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GRADIENT ESTIMATES FOR HARMONIC AND q-HARMONIC FUNCTIONS OF SYMMETRIC STABLE PROCESSES

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ABSTRACT. We give sharp gradient estimates for harmonic functions of rotation invariant stable Lévy processes near the boundary of Lipschitz domains. We also obtain sharp gradient estimates for harmonic functions of corresponding Feynman-Kac semigroups under some assumptions on the potential q.

1. Introduction

The purpose of this paper is to investigate the growth properties of gradients of α-harmonic and q-harmonic functions. Our main result on α-harmonic functions is the following (for definitions see Section 2).

THEOREM 1.1. *Let D be a Lipschitz domain in $\mathbb{R}^d$, $d \in \mathbb{N}$. Let $V \subset \mathbb{R}^d$ be open and let K be a compact subset of V. There exist constants $C = C(D,V,K,\alpha)$ and $\varepsilon = \varepsilon(D,V,K,\alpha)$ such that for every nonnegative function f which is bounded on V, α-harmonic in $D \cap V$, and vanishes in $D^c \cap V$,*

$$C\frac{f(x)}{\delta_D(x)} \leq |\nabla f(x)| \leq d\frac{f(x)}{\delta_D(x)}, \quad x \in K \cap D,\ \delta_D(x) < \varepsilon. \tag{1}$$

Our considerations are motivated by the natural question whether the classical results in this field (see [C], [CZ], [BP]) may be extended to nonlocal operators such as the fractional Laplacian $\Delta^{\alpha/2}$. Further motivation comes from an attempt to understand the role of fractional derivatives in the potential theory of $\Delta^{\alpha/2}$ on regular domains, a problem which may be related to gradient estimates.

To prove Theorem 1.1 we develop a straightforward technique based on Lemmas 4.4 and 4.5 below. It is noteworthy that the technique applies even

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more easily to the classical harmonic functions. In the present context there are additional complications resulting from the fact that the α-harmonic functions we consider need to be globally nonnegative, and the local maximum principle has only certain quantitative substitutes in the present theory (see the proof of Lemma 4.5).

The paper is organized as follows. In Section 2 we introduce the notation and collect some basic facts concerning α-stable symmetric processes and α-harmonic functions. In Section 3 we obtain the upper bound in (1) for an arbitrary open set. In Section 4 we restrict ourselves to Lipschitz domains and obtain the lower bound in (1).

We also give some applications of these estimates. In particular, in Section 5 we derive, for $\alpha > 1$, sharp gradient estimates for nonnegative q-harmonic functions under an appropriate growth condition on the potential function q of the Feynman-Kac semigroup.

2. Preliminaries

Let d be a natural number. By $|\cdot|$ we denote the Euclidean norm in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, $r > 0$ and $A \subset \mathbb{R}^d$ we set $B(x,r) = \{y \in \mathbb{R}^d \colon |x-y| < r\}$, $rA = \{ry \colon y \in A\}$, $\operatorname{diam} A = \sup\{|y-z| \colon y, z \in A\}$, $\operatorname{dist}(A,x) = \inf\{|x-y| \colon y \in A\}$, $\delta_A(x) = \operatorname{dist}(x, A^c)$. A set $D \subset \mathbb{R}^d$ is called a domain if it is open and nonempty. We say that a function f is nontrivial on D, if $f(x) \neq 0$ for some $x \in D$. We generally assume Borel measurability of the sets and functions we consider here.

The notation $c = c(\alpha, \beta, \ldots, \gamma)$ means that c is a constant depending *only* on $\alpha, \beta, \ldots, \gamma$. Constants are always (strictly) positive and finite.

In dimensions $d \geq 2$ a domain $D \subset \mathbb{R}^d$ is called *Lipschitz* if for every $Q \in \partial D$ there are a Lipschitz function $\Gamma_Q \colon \mathbb{R}^{d-1} \to \mathbb{R}$, an orthonormal coordinate system CS_Q, and a number $R_Q > 0$ such that if $y = (y_1, y_2, \ldots, y_{d-1}, y_d)$ in CS_Q coordinates, then

$$D \cap B(Q, R_Q) = \{y \colon y_d > \Gamma_Q(y_1, y_2, \ldots, y_{d-1})\} \cap B(Q, R_Q).$$

Note that we do not assume the connectedness nor the boundedness of D in this definition. We also define a Lipschitz domain on the real line ($d = 1$) as the union of any collection of open (possibly unbounded) intervals such that every bounded subset of $\mathbb{R}$ intersects with only a finite number of these intervals and no two intervals have a common endpoint.

For the rest of the paper, unless stated otherwise, α is a number in $(0, 2)$. By (X_t, P^x) we denote the *standard* (see [BG]) rotation invariant ("symmetric") α-stable, $\mathbb{R}^d$-valued Lévy process (i.e., homogeneous, with independent increments), with index of stability α and characteristic function

$$E^0 e^{i\xi X_t} = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d, \quad t \geq 0.$$

As usual, E^x denotes the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^d$. (X_t, P^x) is a Markov process with transition probabilities given by $P_t(x,A) = P^x(X_t \in A) = \int_A p(t;x,y)\,dy$ and is strong Markov with respect to the so-called "standard filtration" [BG].

For $A \subset \mathbb{R}^d$, we define the *first exit time* from A as $\tau_A = \inf\{t \geq 0\colon X_t \notin A\}$. Given $x \in \mathbb{R}^d$, the P^x distribution of X_{τ_A} is a subprobability measure on A^c (and a probability measure if A is bounded) called the *α-harmonic measure*.

When $r > 0$, $|x| < r$ and $B = B(0,r) \subset \mathbb{R}^d$, the corresponding α-harmonic measure has the density function $P_r(x,\cdot)$ (the *Poisson kernel*) given by the formula

$$P_r(x,y) = C_\alpha^d \left[\frac{r^2 - |x|^2}{|y|^2 - r^2}\right]^{\alpha/2} |y-x|^{-d} \quad \text{for} \quad |y| > r, \tag{2}$$

with $C_\alpha^d = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$, and is equal to 0 otherwise [BGR].

