 in [9]). Let $\pi/2 + k\pi$ be a critical point of g on $\partial\Omega$, with k being some integer. Since $\phi^{-1}((-1,1)) \subset \mathbb{R}$, it follows that $[0, \pi/2 + k\pi) \subset \Omega$. Since g is univalent on Ω, it follows that $k = 0$ or $k = -1$. This implies that $\pi/2 \in \partial\Omega$ or $-\pi/2 \in \partial\Omega$. Since Ω is symmetric, it follows that $\partial\Omega$ contains both $\pi/2$ and $-\pi/2$, and, moreover, that these are the only critical points of g on $\partial\Omega$. This proves Lemma 2. □

Let $c_1 = -\pi/2$ and $c_2 = \pi/2$. Let $\xi = g(c_1) = -1/2 \in g(\partial\Omega)$.

Let $\gamma \subset \mathbb{C}$ be an open curve segment. We say that γ is *real-analytic* if there exists a domain D such that $\gamma \subset D$ and a univalent map $h : D \to U$ such that $h(\gamma) \subset \mathbb{R}$. Now let $C_{1/2} = \{z \mid |z| = 1/2\}$ and

$$\gamma' = \partial g(\Omega) = \phi^{-1}(C_{1/2}).$$

It follows that any open subarc of $\gamma' = g(\partial\Omega)$ is *real-analytic*.

LEMMA 3. *$\partial\Omega$ is a quasi-circle.*

Proof. In fact, $\partial\Omega$ is *real-analytic* everywhere except at the two critical points $\pi/2$ and $-\pi/2$, where $\partial\Omega$ has right angles up to a conformal coordinate transformation. The lemma then follows from Theorem 8.7 of [8]. □

Let Ω' be the unbounded component of $\widehat{\mathbb{C}} \setminus \gamma'$. Recall that $\xi = g(c_1) \in \gamma'$. By the Riemann mapping theorem, it follows that for each $\eta \in \partial\Omega$ there exists a unique conformal map $\mu_\eta : \Omega' \to \widehat{\mathbb{C}} \setminus \overline{\Omega}$ such that $\mu_\eta(\infty) = \infty$ and $\mu_\eta(\xi) = \eta$.

Lemma 4. *μ_η is odd.*

Proof. Since $g(z)$ is odd, it follows from Lemma 1 that both Ω' and $\widehat{\mathbb{C}} \setminus \overline{\Omega}$ are symmetric about the origin. Let $r(z) = -\mu_\eta(-z)$. Then $r : \Omega' \to \widehat{\mathbb{C}} \setminus \overline{\Omega}$ is a conformal isomorphism and $r'(\infty) = \mu_\eta'(\infty)$. It follows that $r(z) = \mu_\eta(z)$. This proves Lemma 4. □

Note that for each $\eta \in \partial\Omega$, the restriction of $F_\eta = \mu_\eta \circ g$ to $\partial\Omega$ is a homeomorphism. Since $\{F_\eta\}_{\eta \in \partial\Omega}$ is a continuous and monotone family of topological circle homeomorphisms as η varies on $\partial\Omega$, by Proposition 11.1.9 [6] we have:

Lemma 5. *There exists a unique $\eta \in \partial\Omega$ such that the rotation number of $F_\eta : \partial\Omega \to \partial\Omega$ is θ.*

The following lemma is a generalized version of the Schwarz symmetry principle (see [1]):

Lemma 6. *Let U be a domain such that $\gamma \subset \partial U$ is an open and real-analytic curve segment. Suppose f is a holomorphic function defined on U such that f can be continuously extended to γ and $f(\gamma)$ is a real-analytic curve segment. Then f can be holomorphically continued to a larger domain which contains γ in its interior.*

Let $\psi : \widehat{\mathbb{C}} \setminus \Delta \to \widehat{\mathbb{C}} \setminus \Omega$ be the Riemann map such that $\psi(\infty) = \infty$ and $\psi(1) = c_1$. Using the same argument as in the proof of Lemma 4, we obtain:

Lemma 7. *ψ is odd.*

From Lemma 7 we get that $\psi(-1) = c_2$. The following lemma plays a key role in the proof of Theorem 1:

Lemma 8. *The circle homeomorphism $f = \psi^{-1} \circ F_\eta \circ \psi : \partial\Delta \to \partial\Delta$ can be analytically extended to an open neighborhood of $\partial\Delta$ such that f has two double critical points at 1 and -1.*

Proof. Take $z \in \partial\Delta$. There are two cases.

