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TOEPLITZ ALGEBRAS AND C*-ALGEBRAS ARISING FROM REDUCED (FREE) GROUP C*-ALGEBRAS

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ABSTRACT. Assume that Γ is a free group on n generators, where $2 \leq n < +\infty$. Let Ω be an infinite subset of Γ such that $\Gamma \setminus \Omega$ is also infinite, and let P be the projection on the subspace $l^2(\Omega)$ of $l^2(\Gamma)$. We prove that, for some choices of Ω, the C*-algebra $C_r^*(\Gamma, P)$ generated by the reduced group C*-algebra $C_r^*\Gamma$ and the projection P has exactly two non-trivial, stable, closed ideals of real rank zero. We also give a detailed analysis of the Toeplitz algebra generated by the restrictions of operators in $C_r^*(\Gamma, P)$ on the subspace $l^2(\Omega)$.

Introduction

Throughout this article, we assume, except otherwise specified, that Γ is a free group of n generators, say $\{g_1, g_2, \ldots, g_n\}$, and e is the unit of Γ, where $2 \leq n < +\infty$. Each element of Γ is a *reduced* word $g_{i_1}^{n_1} g_{i_2}^{n_2} \ldots g_{i_m}^{n_m}$ in the sense that it does not contain any factor of the forms gg^{-1} and $g^{-1}g$, where $n_i \in \mathbb{Z}$ (the group of all integers). Let $\{f_g : g \in \Gamma\}$ be a standard orthonormal basis of the Hilbert space $l^2(\Gamma)$ of all complex valued, square-summable sequences indexed by Γ. Let $\lambda : \Gamma \longrightarrow \mathcal{L}(l^2(\Gamma))$ be the left regular representation of Γ on $\mathcal{L}(l^2(\Gamma))$, where $\mathcal{L}(\mathcal{H})$ denotes the algebra of all bounded operators on a Hilbert space $\mathcal{H}$ as usual, and $\lambda(g) := U(g)$ is a unitary operator defined by $U(g)(f_h) = f_{g^{-1}h}$ for all $g, h \in \Gamma$. The reduced group C*-algebra $C_r^*\Gamma$ is the norm closure of the group ring $\mathbb{C}[\Gamma]$ consisting of all linear combinations $\{\sum_{i=1}^n \alpha_i U(h_i) : h_i \in \Gamma,\ \alpha_i \in \mathbb{C}, \text{and } n \in \mathbb{N}\}$; in other words, $C_r^*\Gamma$ is the C*-subalgebra of $\mathcal{L}(l^2(\Gamma))$ generated by the group $\lambda(\Gamma) = \{U(g) : g \in \Gamma\}$.

The purpose of this article is to investigate the structure of the C*-algebra generated by the *reduced group C*-algebra* $C_r^*\Gamma$ and a projection P onto a subspace of the form $l^2(\Omega)$, denoted by $C_r^*(\Gamma, P)$, where both Ω and $\Gamma \setminus \Omega$ are infinite subsets of Γ. We will consider the specific cases when Ω is equal to one of the following sets:

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(1) $\Gamma_+ := \{g_{i_1}^{n_1} g_{i_2}^{n_2} \dots g_{i_j}^{n_j} : \quad j, n_1, n_2, \dots, n_j \in \mathbb{N}\}$;
(2) $\Gamma'_+ := \Gamma_+ \cup \{e\}$;
(3) Γ_0, a nontrivial subgroup of Γ; and
(4) Γ_A, the union of $\{e, g_1, g_2, \dots, g_n\}$ and the set of all admissible reduced words with respect to A ([8]), where A is an $n \times n$ irreducible matrix with entries in $\{0, 1\}$ ([6]).