Definition 2.1. We say that f defined on $\mathbb{R}^d$ is *α-harmonic* in an open set $D \subset \mathbb{R}^d$ if it has the mean value property

$$f(x) = E^x f(X_{\tau_U}), \quad x \in U, \tag{3}$$

for every bounded open set U with closure contained in D. It is called *regular α-harmonic* in D if (3) holds for $U = D$.

In (3) it is always assumed that the expectation is absolutely convergent. If D is unbounded then by the usual convention $E^x u(X_{\tau_D}) = E^x[\tau_D < \infty;\ u(X_{\tau_D})]$. By the strong Markov property a regular α-harmonic function is necessarily α-harmonic. The converse is not generally true [B2]. An alternative definition of α-harmonic functions by means of the fractional Laplacian

$$\Delta^{\alpha/2} f(x) = \mathcal{A}(d,-\alpha) \lim_{\epsilon \to 0^+} \int_{B(x,\epsilon)^c} \frac{f(y)-f(x)}{|y-x|^{d+\alpha}}\,dy$$

is discussed in [BB1]. Here and below $\mathcal{A}(d,\gamma) = \Gamma[(d-\gamma)/2]/(2^\gamma \pi^{d/2}|\Gamma(\gamma/2)|)$; see [L], [BG]. It follows from (2) and (3) that a function f which is α-harmonic in D satisfies

$$f(x) = \int\limits_{|y-\theta|>r} P_r(x-\theta, y-\theta)\, f(y)\,dy, \quad x \in B(\theta,r), \tag{4}$$

provided $\overline{B(\theta,r)} \subset D$. The integral in (4) is absolutely convergent and by (2) f is smooth on D. If, furthermore, f is nonnegative on $\mathbb{R}^d$ and nontrivial in D, then it is positive in D, regardless of connectedness of D. In fact, the following Harnack inequality holds [B1].

LEMMA 2.1. *Let $x_1, x_2 \in \mathbb{R}^d, r > 0$ and $k \in \mathbb{N}$ with $|x_1 - x_2| < 2^k r$. If f is nonnegative on $\mathbb{R}^d$ and α-harmonic in $B(x_1, r) \cup B(x_2, r)$ then*

$$C_1^{-1} 2^{-k(d+\alpha)} f(x_2) \le f(x_1) \le C_1 2^{k(d+\alpha)} f(x_2),$$

with a constant $C_1 = C_1(\alpha, d)$.

For $\alpha < d$ the *potential operator* U_α of the process X_t is expressed in terms of the Riesz kernel K_α. Namely, for $f \ge 0$ on $\mathbb{R}^d$

$$U_\alpha f(x) = E^x \int_0^\infty f(X_t)\, dt = \int K_\alpha(y - x) f(y)\, dy, \quad x \in \mathbb{R}^d,$$

where

$$K_\alpha(x) = \mathcal{A}(d, \alpha) |x|^{\alpha - d}, \quad x \in \mathbb{R}^d .$$

Whenever $\alpha \ge d$ the process X_t is recurrent (and pointwise recurrent if $\alpha > d = 1$), and it is appropriate to consider the so-called *compensated kernels* [BGR]

$$K_\alpha(y - x) = \int_0^\infty [p(t; x, y) - p(t; 0, x_0)]\, dt\,,$$

where $x_0 = 0$ for $\alpha > d = 1$ and $x_0 = 1$ for $\alpha = d = 1$. Thus, for $\alpha = d = 1$

$$K_\alpha(x) = \frac{1}{\pi} \ln \frac{1}{|x|},$$

and for $\alpha > d = 1$

$$K_\alpha(x) = \frac{\mathcal{A}(1, \alpha)}{|x|^{1-\alpha}} = \frac{|x|^{\alpha-1}}{2\,\Gamma(\alpha)\, \cos(\pi\alpha/2)}, \quad x \in \mathbb{R}^d.$$

Note that $K_\alpha(x) \le 0$ if $\alpha > d = 1$. We say that a domain $D \subset \mathbb{R}^d$ is Greenian if $\alpha < d$ or $\alpha \ge d = 1$ and $\mathbb{R}^d \backslash D$ is nonpolar. If $\alpha > d = 1$ then the only polar set is $\emptyset$, so in our setting nontrivial non-Greenian sets exist only for $\alpha = d = 1$. For a Greenian domain D in $\mathbb{R}^d$ we denote by G_D the Green operator and the Green function for D and X_t, i.e., for $f \ge 0$ we write

$$G_D f(x) = E^x \int_0^{\tau_D} f(X_t)\, dt = \int_D G_D(x, y) f(y)\, dy, \quad x \in \mathbb{R}^d .$$

The Green function satisfies

$$G_D(x, y) = K_\alpha(y - x) - E^x K_\alpha(y - X_{\tau_D}), \quad x, y \in D,\, x \ne y\,, \tag{5}$$

whenever $\alpha < d$ or D is bounded [BGR]. It is well-known that $G_D(x, y) > 0$ on D. Also, G_D is symmetric and for each $y \in D$, $G_D(\cdot, y)$ is α-harmonic in $D \setminus \{y\}$. If $\alpha > d = 1$ and D is a bounded domain then $G_D(\cdot, \cdot)$ is bounded on $D \times D$.

If D is a bounded domain with the *exterior cone property* then (see [IW], [B1])

$$P_D(x, y) = \int_D \frac{\mathcal{A}(d, -\alpha) G_D(x, v)}{|y - v|^{d+\alpha}}\, dv, \quad x \in D,\ y \in \operatorname{int} D^c, \tag{6}$$

where $P_D(x,y)$ denotes the density function (i.e., the Poisson kernel) of the harmonic measure $P^x(X_{\tau_D} \in dy)$.

