Illinois Journal of Mathematics
Volume 46, Number 2, Summer 2002, Pages 533–539
S 0019-2082

A CHARACTERIZATION OF THE DISK ALGEBRA

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ABSTRACT. We prove that a complex unital uniform algebra is isomorphic to the disk algebra if and only if every closed subalgebra with one generator is isomorphic to the whole algebra. Moreover, every such subalgebra of the disk algebra is isometrically isomorphic to the disk algebra. On the way we prove: (1) for a function f in the disk algebra the interior of the polynomial hull of the set $f(\overline{U})$, where $\overline{U}$ is the closed unit disk, is a Jordan domain; (2) if a uniform algebra A on a compact Hausdorff set X containing the Cantor set separates points of X, then there is $f \in A$ such that $f(X) = \overline{U}$.

1. Introduction

The *disk algebra* Δ is the uniform algebra of all continuous complex-valued functions on the closure $\overline{U}$ of the unit disk U in the complex plane $\mathbb{C}$ that are holomorphic on U.

This algebra is probably the most popular uniform algebra after the uniform algebra $C(I)$ of all continuous functions on the interval $I = [0, 1]$. Therefore a new property that characterizes it up to an isomorphism seems to be of interest for a reader.

The property can be stated quite simply:

THEOREM 1.1. *A complex unital uniform algebra $A \neq \mathbb{C}$ is (isometrically) isomorphic to Δ if and only if every closed subalgebra A_f generated by the function 1 and $f \in A$ that is not constant is (isometrically) isomorphic to A. Moreover, any closed subalgebra Δ_f in Δ generated by a non-constant function $f \in \Delta$ is isometrically isomorphic to Δ.*

This theorem follows from two interesting facts that seem to be unknown.

THEOREM 1.2. *Let $f \in \Delta$ be a non-constant function and $K = f(\overline{U})$. Then every component of the complement of K is a Jordan domain.*

Received August 6, 2001; received in final form January 7, 2002.
2000 *Mathematics Subject Classification.* Primary 46J10. Secondary 30D40.

Theorem 1.3. *Let A be a complex uniform algebra on a compact Hausdorff space X containing the Cantor set. If A separates points of X, then there is an element $f \in A$ such that $f(X) = \overline{U}$.*

This theorem immediately implies the following corollary.

Corollary 1.4. *Let A be a complex uniform algebra on a compact Hausdorff space X containing the Cantor set. If A separates points of X, then it contains a closed subalgebra isometrically isomorphic to the disk algebra.*

To prove the corollary we take a mapping f provided by Theorem 1.3 and consider the mapping Φ of Δ into $A(X)$ defined as $\Phi(g) = g \circ f$.

Note that if X is a compact metrizable space and K is the Cantor set, then there is a continuous mapping F from K onto X. If A is a uniform algebra on X, then the mapping $\Psi : A \to C(K)$ defined as $\Psi(g) = g \circ F$ embeds A isometrically into $C(K)$. Thus Δ and $C(K)$ are extreme points on the scale of uniform algebras on compact metrizable spaces.

2. Proof of Theorem 1.2

Let us recall the basics of prime ends (see [CL] or [C]). Let D be a simply connected domain in the plane. A Jordan arc that lies in D except for its two end-points or a Jordan curve that lies in D except for one point is a *cross-cut* of D. A Jordan arc with one end-point in ∂D and all other points in D is called an *end-cut* of D. A point z in ∂D is *accessible* if it is an end-point of an end-cut in D. A cross-cut γ divides D into two domains whose portions of the boundary lying in D are γ. A sequence $\{\gamma_n\}$ of cross-cuts of D is called a *chain* if it satisfies:

(1) No two of them have any point in common.
(2) γ_n separates D into two domains: D_{γ_n} containing γ_{n+1} and D'_{γ_n} containing γ_{n-1}.
(3) The diameter of γ_n tends to zero as $n \to \infty$.