It turns out that the cases (1) and (2) result in the same C*-algebra $C_r^*(\Gamma, P_+)$, which has exactly two nontrivial, stable, closed ideals; one is the algebra $\mathcal{K}(l^2(\Gamma))$ consisting of all compact operators on $l^2(\Gamma)$ and the other is generated by P_+ and denoted by $\mathcal{I}_{P_+}$ (where P_+ is the projection onto $l^2(\Gamma_+)$). Furthermore, $\mathcal{I}_{P_+}/\mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_n \otimes \mathcal{K}$, and $\mathcal{I}_{P_+}$ has real rank zero, where $\mathcal{O}_n$ is the Cuntz algebra. The case (3) yields a C*-algebra $C_r^*(\Gamma, P_0)$ that has a nontrivial, stable, closed ideal, that is, $C_r^*\Gamma_0 \otimes \mathcal{K}$. The case (4) results in a C*-algebra $C_r^*(\Gamma, R)$ that has exactly two non-trivial, stable, closed ideals; one is $\mathcal{K}(l^2(\Gamma))$ and the other is generated by R and denoted by $\mathcal{I}_R$ (where R is the projection onto the subspace $l^2(\Gamma_A)$). In addition, $\mathcal{I}_R/\mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_A \otimes \mathcal{K}$, and $\mathcal{I}_R$ has real rank zero, where $\mathcal{O}_A$ is the Cuntz-Krieger algebra associated with A. Moreover, we will give a necessary and sufficient condition for the equality $\mathcal{I}_P = C_r^*(\Gamma, P)$.

The case $n = +\infty$ (i.e., the case when Γ is the free group on infinitely many generators) and the cases when Γ is any free product of finite and infinite cyclic groups have been studied in [16]; the resulting C*-algebras $C_r^*(\Gamma, P_+)$ have different structures. In [17] we proved that $C_r^*(\Gamma, P)$ can be a purely infinite simple C*-algebra (and hence has real rank zero) for some other choices of P (there Γ can be more general free products of finite or infinite cyclic groups). Thus, there are indeed many interesting C*-algebras in the class

$$\{C_r^*(\Gamma, P_\Omega) : \Omega \subset \Gamma, \quad |\Omega| = |\Gamma \setminus \Omega| = +\infty\}.$$

It appears to be an interesting, but difficult problem to classify, up to *-isomorphism, all C*-algebras of the form $C_r^*(\Gamma, P_\Omega)$.

This article is self-contained with only few references needed. More references are provided only for the convenience of the reader in searching for some relevant literature.

0. Preliminaries

Let Ω be an infinite subset of Γ such that $\Gamma \setminus \Omega$ is also an infinite subset of Γ, and let P be the projection in $\mathcal{L}(l^2(\Gamma))$ onto the subspace $l^2(\Omega)$ of $l^2(\Gamma)$. It easily follows from the definition that $U(h)^* = U(h^{-1})$ for $h \in \Gamma$,

$U(h_1h_2) = U(h_2)U(h_1)$ for $h_1, h_2 \in \Gamma$, and for any $g \in \Gamma$

$$U(g)^*PU(g)f_h = \begin{cases} f_h & \text{if } h \in g\Omega, \\ 0 & \text{if } h \notin g\Omega; \end{cases}$$

$$U(g)PU(g)^*f_h = \begin{cases} f_h & \text{if } h \in g^{-1}\Omega, \\ 0 & \text{if } h \notin g^{-1}\Omega. \end{cases}$$

Hence $U(g)^*PU(g)$ and $U(g)PU(g)^*$ are the projections onto the subspaces $l^2(g\Omega)$ and $l^2(g^{-1}\Omega)$, respectively. As a natural analogue of the classic Toeplitz operators associated with $\Omega := \mathbb{Z}^+ \subset \Gamma := \mathbb{Z}$, for each $g \in \Gamma$ one defines a Toeplitz operator T_g as follows:

$$T_g := PU(g)P \ \in \mathcal{L}(l^2(\Omega)).$$

Obviously,

$$T_g(f_h) = Pf_{g^{-1}h} = \begin{cases} f_{g^{-1}h} & \text{if } h \in g\Omega \cap \Omega, \\ 0 & \text{if } h \notin g\Omega \cap \Omega. \end{cases}$$

Thus, $\{T_g : g \in \Gamma\}$ is a set of partial isometries on $l^2(\Omega)$ such that

$$\begin{aligned} &T_g^* = T_{g^{-1}}, \\ &T_g^*T_g \text{ is the projection onto } \ l^2(g\Omega \cap \Omega), \text{ and} \\ &T_gT_g^* \text{ is the projection onto } \ l^2(g^{-1}\Omega \cap \Omega). \end{aligned}$$

The C*-algebra $\mathcal{T}_P$ generated by $\{T_g : g \in \Omega\}$ is called the *Toeplitz C*-algebra associated with* Ω (cf. [7], [8], [9]). The hereditary C*-subalgebra $\mathcal{A}_P := PC_r^*(\Gamma, P)P$ is often called a *corner algebra supported by* P. It is obvious that $\mathcal{A}_P$ is generated by $\{T_g : g \in \Gamma\}$ and hence contains $\mathcal{T}_P$. We will later prove that in some cases the corner $\mathcal{A}_P$ is actually equal to $\mathcal{T}_P$.