By letting $|y| \to \infty$ in $|y|^{d+\alpha} P_r(x,y)$, we obtain for $B = B(0,r)$, $r > 0$,

$$E^x \tau_B = \int_B G_B(x,y)\, dy = \frac{C_\alpha^d}{\mathcal{A}(d,-\alpha)} (r^2 - |x|^2)^{\alpha/2}, \quad |x| < r. \tag{7}$$

3. The upper bound

For $r > 0$ and $x, y \in \mathbb{R}^d$ we set $\nabla P_r(x,y) = (D_i P_r(x,y))_{i=1}^d$, where

$$D_i P_r(x,y) = \frac{\partial}{\partial x_i} P_r(x,y)\,, \quad |x| < r\,,\ |y| > r\,,\ i = 1, \ldots, d\,.$$

Lemma 3.1. *For $r > 0$ and $B = B(0,r) \subset \mathbb{R}^d$ we have*

$$|\nabla P_r(x,y)| \le (d+\alpha) \frac{P_r(x,y)}{r - |x|}, \quad x \in B,\ y \in \operatorname{int} B^c.$$

Proof. Since

$$\frac{\partial}{\partial x_i} P_r(x,y) = P_r(x,y) \left[\frac{-\alpha x_i}{r^2 - |x|^2} + d \frac{y_i - x_i}{|y-x|^2} \right] , \tag{8}$$

we have

$$\begin{aligned} |\nabla P_r(x,y)| &\le P_r(x,y) \left[\frac{\alpha |x|}{r^2 - |x|^2} + \frac{d}{|y-x|} \right] \\ &\le (d+\alpha) \frac{P_r(x,y)}{r - |x|}\,. \end{aligned} \tag{9}$$

□

Assume that f is as in (4). By Lemma 3.1 and the bounded convergence theorem,

$$\frac{\partial}{\partial x_i} f(x) = \int\limits_{|y-\theta|>r} D_i P_r(x-\theta, y-\theta) f(y)\, dy\,,\ x \in B(\theta, r)\,,\ i = 1, \ldots, d\,. \tag{10}$$

Lemma 3.2. *Let D be an arbitrary open set in $\mathbb{R}^d$. For every nonnegative function f which is α-harmonic in D we have*

$$|\nabla f(x)| \le d \frac{f(x)}{\delta_D(x)}, \quad x \in D\,.$$

Proof. Let $x \in D$ and $0 < r < \delta_D(x)$. By (10) with $\theta = x$ and (9) we have

$$\begin{aligned} |\nabla f(x)| &\le \int\limits_{|y-x|>r} |\nabla P_r(0, y-x)|\, f(y)\, dy \le \frac{d}{r} \int\limits_{|y-x|>r} P_r(0, y-x)\, f(y)\, dy \\ &= d\frac{f(x)}{r} \ \to\ d\frac{f(x)}{\delta_D(x)} \quad \text{as } r \to \delta_D(x)\,. \end{aligned}$$

□

Lemma 3.2 applied to $\mathbb{R}^d$ gives a quick proof of the fact that the only functions bounded from below (or above) and α-harmonic on the whole space $\mathbb{R}^d$ are constants. The next result follows by an application of Lemma 3.2 to $D \setminus \{y\}$.

Corollary 3.3. *Let D be a Greenian domain in $\mathbb{R}^d$. Then*

$$|\nabla_x G_D(x,y)| \le d\frac{G_D(x,y)}{\min\{|x-y|, \delta_D(x)\}}, \quad x, y \in D,\ x \ne y.$$

We note that the inequality in Corollary 3.3 may be stated more explicitly in more regular domains (e.g., of class $C^{1,1}$), because sharp estimates for the Green function of such domains are known ([CS1], [K1]; see also [CS2], [B3]).

4. The lower bound

We introduce some auxiliary notation. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we write $x = (\tilde{x}, x_d)$, where $\tilde{x} = (x_1, \dots x_{d-1})$. In order to include the case $d = 1$ in the considerations below, we make the convention that for $x \in \mathbb{R}$, $\tilde{x} = 0$, and we set $\mathbb{R}^0 = \{0\}$.

For the remainder of the section we fix a Lipschitz function $\Gamma\colon \mathbb{R}^{d-1} \to \mathbb{R}$, with Lipschitz constant λ, so that $|\Gamma(\tilde{x}) - \Gamma(\tilde{y})| \le \lambda |\tilde{x} - \tilde{y}|$ for $\tilde{x}, \tilde{y} \in \mathbb{R}^{d-1}$. We put $\rho(x) = x_d - \Gamma(\tilde{x})$. Unless stated otherwise, D denotes the *special Lipschitz domain* defined by $D = \{x \in \mathbb{R}^d \colon \rho(x) > 0\}$. The function $\rho(x)$ serves as vertical distance from $x \in D$ to ∂D; it satisfies

$$\rho(x)/\sqrt{1+\lambda^2} \le \delta_D(x) \le \rho(x), \quad x \in D. \tag{11}$$

We define the "box" $\Delta(x, a, r) = \{y \in \mathbb{R}^d \colon 0 < \rho(y) < a, |\tilde{x} - \tilde{y}| < r\}$, where $x \in \mathbb{R}^d$ and $a, r > 0$. We note that $\Delta(x, a, r)$ is a Lipschitz domain (with "bottom" on ∂D) and depends on x only through $\tilde{x}$. We also define the "inverted box" $\nabla(x, a, r) = \{y \in \mathbb{R}^d \colon -a < \rho(y) \le 0, |\tilde{x} - \tilde{y}| < r\}$. (The same symbol ∇ is used for the gradient, but the meaning will be clear from the context.)

The following version of the boundary Harnack principle (BHP) for α-harmonic functions follows from [B1, Lemma 16 and the proof of Theorem 1]. Note that the case $d = 1$ is a consequence of (2) and (4).