Two chains $\{\gamma_n\}$ and $\{\delta_n\}$ in D are *equivalent* if for every n there is m such that D_{γ_n} contains all δ_j and D_{δ_n} contains all γ_j when $j > m$. A *prime end* is an equivalence class of chains.

Let $\widetilde{D}$ be the union of D and all prime ends. A subset E of $\widetilde{D}$ is *open* if $E \cap D$ is open in D and for every $p \in E \setminus D$ there is a chain $\{\gamma_n\}$ in p and an integer n such that $D_{\gamma_n} \subset E \cap D$.

The classical Carathéodory's correspondence theorem states:

Theorem 2.1. *If D is a bounded simply connected domain in the plane and $f : U \to D$ is a conformal equivalence, then f extends to a homeomorphism of $\overline{U}$ onto $\widetilde{D}$.*

In particular, the set $\widetilde{D} \setminus D$ is homeomorphic to the unit circle S, and the prime end topology on $\widetilde{D}$ is Hausdorff.

Proof of Theorem 1.2. The complement to the set $K = f(\overline{U})$ in $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the union of connected open sets $D_0, D_1, \dots$, where D_0 contains ∞ and all other sets D_j are bounded. The polynomial hull $\hat{K}$ of K is $\mathbb{C}_\infty \setminus D_0$ and by [C, Prop. 13.1.1] the set $\partial\hat{K}$ is connected.

We will prove that the interior D of $\hat{K}$ is a Jordan domain. This immediately implies that D_0 is also a Jordan domain. To prove that D_j, $j \geq 1$, is Jordan, take a point w_0 in D_j and replace f by $1/(f - w_0)$.

Let us show that $\partial\hat{K}$ belongs to $f(S)$. If $z \in \partial\hat{K}$ but z is not in $f(S)$, then z is not in $f(U)$ either and, therefore, z belongs to one of the sets D_j, $j \geq 1$. But $D_j \subset \hat{K}$ is open. Hence z belongs to the interior D of $\hat{K}$, and by contradiction it follows that $z \in f(S)$.

Let V be the connected open component of the interior D of $\hat{K}$ that contains $f(U)$. Then $f(S) \subset \overline{V}$ and, therefore, $\partial V = \partial\hat{K}$. Since $\partial\hat{K}$ is connected, by [C, Prop. 13.1.1] V is simply connected. Thus V is connected and simply connected.

Let us study the topology of the space of prime ends of V. Suppose that a point $w_0 \in \partial V$ is accessible in V by two end-cuts γ_1 and γ_2. As was proved in [CL, p. 177], every end-cut γ with $\gamma(1) = w_0$ converges to a prime end P_γ. Let us show that $P_{\gamma_1} = P_{\gamma_2}$.

Let $\{\alpha_n\}$ be a chain in P_{γ_1}. If we prove that for every n there is $0 < t_n < 1$ such that $\gamma_2(t) \in V_{\alpha_n}$ when $t_n < t < 1$, then it will follow that $\gamma_2(t)$ converges to P_{γ_1} and, since the topology on $\widetilde{V}$ is Hausdorff, $P_{\gamma_1} = P_{\gamma_2}$.

Suppose that the latter statement does not hold for some integer n. Let us take a chain $\{\beta_j\}$ in P_{γ_2}. Since the prime end topology is Hausdorff there is a natural number m such that V_{β_m} does not intersect V_{α_n}. We may assume that neither α_n nor β_m contain w_0. Let t_1 and t_2 denote the maximal numbers t such that $\gamma_1(t) \in \alpha_n$ and $\gamma_2(t) \in \beta_m$, respectively. Since both α_n and β_m lie at some positive distance from w_0 and $\gamma_1(1) = \gamma_2(1) = w_0$, these numbers are strictly less than 1. The intersection of V'_{α_n} and V'_{β_m} is connected and therefore there is a Jordan arc $\gamma_3 : [0,1] \to V$ that connects the points $\gamma_1(t_1)$ and $\gamma_2(t_2)$ and lies in the intersection except for the points $\gamma_3(0) = \gamma_1(t_1)$ and $\gamma_3(1) = \gamma_2(t_2)$. Therefore the curve γ consisting of γ_1 restricted to $[1, t_1]$, γ_3, and γ_2 restricted to $[t_2, 1]$, is Jordan.