In the first case, $z \notin \{1, -1\}$. Then f is holomorphic in a half neighborhood N_1' of z which is attached to the unit circle from the outside. We can take N_1' small enough such that f maps N_1' homeomorphically to a half neighborhood N_2' of $f(z)$ which is also attached to the unit circle from the outside. By the Schwarz reflection lemma, f can be holomorphically extended to an open

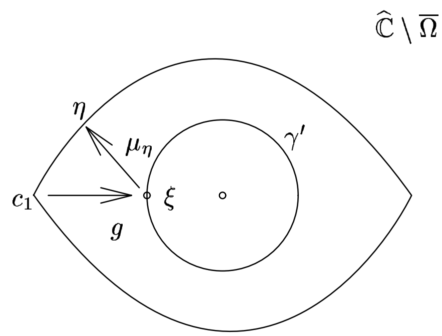

FIGURE 1. The construction of $F_\eta = \mu_\eta \circ g : \partial\Omega \to \partial\Omega$

neighborhood N_1 of z such that f maps N_1 homeomorphically to an open neighborhood N_2 of $f(z)$. This proves Lemma 8 in the first case.

In the second case, we have $z = 1$ or $z = -1$. Say $z = 1$; the case for $z = -1$ can be proved by the same argument. Write $f = (\psi^{-1} \circ \mu_\eta) \circ (g \circ \psi)$. Take a small half neighborhood N_1' of 1 as in the first case. Note that if N_1' is small enough, the boundary segment of N_1' which lies on the unit circle is mapped by $g \circ \psi$ to a *real-analytic* curve segment on γ'. Applying Lemma 6 to $g \circ \psi$, we see that $g \circ \psi$ can be holomorphically extended to an open neighborhood N_1 of 1 such that $g \circ \psi$ maps N_1 $3:1$ to an open neighborhood N_2 of $\xi = (g \circ \psi)(1)$. We may take N_1 small enough so that the following holomorphic continuation is valid. Let $N_2' \subset \Omega'$ be the half neighborhood of N_2. Note that the boundary segment of N_2' which lies on γ' is *real-analytic* and is mapped by $\psi^{-1} \circ \mu_\eta$ to an Euclidean arc segment, so by Lemma 6 again $\psi^{-1} \circ \mu_\eta$ can be holomorphically continued to N_2 and maps N_2 homeomorphically to some neighborhood of $f(1) = \psi^{-1} \circ \mu_\eta(\xi)$. This proves the second case and Lemma 8 follows. □

By Lemma 8 we know that f is a real-analytic critical circle homeomorphism with rotation number θ of *bounded type*. We now apply the Herman-Swiatek quasisymmetric linearization theorem to f (see [7], [11]).

LEMMA 9. *Let $f : \partial\Delta \to \partial\Delta$ be a real-analytic critical circle homeomorphism of rotation number θ. Then f is quasisymmetrically conjugate to the rigid rotation R_θ if and only if θ is of bounded type.*

It follows that $f = \psi^{-1} \circ F_\eta \circ \psi : \partial\Delta \to \partial\Delta$ is quasi-symmetrically conjugate to the rigid rotation R_θ. Let $h : \partial\Delta \to \partial\Delta$ be the quasi-symmetric homeomorphism such that $h(1) = 1$, and $f = h \circ R_\theta \circ h^{-1}$. Note that h is unique.

LEMMA 10. *h is odd.*

Proof. First let us show that $h(-1) = -1$. Let $U(N)$ be the number of points in $\{f^k(1) \mid k = 1, \dots, N\}$ which lie in the upper half circle. Let $L(N)$ be the number of the points in $\{f^k(-1) \mid k = 1, \dots, N\}$ which lie in the lower half circle. Since f is odd, it follows that $U(N) = L(N)$. Since the angle length of the image of the upper half circle under h is equal to the limit of $2\pi U(N)/N$ as $N \to \infty$, and the angle length of the image of the lower half circle under h is equal to the limit of $2\pi L(N)/N$ as $N \to \infty$, it follows that the angle length of the images of the upper half circle and the lower half circle under h are equal to each other. This implies that $h(-1) = -1$.

To show that h is odd, let $t(z) = -h(-z)$. We have $t(1) = 1 = h(1)$. Since

$$t \circ R_\theta \circ t^{-1}(z) = -f(-z) = f(z),$$

it follows that $t = h$. This proves Lemma 10. □

LEMMA 11.

(1) *μ_η can be extended to a quasiconformal homeomorphism $\tilde{\mu}_\eta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\tilde{\mu}_\eta(-z) = -\tilde{\mu}_\eta(z)$.*

(2) *ψ can be extended to a quasiconformal homeomorphism $\tilde{\psi} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\tilde{\psi}(-z) = -\tilde{\psi}(z)$.*

(3) *h can be extended to a quasiconformal homeomorphism $H : \Delta \to \Delta$ such that $H(-z) = -H(z)$.*

In particular, $\tilde{\psi}(0) = \tilde{\mu}_\eta(0) = H(0) = 0$.