Notice that all of the above observations remain valid when Γ is any free product of cyclic groups of finite or infinite order, consisting of all reduced words of elements in the groups.

1. A criterion for $\mathcal{I}_\mathbf{P} = \mathbf{C}_\mathbf{r}^*(\mathbf{\Gamma}, \mathbf{P})$

In this section, we investigate under what condition on Ω the closed ideal $\mathcal{I}_P$ of $C_r^*(\Gamma, P)$ generated by P is equal to $C_r^*(\Gamma, P)$. The following is a necessary and sufficient condition for this equality.

1.1. Theorem. *Let* Γ *be any free product of cyclic groups with finite or infinite order. Then* $\mathcal{I}_P = C_r^*(\Gamma, P)$ *if and only if there exist finitely many elements* $h_1, h_2, \ldots, h_m \in \Gamma$ *such that* $\Gamma = \bigcup_{j=1}^m h_j\Omega$.

Before proving this criterion, we need to deal with some preliminary matters. The two operations $\vee$ and $\wedge$ on projections are defined in a von Neumann algebra but not in a C*-algebra in general, for the resulting projections $Q_1 \vee Q_2$

and $Q_1 \wedge Q_2$ may lie outside the C*-algebra. Nevertheless, $\vee$ and $\wedge$ can be partially executed in this particular C*-algebra $C_r^*(\Gamma, P)$.

1.2. Lemma.

(i) *The projections* $U(h_1)PU(h_1)^*$ *and* $U(h_2)PU(h_2)^*$ *commute for any two elements* $h_1, h_2 \in \Gamma$.

(ii) $U(h_1)PU(h_1)^* \vee \cdots \vee U(h_m)PU(h_m)^*$ *and* $U(h_1)PU(h_1)^* \wedge \cdots \wedge U(h_m)PU(h_m)^*$ *are projections in* $C_r^*(\Gamma, P)$ *for any finitely many elements* $h_1, h_2, \ldots, h_m \in \Gamma$.

Proof. (i) This is immediate, since $U(h_1)PU(h_1)^*$ and $U(h_2)PU(h_2)^*$ are projections onto the subspaces $l^2(h_1^{-1}\Omega)$ and $l^2(h_2^{-1}\Omega)$.

(ii) $U(h_1)PU(h_1)^*U(h_2)PU(h_2)^*$ is the projection onto $l^2(h_1^{-1}\Omega \cap h_2^{-1}\Omega)$, that is in $C_r^*(\Gamma, P)$. By definition,

$$\begin{aligned} U(h_1)PU(h_1)^* \vee U(h_2)PU(h_2)^* &= U(h_1)PU(h_1)^* + U(h_2)PU(h_2)^* \\ &\quad - U(h_1)PU(h_1)^*U(h_2)PU(h_2)^*, \\ U(h_1)PU(h_1)^* \wedge U(h_2)PU(h_2)^* &= U(h_1)PU(h_1)^*U(h_2)PU(h_2)^*, \end{aligned}$$

which are both projections in $C_r^*(\Gamma, P)$. The general conclusion follows by induction. ☐

1.3. Proof of Theorem 1.1. First, assume that $\Gamma = \bigcup_{j=1}^m h_j\Omega$, where $h_1, h_2, \ldots, h_m \in \Gamma$. We show that the identity I of $C_r^*(\Gamma, P)$ is in $\mathcal{I}_P$, and hence $\mathcal{I}_P = C_r^*(\Gamma, P)$. Clearly,

$$U(h_1)PU(h_2)^* \vee U(h_2)PU(h_2)^* \vee \cdots \vee U(h_m)PU(h_m)^* = I,$$

since the projection on the left-hand side of the above equality is onto the subspace $l^2(\bigcup_{j=1}^m h_j\Omega)$, that is, the whole space $l^2(\Gamma)$. Thus, $\mathcal{I}_P = C_r^*(\Gamma, P)$ by the above lemma.