Since γ lies in $\hat{K}$ the interior component E of γ also lies in $\hat{K}$. But $\gamma \in V$ and hence $E \subset V$. Thus $\partial E \cap \partial V = \{w_0\}$. But the cross-cuts α_n and β_m intersect ∂E only at the points $\gamma_1(t_1)$ and $\gamma_2(t_2)$, respectively, and must end at ∂V. Hence they do not intersect E. The open sets $V'_{\alpha_n} = V_{\alpha_n} \cap E$ and $V'_{\beta_m} = V_{\beta_m} \cap E$ are non-empty and have no boundaries in E. So $V'_{\alpha_n} = V'_{\beta_m}$, and this contradicts the assumption that $V_{\alpha_n} \cap V_{\beta_m} = \emptyset$. Therefore $P_{\gamma_1} = P_{\gamma_2}$.

Let F be the set of all prime ends $P = \{\gamma_n\}$ in V whose impressions

$$I(P) = \bigcap_{n=1}^{\infty} \overline{D}_{\gamma_n}$$

contain a point w_P that is accessible relative to P, i.e., there is an end-cut γ of V ending at w_P and converging to P in the prime end topology. Since by [CL, Theorem 9.7] for any prime end P its impression $I(P)$ has at most one point that is accessible relative to P, the mapping $\tau : P \to w_P$ is well defined on F.

Let us show that τ is onto. If $w \in \partial V$, then there is $z \in S$ such that $f(z) = w$. The curve $\gamma(t) = f(tz)$, $0 \le t \le 1$, need not to be a Jordan arc. But it is an analytic curve and it is easy to see that by cutting out loops we can modify it into a Jordan arc γ' in V that becomes an end-cut of V. So w is accessible relative to $P_{\gamma'}$ and $w = \tau(P_{\gamma'})$.

This mapping is also one-to-one because, as we proved above, if w is accessible relative to P_1 and P_2, then $P_1 = P_2$.

Thus the inverse mapping ρ of τ is defined. Let us show that ρ is continuous. Let $\{w_j\}$ be a sequence of points in ∂V converging to w and let $P_j = \rho(w_j)$. For every point w_n we choose a point $z_n \in S$ such that $w_n = f(z_n)$. We assume that there is a neighborhood W of $P = \rho(w)$ in $\widetilde{V}$ such that all prime ends $\rho(w_j)$ are not in W, but the sequence of z_n converges to a point $z \in S$ such that $f(z) = w$. Let $\{\alpha_n\}$ be a chain in P. For all sufficiently large n the point $f(0)$ lies outside V_{α_n}. Among such numbers n we select one so that the cross-cut α_n and V_{α_n} are in W. Then the closure of V_{α_n} in the space of prime ends also lies in W. Let t_n be the last instant when the curve $\gamma(t) = f(tz)$ meets α_n. We take a number s_n strictly between t_n and 1 and find an integer j_0 such that $f(s_n z_j) \in V_{\alpha_n}$ and the curves $f(tz_j)$ never meet α_n when $j \ge j_0$ and $s_n \le t < 1$. Hence $f(tz_j) \in V_{\alpha_n}$ whenever $j \ge j_0$ and $s_n \le t < 1$. But as t goes to 1 the limit of $f(tz_j)$ in the prime ends topology is P_j and therefore P_j belongs to the closure $\widetilde{V}_{\alpha_n}$ of V_{α_n} in the prime ends topology. Thus $P_j \in W$ and this contradiction shows that the mapping ρ is continuous.

Therefore the set F is closed in $\widetilde{V}$ as the image of a compact set under a continuous mapping. But F is dense in the set of all prime ends and therefore $F = \widetilde{V} \setminus V$.