Lemma 4.1 (BHP). *For all $Q \in \partial D$, $r > 0$, and nonnegative functions u, v which are regular α-harmonic in $\Delta(Q, 2r, 2r)$, vanish on $\nabla(Q, 2r, 2r)$ and satisfy $u(y_0) = v(y_0) > 0$ for some $y_0 \in \Delta(Q, r, r)$, the ratio $h(x) = u(x)/v(x)$ is Hölder continuous in $\Delta(Q, r, r)$. In fact, there exist constants $C_2 = C_2(\alpha, d, \lambda)$ and $\xi = \xi(\alpha, d, \lambda)$ such that*

$$|h(x) - h(y)| \le C_2(|x-y|/r)^\xi, \quad x, y \in \Delta(Q, r, r).$$

In particular, there is a constant $C_3 = C_3(\alpha, d, \lambda)$ such that

$$C_3^{-1} \le \frac{u(x)}{v(x)} \le C_3, \quad x \in \Delta(Q, r, r).$$

If $Q \in \partial D$ and $r > 0$ then $A_r(Q)$ denotes the unique point "above" Q such that $|A_r(Q) - Q| = (A_r(Q))_d - Q_d = r/2$. For convenience we state the following useful estimate (see [B1, Lemma 5]).

Lemma 4.2. *Under the same assumptions on Q, r and u as in Lemma 4.1 let $A = A_r(Q)$. There are constants $C_4 = C_4(\alpha, d, \lambda)$ and $\gamma = \gamma(\alpha, d, \lambda)$ such that*

$$u(x) \ge C_4 \, u(A) \left[\frac{\rho(x)}{\rho(A)} \right]^{\alpha - \gamma}, \quad x \in \Delta(Q, r, r)\,.$$

In the case $d = 1$, by (2) and (4) we have $\gamma = \alpha/2$. This is also true for $C^{1,1}$ functions Γ (see [CS1]), but not for general Lipschitz Γ ([K2]; see also [M]).

We consider a particular Lipschitz "box" $\Delta = \Delta(0, 1, 1)$ and define

$$g(x) = P^x\{X_{\tau_\Delta} \notin \nabla(0, \infty, 1)\}, \quad x \in \mathbb{R}^d. \tag{12}$$

Clearly, g is regular α-harmonic on Δ, $g = 0$ on $\nabla(0, \infty, 1)$ and $g = 1$ on $(\Delta \cup \nabla(0, \infty, 1))^c$.

Lemma 4.3. *The function $g(x)$ is nondecreasing in x_d.*

Proof. Note that $g(x) = 1 - P^x\{X_{\tau_\Delta} \in \nabla(0, \infty, 1)\}$ for $x \in \mathbb{R}^d$. Take $x, y \in \Delta$ such that $\tilde{x} = \tilde{y}$, $x_d \le y_d$ (i.e., y is "above" x). Consider $\omega + x$ and $\omega + y$; the trajectories X_t are of the same shape, but start at x and y, respectively. Observe that if $\omega + y$ exits Δ by going into $\nabla(0, \infty, 1)$, then so does $\omega + x$. $\square$

Lemma 4.4. *There is a constant $C_5 = C_5(d, \alpha, \lambda)$ such that*

$$\frac{\partial}{\partial x_d} g(x) \ge C_5 \frac{g(x)}{\delta_D(x)}, \quad x \in \Delta(0, 1/4, 1/2)\,.$$

Proof. Choose $x \in \Delta(0, 1/4, 1/2)$ and set $\eta = \rho(x)$. Let $r = \eta/(2\sqrt{1 + \lambda^2})$. Put $B_1 = B(x, r)$, $B_2 = B(\hat{x}, r)$ and $B_3 = B(\check{x}, r)$, where $\hat{x} = x + (0, \dots, 0, 2\eta)$, $\check{x} = x - (0, \dots, 0, 2\eta)$. By (11) we have $B_1 \subset B(x, 2r) \subset \Delta$, $B_2 \subset B(\hat{x}, 2r) \subset \Delta$ and $B_3 \subset \nabla(0, \infty, 1)$. Note that B_2 and B_3 are symmetric to each other with respect to the hyperplane $\Pi = \{y \in \mathbb{R}^d \colon y_d = x_d\}$. Using (4) and (10) with

$\theta = x$ and (9) we get

$$\begin{aligned}\frac{\partial}{\partial x_d} g(x) &= \int_{|y-x|>r} D_d P_r(0, y-x)\, g(y)\, dy \\ &= d \int_{|y-x|>r} P_r(0, y-x)\, \frac{y_d - x_d}{|y-x|^2}\, g(y)\, dy.\end{aligned}$$

The function $y \mapsto P_r(0, y-x)(y_d - x_d)/|y-x|^2$ is antisymmetric with respect to the hyperplane Π, and positive in the half-space "above" Π. From this and Lemma 4.3 we obtain

$$\frac{\partial}{\partial x_d} g(x) \geq d \int_{B_2 \cup B_3} P_r(0, y-x)\, \frac{y_d - x_d}{|y-x|^2}\, g(y)\, dy. \tag{13}$$

Since $g \equiv 0$ on B_3, the domain of integration $B_2 \cup B_3$ here may be replaced by B_2.

We consider an arbitrary point $y \in B_2$. We have $B(y, r) \subset \Delta$ and $\eta < |y - x| < 3\eta$. Lemma 2.1 yields $g(y) \geq c_1 g(x)$, with $c_1 = c_1(d, \alpha, \lambda)$. Since $y_d - x_d > \eta$, using (13) we obtain

$$\begin{aligned}\frac{\partial}{\partial x_d} g(x) &\geq d \int_{B_2} C_\alpha^d \left[\frac{r^2}{|y-x|^2 - r^2}\right]^{\alpha/2} \frac{y_d - x_d}{|y-x|^{d+2}}\, g(y)\, dy \\ &\geq c_1 d \int_{B_2} \left[\frac{\eta^2/(4+4\lambda^2)}{9\eta^2}\right]^{\alpha/2} \frac{\eta}{(3\eta)^{d+2}}\, g(x)\, dy \\ &= c_1 d \left(6\sqrt{1+\lambda^2}\right)^{-\alpha} 3^{-d-2}\, m(B_2) \eta^{-d-1} g(x) \\ &= c_2 g(x)/\rho(x),\end{aligned}$$

where $m(B_2)$ is the Lebesgue measure of B_2 and $c_2 = c_2(d, \alpha, \lambda)$. The lemma follows from (11). $\square$

LEMMA 4.5. *Assume that f is nonnegative on $\mathbb{R}^d$ and regular α-harmonic in Δ and vanishes on $\nabla(0, 1, 1)$. Then*

$$\frac{\partial}{\partial x_d} f(x) \geq C_6 \frac{f(x)}{\delta_D(x)}, \quad x \in \Delta(0, \eta, 1/2)\,,$$

with some constants $C_6 = C_6(d, \alpha, \lambda)$ and $\eta = \eta(d, \alpha, \lambda)$.