Proof. We only prove (1); (2) and (3) can be proved by the same argument. Let Ω'' be the bounded component of $\widehat{\mathbb{C}} \setminus \gamma'$. Clearly, Ω'' is symmetric about the origin. Let $\phi_1 : \Delta \to \Omega''$, $\phi_2 : \Delta \to \Omega$ be the conformal isomorphisms such that $\phi_1(0) = \phi_2(0) = 0$. Since both of Ω and Ω'' are symmetric about the origin, it follows that both of ϕ_1 and ϕ_2 are odd. Then the map $s = \phi_2^{-1} \circ \mu_\eta \circ \phi_1 : \partial\Delta \to \partial\Delta$ is a homeomorphism. Since μ_η is odd, we have

$$s(-z) = -s(z).$$

By the Douady-Earle extension [3] the map s can be quasiconformally extended to a homeomorphism $\tilde{s} : \Delta \to \Delta$ such that $\tilde{s}(-z) = -\tilde{s}(z)$. Now let $\tilde{\mu_\eta}(z) = \mu_\eta(z)$ for $z \in \widehat{\mathbb{C}} \setminus \Omega''$ and $\tilde{\mu_\eta}(z) = \phi_2 \circ \tilde{s} \circ \phi_1^{-1}(z)$ for $z \in \Omega''$. Clearly, $\tilde{\mu_\eta}$ is a desired extension. □

Let $\Omega_k = \{z + k\pi \mid z \in \Omega\}$ for $k \in \mathbb{Z}$. Note that $\Omega_0 = \Omega$.

LEMMA 12. *The sets $\Omega_k, k \in \mathbb{Z}$, are disjoint.*

Proof. Assume this is not true. Then $\Omega_0 \cap \Omega_l \neq \emptyset$ for some $l \in \mathbb{Z}$. Take $x \in \Omega_0 \cap \Omega_l$. There are two cases. In the first case, l is even. It follows that $g(x) = g(x - l\pi)$. Since $x, x - l\pi \in \Omega_0$, and g is univalent on Ω_0, we get a contradiction. In the second case, l is odd. Then $g(-x) = g(x - l\pi)$. Since Ω_0 is symmetric about the origin, it follows that $-x \in \Omega_0$. Since $x - l\pi \in \Omega_0$, this is again a contradiction to the fact that g is univalent on Ω_0. □

Define

$$\tilde{f}_\theta(z) = \begin{cases} (\tilde{\mu}_\eta \circ g)(z) & \text{for } z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} \Omega_k, \\ \tilde{\psi} \circ H \circ R_\theta \circ H^{-1} \circ \tilde{\psi}^{-1}(z - k\pi) & \text{for } z \in \Omega_k,\ k \text{ even}, \\ -\tilde{\psi} \circ H \circ R_\theta \circ H^{-1} \circ \tilde{\psi}^{-1}(z - k\pi) & \text{for } z \in \Omega_k,\ k \text{ odd}. \end{cases}$$

From the definition we obtain:

Lemma 13. *$\tilde{f}_\theta$ is odd and $\tilde{f}_\theta(z + \pi) = -\tilde{f}_\theta(z)$. Moreover, the set of the zeros of $\tilde{f}_\theta$ is $\{k\pi \mid k \in \mathbb{Z}\}$.*

Now let us define a $\tilde{f}_\theta$−invariant complex structure ν as follows. For $z \in \Omega$, define ν to be the complex structure given by $(\tilde{\psi} \circ H)^*(\nu_0)$, where ν_0 is the standard complex structure. For $z \in \mathbb{C} \setminus \Omega$, there are two cases. In the first case, there is an $m \geq 1$ such that $x = \tilde{f}_\theta^{\,m}(z) \in \Omega$. In this case, we define $\nu(z)$ to be the pull-back of the complex structure at x by $\tilde{f}_\theta^{\,m}$. In the second case, the forward orbit of z under $\tilde{f}_\theta$ does not enter Ω. In this case, we define $\nu(z) = 0$. Clearly, the complex structure ν defined in this way is $\tilde{f}_\theta$−invariant with $\|\nu\|_\infty < 1$. By the measurable Riemann mapping theorem (see [2]), there exists a unique quasiconformal homeomorphism of the sphere $\omega : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes 0, 2π and ∞ and solves the Beltrami equation given by ν.