Thus ρ is a continuous one-to-one mapping of a compact space ∂V onto a Hausdorff space $\widetilde{V} \setminus V$. Hence ρ is a homeomorphism. Since $\widetilde{V} \setminus V$ is homeomorphic to a circle, ∂V is a Jordan curve and V is a Jordan domain. Consequently, $\partial \hat{K} = \partial V$ is a Jordan curve and therefore it divides $\mathbb{C}_\infty$ into two domains D_0 and V. Thus the interior of $\hat{K}$ is the Jordan domain V. $\square$

3. Proof of Theorem 1.3

We will need some auxiliary results. Given a uniform algebra A defined on a compact Hausdorff space X, a set $E \subseteq X$ is a *peak interpolation set* (PI set) if it satisfies:

(1) There exists $f \in A$ with $f = 1$ on E and $|f| < 1$ off E.
(2) For each $g \in C(E)$, there exists $f \in A$ with $f|_E = g$ and $\|f\| = \|g\|$.

The next result (due to Glicksberg) can be found in [G]

PROPOSITION 3.1. *Let A be a uniform algebra defined on a compact Hausdorff space X. Then E is a PI set for A if and only if E is a closed G_δ-set so that $|\nu|(E) = 0$ for all measures ν on X with $\nu \perp A$.*

The main step in the proof of Theorem 1.3 is the next result. For a compact subset L of $\mathbb{C}$, we let $P(L)$ denote the closure of all polynomials in $C(L)$.

PROPOSITION 3.2. *Let L be a compact subset of $\mathbb{C}$ so that $P(L) \neq C(L)$. Then there exists a PI set $E \subset L$ for $P(L)$ with E homeomorphic to the Cantor set.*

Proof. We make use of several standard results concerning uniform algebras; see [G].

Without loss of generality we can assume that L is the Shilov boundary of $P(L)$. By a theorem of Walsh, $P(L)$ is a Dirichlet algebra on L. This implies that every point z in the space $M_{P(L)} = \hat{L}$ of maximal ideals of $P(L)$ has a unique representing measure supported by L. In particular, this measure for a point $z \in L$ is the Dirac measure δ_z. Since $P(L) \neq C(L)$, Lavrentiev's Theorem implies the existence of non-trivial Gleason parts. It follows from [G, Theorem 6.2.2] that every representing measure for a point in a non-trivial Gleason part has no atoms. By Wermer's embedding theorem every Gleason part of $\hat{L}$ is either a unit disk or a point. Thus the set of non-trivial Gleason parts is at most countable. Select a sequence $\{z_n\}$ containing one point from each such part. Let μ_n be a representing measure for z_n. Set $m = \sum(1/2^n)\mu_n$. Since m has no atoms, there exists a compact set $E \subseteq L$ which is homeomorphic to the Cantor set and satisfies $m(E) = 0$. If ν is a measure on L with $\nu \perp P(L)$, then by [G, Theorem 6.2.3] ν is absolutely continuous with respect to m. Thus $\nu(E) = 0$ and by Proposition 3.1 E is a PI set for $P(L)$. $\square$

Proof of Theorem 1.3. If $A \neq C(X)$, there exists $g \in A$ so that $\bar{g} \notin A$. Let $L = g(X)$. The choice of g guarantees that $\bar{z} \notin P(L)$. Hence $P(L) \neq C(L)$.

For $h \in P(L)$ define $\phi(h) = h \circ g$. It is easily seen that $\phi\colon P(L) \to A$ is an isometric isomorphic embedding.

By Proposition 3.2, we can select a set E that is a PI set for $P(L)$ and homeomorphic to the Cantor set. Let h be a continuous map h from E onto

$\overline{U}$. Since E is a PI set, there exists $f \in P(L)$ with $\|f\| = 1$ and $f|_E = h$. Set $F = \phi(f)$. Then $F \in A$, $\|F\| = 1$, and $F(X) = f(L) \supseteq h(E) = \overline{U}$, and hence $F(X) = \overline{U}$.