Proof. Let $x \in \Delta(0, 1/16, 1/2)$. Let $Q = (\tilde{x}, \Gamma(\tilde{x}))$ be the point on ∂D "below" x. We define $u(y) = cg(y)$, $y \in \mathbb{R}^d$, where $c = \lim_{D \ni y \to Q} f(y)/g(y)$, so that $h(y) = f(y)/u(y) \to 1$ as $D \ni y \to Q$ (see Lemma 4.1). By Lemma 4.4 and BHP in $\Delta(Q, 1/4, 1/4)$ (taking $r = 1/4$ in Lemma 4.1) we have

$$\begin{aligned}\frac{\partial}{\partial x_d} f(x) &\geq \frac{\partial}{\partial x_d} u(x) - |\nabla(f-u)(x)| \\ &\geq C_5 C_3^{-1} \frac{f(x)}{\delta_D(x)} - |\nabla(f-u)(x)|\,.\end{aligned} \tag{14}$$

Let $\mu = 2\rho(x)$. Note that $\mu \in (0, 1/8)$ and consider an arbitrary $r \in (2\mu, 1/4]$, to be specified later. We put $\Delta_r = \Delta(Q, r, r)$ and $\Delta_\mu = \Delta(Q, \mu, \mu)$. For clarity we note that, e.g., $\Delta_r \subset B(Q, 2r\sqrt{1+\lambda^2})$. Recall that $v = f - u$ is regular α-harmonic in Δ_μ and let $V(y) = E^y|v(X_{\tau_{\Delta_\mu}})|$, $y \in \mathbb{R}^d$. Clearly, $|v| \le V$. By Lemma 3.2,

$$
\begin{aligned}
(15) \qquad |\nabla(f-u)(x)| &\le |\nabla V(x)| + |\nabla(V-v)(x)| \\
&\le 3d\frac{V(x)}{\delta_{\Delta_\mu}(x)} \le 3d\sqrt{1+\lambda^2}\,\frac{V(x)}{\delta_D(x)}\,.
\end{aligned}
$$

To estimate $V(x)$ we note that, by BHP in $\Delta(Q, 1/4, 1/4)$,

$$
|(f-u)(y)| = u(y)|h(y) - 1| \le C_2C_3(4|y-Q|)^\xi f(y)\,, \quad y \in \Delta(Q, 1/4, 1/4)\,.
$$

By the mean value property,

$$
\begin{aligned}
(16) \qquad V(x) &\le E^x\{X_{\tau_{\Delta_\mu}} \in \Delta_r\,;\ |(f-u)(X_{\tau_{\Delta_\mu}})|\} + E_f + E_u \\
&\le C_2C_3(8r\sqrt{1+\lambda^2})^\xi f(x) + E_f + E_u\,,
\end{aligned}
$$

where the terms $E_f = E^x\{X_{\tau_{\Delta_\mu}} \in \Delta_r^c;\ f(X_{\tau_{\Delta_\mu}})\}$ and $E_u = E^x\{X_{\tau_{\Delta_\mu}} \in \Delta_r^c;\ u(X_{\tau_{\Delta_\mu}})\}$ result from the jumps of the trajectories of X_t and can be estimated as follows.

Let G_μ be the Green function of Δ_μ. By (6),

$$
E_f = \int_{\Delta_r^c}\int_{\Delta_\mu} G_\mu(x,v)\frac{\mathcal{A}(d,-\alpha)}{|y-v|^{d+\alpha}}f(y)\,dvdy\,.
$$

Let $A = A_r(Q)$. For $v \in \Delta_\mu$ and $y \in \overline{\Delta_r}^c \cap \operatorname{supp} f$ we have

$$
\begin{aligned}
|y-v| &\ge (|y-A| - |A-v|) \vee \frac{r}{2\sqrt{1+\lambda^2}} \\
&\ge (|y-A| - 2r\sqrt{1+\lambda^2}) \vee \frac{r}{2\sqrt{1+\lambda^2}} \ge \frac{|y-A|}{8(1+\lambda^2)}\,.
\end{aligned}
$$

It follows that

$$
E_f \le [8(1+\lambda^2)]^{d+\alpha}\int_{\Delta_\mu} G_\mu(x,v)\,dv \cdot \int_{\Delta_r^c}\frac{\mathcal{A}(d,-\alpha)}{|y-A|^{d+\alpha}}f(y)\,dy\,.
$$

We have by (7)

$$
\int_{\Delta_\mu} G_\mu(x,v)\,dv = E^x\tau_{\Delta_\mu} \le E^x\tau_{B(Q,2\mu\sqrt{1+\lambda^2})} \le \frac{C_\alpha^d}{\mathcal{A}(d,-\alpha)}(2\mu\sqrt{1+\lambda^2})^\alpha\,.
$$

Let $B = B(A, r/(2\sqrt{1+\lambda^2}))$. For $y \in \Delta_r^c \subset B^c$ we have

$$
\frac{C_\alpha^d}{|y-A|^{d+\alpha}} \le \left(r/\left(2\sqrt{1+\lambda^2}\right)\right)^{-\alpha} P_{r/(2\sqrt{1+\lambda^2})}(0, y-A)
$$

(see (2)). Thus by the mean value property and Lemma 4.2

$$\begin{aligned} E_f &\leq [8(1+\lambda^2)]^{d+\alpha} 2^{2\alpha}(1+\lambda^2)^\alpha (\mu/r)^\alpha E^A f(X_{\tau_B}) \\ &\leq C_4 [8(1+\lambda^2)]^{d+\alpha} 2^{2\alpha}(1+\lambda^2)^\alpha (\mu/r)^\gamma f(x)\,. \end{aligned}$$