Since $\tilde{\psi} \circ H$ is odd, the infinitesimal ellipse field in Ω given by $\tilde{\psi} \circ H$ is symmetric about the origin. Since $\tilde{f}_\theta$ is odd and $\tilde{f}_\theta(z + \pi) = -\tilde{f}_\theta(z)$, we obtain:

Lemma 14. *$\nu(z) = \nu(-z)$ and $\nu(z + \pi) = \nu(z)$.*

Lemma 15. *$\omega(z + \pi) = \omega(z) + \pi$.*

Proof. Consider $r(z) = \omega(z+\pi)$. Let $\nu_r(z)$ be the Beltrami coefficient of r. It follows that $\nu_r(z) = \nu(z + \pi) = \nu(z)$. Since $r(\infty) = \omega(\infty) = \infty$, it follows that $r(z) = a\omega(z) + b$ for some constants a and b.

Let us first show that $a = 1$. To see this, note that for $|z|$ large enough, the annulus

$$A_z = \{w \mid \pi < |w - (z + \pi/2)| < |z|/2\}$$

separates $\{z, z + \pi\}$ and $\{0, \infty\}$, and $\mod(A_z) \to \infty$ as $z \to \infty$. It follows that the annulus $\omega(A_z)$ separates $\{\omega(z), \omega(z + \pi)\}$ and $\{0, \infty\}$. Moreover, since $\|\nu\|_\infty = K < 1$, we have $\mod(\omega(A_z)) \to \infty$ as $z \to \infty$. This implies $\omega(z+\pi)/\omega(z) \to 1$ as $z \to \infty$. It follows that $a = 1$. Since, by assumption, $\omega(2\pi) = 2\pi$ and $\omega(0) = 0$, we have $b = \pi$ and $\omega(z+\pi) = \omega(z) + \pi$. □

LEMMA 16. *ω is odd.*

Proof. Let $t(x) = -\omega(-x)$. Let ν_t be the Beltrami coefficient of t. From Lemma 14 it follows that $\nu_t = \nu$. Since $t(0) = \omega(0)$, it follows that $t(x) = a\omega(x)$. On the other hand, by Lemma 15 we have $\omega(-\pi) = -\pi$. It follows that $t(\pi) = -\omega(-\pi) = \pi = \omega(\pi)$. This implies that $a = 1$ and Lemma 16 follows. □

LEMMA 17. *$\omega(\pi/2) = \pi/2$, and $\omega(-\pi/2) = -\pi/2$.*

Proof. By Lemma 16 we have $\omega(-\pi/2) = -\omega(\pi/2)$. By Lemma 15, we have $\omega(\pi/2) = \omega(-\pi/2+\pi) = \omega(-\pi/2) + \pi$. It follows that $\omega(\pi/2) = \pi/2$ and $\omega(-\pi/2) = -\pi/2$. □

LEMMA 18. *$T = \omega \circ \tilde{f}_\theta \circ \omega^{-1}$ is odd and periodic of period 2π.*

Proof. From Lemmas 13 and 16 it follows that T is odd. From Lemmas 13 and 15 it follows that T is periodic of period 2π. □

LEMMA 19. *The set of the zeros of T is $\{k\pi \mid k \in \mathbb{Z}\}$.*

Proof. From the definition of T and Lemma 13 it follows that $T(z) = 0$ if and only if $\omega(z) \in \{k\pi \mid k \in \mathbb{Z}\}$. So the set of the zeros of T is $\{\omega^{-1}(k\pi) \mid k \in \mathbb{Z}\}$, and this set is equal to $\{k\pi \mid k \in \mathbb{Z}\}$ by Lemma 15. □

Proof of the Main Theorem. Applying Mori's theorem to $T(z)$ in a neighborhood of ∞, we get

$$|T(\omega(z))| \le Ce^{|\omega(z)|^K},$$

where C and K are some constants dependent only on $\|\nu\|_\infty$. It follows that T is of finite order. From Lemma 19 we have

$$T(z) = Ce^{P(z)} \sin z,$$

where $P(z)$ is some polynomial and C is some constant. Since $T(z)$ is periodic of period 2π, for each z there is an integer k such that

$$P(z+2\pi) - P(z) = 2k\pi i.$$

Since $T(z)$ varies continuously as z varies, there is a fixed k such that for all z,

$$P(z+2\pi) - P(z) = 2k\pi i.$$

This can only hold when $P(z) = ikz + b$ for some constant b. On the other hand, Lemma 18 implies that e^{ikz+b} is even. This can be true only when $k = 0$.

The above observations imply that $T(z) = C\sin z$. Since $T(z)$ has a Siegel disk centered at the origin which has rotation number θ, it follows that $C = \lambda = e^{2\pi i\theta}$. Therefore, $T(z) = \lambda \sin z$. It follows that the boundary of the Siegel disk of $\lambda \sin z$ is a quasi-circle, and by Lemmas 17 and 2 it passes through exactly two critical points $\pi/2$, and $-\pi/2$. This finishes the proof of the Main Theorem. □

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