If $A = C(X)$, then we take the Cantor set $K \subset X$ and a continuous function F that maps K onto $\overline{U}$. A continuous extension of F to X preserving the sup-norm gives us the desirable mapping. □

4. Proof of Theorem 1.1

Let us show that an algebra Δ_f, where f is not a constant, is isometrically isomorphic to Δ. Let $\hat{K}$ be the polynomial hull of $K = f(\overline{U})$. Let us show that the algebra Δ_f is isometric to the algebra B of functions that are continuous on $\hat{K}$ and holomorphic in the interior D of $\hat{K}$. Let $\Phi : B \to \Delta$ be a mapping defined as $\Phi(g) = g \circ f$. The mapping Φ is an isometry because by the maximum principle the function $g_1 - g_2$ attains the maximum of its absolute value on $\partial\hat{K} \subset f(S)$ and therefore $||g_1 - g_2|| = ||\Phi(g_1) - \Phi(g_2)||$. If $h \in \Delta_f$, then h is the uniform limit on $\overline{U}$ of a sequence of functions $P_n(f)$, where P_n are polynomials. Since $\hat{K}$ is the polynomial hull of $f(\overline{U})$, it follows that the polynomials P_n converge uniformly on $\hat{K}$ to a function g. Clearly $\Phi(g) = h$. Hence $\Delta_f \subset \Phi(B)$.

Since $\hat{K}$ is polynomially convex, every function $g \in B$ is the uniform limit on $\hat{K}$ of a sequence of polynomials P_n. The functions $P_n(f)$ converge uniformly on $\overline{U}$ to a function $h \in \Delta_f$, and $\Phi(g) = h$. Thus $\Phi(B) = \Delta_f$.

But the interior D of $\hat{K}$ is a Jordan domain. Therefore a conformal equivalence e of D and U produces the isometry Ψ of B and Δ as $\Psi(g) = g \circ e$. Hence Δ and Δ_f are isometric.

Thus if an algebra A is (isometrically) isomorphic to Δ, then any subalgebra A_f is (isometrically) isomorphic to A.

Suppose now that any subalgebra A_f, where f is not a constant, is (isometrically) isomorphic to $A \neq \mathbb{C}$. It is easy to see that A is infinitely dimensional. If the algebra A contains an idempotent f, then the algebra generated by f is finite-dimensional. Thus A does not contain idempotents and therefore the space M_A of its maximal ideals is connected (see [B, 1.4.10]).

Since A is a unital uniform algebra, the space M_A is compact and the Gelfand transform G is an isometry. Hence M_A contains more than one point because A is not isomorphic to $\mathbb{C}$. If f is a generator of A, then $\hat{f} = G(f)$ continuously maps a compact space M_A onto a Hausdorff space $K_f = \hat{f}(M_A)$. Since f is a generator, $\hat{f}$ is one-to-one and, therefore, a homeomorphism. As a connected closed set in the plane with more than one point the set K_f and consequently also M_A, contain the Cantor set.

The Gelfand transform maps A onto a closed subalgebra $\hat{A}$ of $C(M_A)$ that separates points of M_A. By Theorem 1.3 there is an element $\hat{f} \in \hat{A}$ such that $\hat{f}(M_A) = \overline{U}$.

Let $\hat{f} = G(f)$, $f \in A$, and let A_f be the subalgebra of A generated by f. If $p(z)$ is a polynomial of the complex variable z and $g = p(f)$, then

$$\|g\| = \|\hat{g}\|_{M_A} = \|p(\hat{f})\|_{M_A} = \|p\|_{\overline{U}}.$$

So the mapping $p(f) \to p(z)$ is an isometry and, therefore, it extends to an isometrical isomorphism of A_f and Δ. But A is (isometrically) isomorphic to A_f and, consequently, to Δ. $\square$

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