By a similar reasoning and BHP

$$E_u \leq C_3 C_4 [8(1+\lambda^2)]^{d+\alpha} 2^{2\alpha}(1+\lambda^2)^\alpha (\mu/r)^\gamma f(x)\,.$$

Recall that $\mu = 2\rho(x)$. We now define $r = (2\mu \cdot \mu^{-\xi/(\gamma+\xi)}) \wedge (1/4)$. Then $(\mu/r)^\gamma \leq 4^\gamma \mu^{\gamma\xi/(\gamma+\xi)}$. Since $r \leq 2\mu^{\gamma/(\gamma+\xi)}$, by (16), there exists $c = c(d,\alpha,\lambda)$ such that $V(x) \leq c\rho(x)^{\gamma\xi/(\gamma+\xi)} f(x)$. The lemma now follows from (14) and (15) provided we choose η so that $3d\sqrt{1+\lambda^2}c\eta^{\gamma\xi/(\gamma+\xi)} \leq C_5 C_3^{-1}/2$. □

In the case $d = 1$ a more explicit estimate easily follows from (8) and (10).

Lemma 4.6. *For every nonnegative function f on $\mathbb{R}$ which is regular α-harmonic in $(-1,1)$ and vanishes on $(-3,-1]$*

$$f'(x) \geq \frac{\alpha}{6}\frac{f(x)}{1-|x|}, \quad x \in (-1, -1+\alpha/6)\,.$$

Proof of Theorem 1.1. The upper bound in (1) was stated more generally in Lemma 3.2. To prove the lower bound we observe that its validity is not affected by a translation or a unitary transformations of $\mathbb{R}^d$. We also note that a nonnegative function which is bounded and α-harmonic on a Lipschitz domain is regular α-harmonic on this domain (see [B1]). Thus, we can use Lemma 4.5 and the result follows from the inequality $|\nabla f| \geq \left|\frac{\partial}{\partial x_d} f\right|$, the scaling properties of α-harmonic functions and the compactness of $\partial D \cap K$. □

Example 4.1. Under the notation and the assumptions of Theorem 1.1 the function f has no local extremum on the set $\{x \in D \cap K : \delta_D(x) < \varepsilon\}$. In this connection consider the set $D = (1/2,1) \cup (1/8,1/4) \cup (1/32,1/16) \cup \cdots \subset \mathbb{R}$. Define $f(x) = P^x\{X_{\tau_D} > 1\}$, $x \in \mathbb{R}$. On each interval $(4^{-n}/2, 4^{-n})$, f has a local maximum, so the lower gradient estimate does not hold near $0 \in \partial D$ even though D is rather "fat" at 0.

We now consider the cones $C_h = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d \,:\, |\tilde{x}| < hx_d\}$, $h > 0$. Each C_h is a Lipschitz domain. By [B2] there is a Martin kernel M for C_h corresponding to the point at infinity. M vanishes continuously outside C_h and is α-harmonic in C_h. By the uniqueness of M and the homogeneity and symmetry of C_h we obtain $M(x) = |x|^\beta \phi(x/|x|)$, where ϕ is symmetric with respect to the axis of C_h and $0 < \beta < \alpha$. Thus for $x = (\tilde{0}, x_d)$ with $x_d > 0$ we have

$$|\nabla M(x)| = \frac{\partial}{\partial x_d} M(x) = \beta \frac{M(x)}{|x|} = \beta \frac{M(x)}{\delta_{C_h}(x)} \frac{h}{\sqrt{1+h^2}}\,.$$

This shows that $C_6 \to 0$ in Lemma 4.5 as $\lambda \to \infty$, and the same behavior may be expected for domains with narrow thorns.

We state two results which are gradient analogs of the boundary Harnack principle and the Harnack inequality. The first is a direct consequence of Theorem 1.1 and BHP in a global version given in [B1]; the second follows from Theorem 1.1 and Lemma 2.1.

Corollary 4.7 (BHP). *Under the assumptions of Lemma 4.1 there is a constant $C_7 = C_7(V, K, D, \alpha)$ such that*

$$C_7^{-1}|\nabla u(x)| \le |\nabla v(x)| \le C_7|\nabla u(x)|, \quad x \in D \cap K,\ \delta_D(x) < \varepsilon\,,$$

where $\varepsilon = \varepsilon(V, K, D, \alpha)$ is the constant of Theorem 1.1.

Corollary 4.8. *Under the assumptions of Theorem 1.1 let $x_1, x_2 \in K \cap D$, $r > 0$ and $k \in \mathbb{N}$ be such that $|x_1 - x_2| < 2^k r$ and $B(x_1, r) \cup B(x_2, r) \subset D \cap V$. There exists a constant $C_8 = C_8(D, V, K, \alpha)$, such that*

$$C_8^{-1} 2^{-k(d+\alpha+1)}|\nabla f(x_2)| \le |\nabla f(x_1)| \le C_8 2^{k(d+\alpha+1)}|\nabla f(x_2)|\,,$$

provided $\delta_D(x_1) < \varepsilon$ and $\delta_D(x_2) < \varepsilon$, where $\varepsilon = \varepsilon(V, K, D, \alpha)$ is the constant of Theorem 1.1.

5. q-harmonic functions

In this section we derive gradient estimates for q-harmonic functions from gradient estimates for α-harmonic functions. We will use the properties of nonnegative q-harmonic functions established in [BB1] (for $\alpha < d$) and [BB2] (for all $\alpha \in (0, 2)$ and $d \in \mathbb{N}$). We first give some necessary definitions.

A function q on $\mathbb{R}^d$ belongs to the Kato class $\mathcal{J}^\alpha$ if

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|y-x|<r} |q(y)K_\alpha(y-x)|\, dy = 0, \tag{17}$$

where K_α is the function defined in Section 2. Clearly, if $\alpha < \beta < 2$, then $\mathcal{J}^\alpha \subset \mathcal{J}^\beta$. If $\alpha > d = 1$ then (17) is equivalent to

$$\sup_{x\in\mathbb{R}^d} \int_{|y-x|<1} |q(y)|\, dy < \infty\,. \tag{18}$$

For $q \in \mathcal{J}^\alpha$ we define the Feynman-Kac functional $e_q(t) = \exp\left(\int_0^t q(X_s)\, ds\right)$, $t \ge 0$.

Definition 5.1. Let $q \in \mathcal{J}^\alpha$. We say that a function u on $\mathbb{R}^d$ is *q-harmonic* in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = E^x[e_q(\tau_U)u(X_{\tau_U})], \quad x \in U, \tag{19}$$

for every bounded open set U with $\overline{U} \subset D$. The function u is called *regular q-harmonic* in D if (19) holds for $U = D$.

In the latter case, for unbounded D, the expectation in (19) is to be understood as $E^x[\tau_D < \infty\,;\; e_q(\tau_D)u(X_{\tau_D})]$. It is known [BB2] that if a function u is q-harmonic on D, then it is continuous on D and satisfies

$$u(x) = E^x[u(X_{\tau_U})] + G_U(qu)(x)\,, \qquad x \in U \tag{20}$$

for every bounded open U with $\overline{U} \subset D$. For nonnegative u the converse is also true [BB2]. If, moreover, D is a Lipschitz domain and u is nonnegative and bounded on D, then u is regular q-harmonic on D [BB1, Lemma 5.4]. This identification will be used in the sequel without further comments.

To obtain gradient estimates for q-harmonic functions it is appropriate to impose a more stringent assumption on q, namely $q \in \mathcal{J}^{\alpha-1}$, where $\alpha > 1$. The case $\alpha \le 1$ seems to require a modification of our arguments and definitions and will not be discussed. The main result of this section is the following.

THEOREM 5.1. *Let D be a Lipschitz domain in $\mathbb{R}^d$, $d \in \mathbb{N}$, $\alpha \in (1,2)$ and $q \in \mathcal{J}^{\alpha-1}$. Let $V \subset \mathbb{R}^d$ be open and let K be a compact subset of V. There exist constants $C_9 = C_9(D,V,K,\alpha,q)$ and $\varepsilon = \varepsilon(D,V,K,\alpha,q)$ such that for every nonnegative function f which is bounded on V, q-harmonic in $D \cap V$, and vanishes in $D^c \cap V$, we have*

$$C_9^{-1}\frac{u(x)}{\delta_D(x)} \le |\nabla u(x)| \le C_9\frac{u(x)}{\delta_D(x)}\,, \quad x \in K \cap D,\ \delta_D(x) < \varepsilon\,.$$

For the rest of this section, unless stated otherwise, we fix $d \in \mathbb{N}$, $\alpha \in (1,2)$ and $q \in \mathcal{J}^{\alpha-1}$.

LEMMA 5.2. *Consider a bounded domain $B \subset \mathbb{R}^d$ and a bounded function u on B. We have*

$$\frac{\partial}{\partial x_i}G_B(qu)(x) = \int_B \frac{\partial}{\partial x_i}G_B(x,y)q(y)u(y)\,dy, \quad x \in B\,, i = 1,2,\ldots,d\,.$$

Proof. We assume, as we may, that $i = d$. Let $x_0 \in B$, $0 < h < \delta_B(x_0)/2$ and $h_d = (0,\ldots,0,h)$. By (5) we have

$$\begin{aligned}\frac{\partial}{\partial x_d}G_B(qf)(x_0) &= \lim_{h\to 0}\int_B \frac{K_\alpha(x_0+h_d-y) - K_\alpha(x_0-y)}{h}q(y)u(y)\,dy \\ &- \lim_{h\to 0}\int_B \frac{H(x_0+h_d,y) - H(x_0,y)}{h}q(y)u(y)\,dy = I - II,\end{aligned}$$

where $H(x,y) = E^x K_\alpha(X_{\tau_B} - y)$. Since

$$\frac{|K_\alpha(x_0+h_d-y) - K_\alpha(x_0-y)|}{h} \le c(\alpha,d)(|x_0+h_d-y| \wedge |x_0-y|)^{\alpha-d-1}\,,$$

the integrand in I is uniformly in h integrable on B. The same is true for II by Lemma 3.2, the Harnack inequality, and the boundedness of $H(x_0,y)$ in $y \in B$. $\square$

As in Section 4, we first consider the special Lipschitz domain D given by a Lipschitz function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant λ. For $r > 0$ and $Q \in \partial D$ we set $\Delta_r = \Delta(Q, r, r)$ and $G_r = G_{\Delta_r}$. For a (nonnegative) function u we put $u^{\Delta_r}(x) = E^x u(X_{\tau_{\Delta_r}}), \quad x \in \mathbb{R}^d$.

LEMMA 5.3. *For every $\varepsilon > 0$ there exists a constant $r_0 = r_0(d, \lambda, \alpha, q, \varepsilon)$ such that if $r \le r_0$ and u is nonnegative in $\mathbb{R}^d$ and q-harmonic and bounded in $\Delta_r = \Delta(Q, r, r)$, then*

$$(1 - \varepsilon) u^{\Delta_r}(x) \le u(x) \le (1 + \varepsilon) u^{\Delta_r}(x) , \quad x \in \mathbb{R}^d . \tag{21}$$

Furthermore,

$$G_r(|q|u)(x) \le \varepsilon u^{\Delta_r}(x), \quad x \in \mathbb{R}^d . \tag{22}$$

For $d = 1$, when Δ_r is an interval, (21) follows from the estimate for the *conditional gauge function* given in Lemma 3.5 of [BB2] (see also (2.15) there). This estimate in turn is a simple consequence of the Khasminskii's Lemma and the 3G Theorem for the ball [BB2]. In dimensions $d > 1$ the same argument works by the version of the 3G Theorem stated in [BB1] and [CS3] for Lipschitz domains (see also the earlier paper [CS1] for the case of $C^{1,1}$ domains), and by scaling. The estimate (22) follows from (21) and (20), when applied to $|q|$ and q.

Since $q \in \mathcal{J}^{\alpha-1} \subset \mathcal{J}^{\alpha}$, given $\varepsilon > 0$ we have, by choosing a smaller value for $r_0 = r_0(d, \lambda, \alpha, q, \varepsilon) \le 1$ if necessary, for every $Q \in \partial D$ and $r \le r_0$,

$$\sup_{x \in \Delta_r} \int_{\Delta_r} |q(y) K_{\alpha-1}(y - x)| \, dy \le \varepsilon , \tag{23}$$

and

$$\sup_{x \in \Delta_r} \int_{\Delta_r} |q(y) K_{\alpha}(y - x)| \, dy \quad \le \quad \varepsilon , \qquad \text{if } d > 1 , \tag{24}$$

$$\sup_{x \in \Delta_r} \int_{\Delta_r} |q(y)| \, dy \quad \le \quad \varepsilon , \qquad \text{if } d = 1 . \tag{25}$$

LEMMA 5.4. *Let $\varepsilon \le 1/2$, $Q \in \partial D$ and $r \le r_0(d, \lambda, \alpha, q, \varepsilon)$. Assume that u is nonnegative in $\mathbb{R}^d$, and q-harmonic and bounded in $\Delta_r = \Delta(Q, r, r)$. There exists a constant $C_{10} = C_{10}(d, \alpha)$ such that*

$$|\nabla G_r(qu)(x)| \le \varepsilon \, C_{10} \frac{u(x)}{\delta_{\Delta_r}(x)}, \quad x \in \Delta_r.$$

Proof. By Lemma 5.2,

$$|\nabla G_r(qu)(x)| \le \int_{\Delta_r} |\nabla_x G_r(x, y)| |q(y)| u(y) \, dy , \qquad x \in \Delta_r.$$

Let $H(x,y) = E^x K_\alpha(X_{\tau_{\Delta_r}} - y)$, $x, y \in \Delta_r$. We fix $x \in \Delta_r$. Let $B = B(x, \delta_{\Delta_r}(x)/2)$. By (5) and Lemma 3.2, for $y \in B$ we have

$$|\nabla_x G_r(x,y)| \le c_1 \left[K_{\alpha-1}(y-x) + |H(x,y)|/\delta_{\Delta_r}(x)\right], \quad y \neq x\,, \tag{26}$$

where $c_1 = c_1(d,\alpha)$. Recall that $\alpha > 1$. For $d \ge 2$ we have $H(x,y) \le K_\alpha(x-y)$, and if $d = 1 < \alpha$ then, by scaling,

$$H(x,y) \le c_3 r^{\alpha-1}\,, \qquad x, y \in \Delta_r\,,$$

where $c_3 = c_3(\alpha)$. By (21) and the Harnack inequality,

$$u(y) \le 3/2 u^{\Delta_r}(y) \le 3/2 c_2 u^{\Delta_r}(x) \le 3 c_2 u(x)\,, \tag{27}$$

where $c_2 = c_2(d,\alpha)$ results from Lemma 2.1. This, together with (26) and (23), implies

$$\int_B |\nabla_x G_r(x,y)||q(y)|u(y)\,dy \le 3c_1c_2u(x)\varepsilon[1 + \delta_{\Delta_r}^{-1}(x)]\,, \qquad \text{if } d > 1, \tag{28}$$

$$\int_B |\nabla_x G_r(x,y)||q(y)|u(y)\,dy \le 3c_1c_2u(x)\varepsilon\left[1 + \frac{c_3 r^{\alpha-1}}{\delta_{\Delta_r}(x)}\right]\,, \qquad \text{if } d = 1. \tag{29}$$

By Corollary 3.3 we also have

$$\int_{\Delta_r \setminus B} |\nabla_x G_r(x,y)||q(y)|u(y)\,dy \le 2d G_r(|q|u)(x)/\delta_{\Delta_r}(x)\,.$$

The lemma follows from (21), (22), (28) and (29) because $\delta_{\Delta_r}(x) < r_0 \le 1$. □

As in the case of α-harmonic functions, the upper bound below holds for every domain.

Lemma 5.5. *Let B be an arbitrary domain in $\mathbb{R}^d$. There exists a constant $C_{11} = C_{11}(d,\alpha,q)$ such that for every function u that is nonnegative in $\mathbb{R}^d$ and q-harmonic in B we have*

$$|\nabla u(x)| \le C_{11} \frac{u(x)}{\delta_B(x) \wedge 1}, \quad x \in B.$$

Proof. Fix $x \in B$ and $\varepsilon = 1/2$. For $r > 0$ we consider the particular Lipschitz box $\Delta_r = \{y \in \mathbb{R}^d : |x_d - y_d| < r/2, |\tilde{x} - \tilde{y}| < r\}$. Let $r = r_0(d, 0, \alpha, q, 1/2) \wedge (\delta_B(x)/2)$. By (20) we have

$$|\nabla u(x)| \le |\nabla u^{\Delta_r}(x)| + |\nabla G_r(qu)(x)|.$$

The assertion follows from Lemma 3.2, Lemma 5.4 and (21). □

Lemma 5.6. *There are constants $C_{12} = C_{12}(d,\alpha,\lambda,q)$ and $\kappa = \kappa(d,\alpha,\lambda,q)$ such that if $0 < r \le \kappa$, $Q \in \partial D$ and u is nonnegative in $\mathbb{R}^d$, q-harmonic and bounded in $\Delta(Q,2r,2r)$, and vanishes in $\nabla(Q,2r,2r)$, then*

$$|\nabla u(x)| \ge C_{12} \frac{u(x)}{\delta_D(x)}, \quad x \in \Delta(Q,r,r).$$

Proof. The function u satisfies (20) with $U = \Delta(Q, 2r, 2r)$. Using Lemmas 4.5, 5.4 and 5.3 we obtain the result by an appropriate choice of (ε and) κ. $\square$

Proof of Theorem 5.1. The upper bound follows from Lemma 5.5. Note that the class $\mathcal{J}^\alpha$ and the estimate in Lemma 5.6 are rotation invariant. Thus the lower bound follows from this lemma and the compactness of $\partial D \cap K$. $\square$

We note that the techniques presented in this paper apply even more easily to the classical harmonic and q-harmonic functions and give the estimates of [C] and [BP].

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