Secondly, assume that $\Gamma \neq \bigcup_{j=1}^m h_j\Omega$ for any finitely many elements $h_1, h_2, \ldots, h_m$ of Γ. Then $\Gamma \setminus \bigcup_{j=1}^m h_j\Omega$ must be an infinite subset of Γ. We show that the identity I is not in $\mathcal{I}_P$. To do so, we suppose $I \in \mathcal{I}_P$ and then reach a contradiction.

Since the linear span of $\{U(g) : g \in \Gamma\}$ is norm dense in $C_r^*\Gamma$, it is clear that the linear span $\mathcal{L}'$ of all products of elements in

$$\{PU(g), U(g)P, PU(g)(I-P), (I-P)U(g)P : g \in \Gamma\}$$

is norm dense in $\mathcal{I}_P$. Take a linear combination X from $\mathcal{L}'$ such that

$$\|X - I\| < \delta < 1.$$

Then

$$\|(I-P)X(I-P) - (I-P)\| < \delta.$$

Obviously, the ith term of $(I-P)X(I-P)$ can be written in the form

$$X_i := \alpha_i(I-P)U(k_i)P_1U(h_{i1})P_2U(h_{i2})P_3\dots P_{l_i}U(h_{il_i})P_{l_{i+1}}U(k_i')(I-P),$$

where $\alpha_i \in \mathbb{C}$ and P_j is equal to either P or $I-P$ for $1 \le j \le i+1$ and at least one of the P_j's is equal to P, and $k_i, k_i', h_{ij} \in \Gamma$ for $0 \le j \le l_i$. Let P_{j_0} be the first term P from the left occurring in the above product. It is clear that the range projection of X_i is a subprojection of the range projection of

$$\begin{aligned}&(I-P)U(k_i)P_1U(h_{i1})P_2U(h_{i2})P_3\dots P_{j_0-1}U(h_{j_0-1,l_{j_0-1}})P_{j_0}\\ &\qquad = (I-P)U(k_i)(I-P)U(h_{i1})(I-P)\dots(I-P)U(h_{j_0-1,l_{j_0-1}})P_{j_0}.\end{aligned}$$

Let k_i' be the reduced form of the product $h_{j_0-1,l_{j_0-1}}\dots h_{i2}h_{i1}k_i$. Clearly, the range projection of

$$U(k_i)(I-P)U(h_{i1})(I-P)\dots(I-P)U(h_{j_0-1,l_{j_0-1}})P_{j_0}$$

is a subprojection of $U(k_i')PU(k_i')^*$ for $1 \le i \le m$. Take an element $h \in \Gamma \setminus \bigcup_{j=1}^m k_j'^{-1}\Omega \setminus \Omega$ and any element $g \in \Omega$. Then $h \in hg^{-1}\Omega$. It follows that

$$U(gh^{-1})^*PU(gh^{-1})(I - U(k_1')PU(k_1')^* \vee \dots \vee U(k_m')PU(k_m')^* \vee P)$$

is a nonzero projection in $C_r^*(\Gamma, P)$, denoted by R. Furthermore, by the construction R is a projection in $\mathcal{I}_P$ such that $R(I-P) = R$ and $R(I-P)X(I-P) = 0$. We immediately reach the following contradiction:

$$1 = \|R\| = \|R((I-P)X(I-P) - (I-P))\| < \delta < 1.$$

Therefore, $I \notin \mathcal{I}_P$, and hence $\mathcal{I}_P \ne C_r^*(\Gamma, P)$.

For a subgroup Γ_0 of Γ, one denotes as usual the *index of* Γ_0 *in* Γ by $[\Gamma : \Gamma_0]$, the cardinality of the set of all left cosets $\{h\Gamma_0 : h \in \Gamma\}$. Let P_0 be the projection onto the subspace $l^2(\Gamma_0)$.

1.4. Corollary. *Let Γ_0 be an infinite subgroup of Γ such that $\Gamma \setminus \Gamma_0$ is infinite. Then $\mathcal{I}_{P_0} = C_r^*(\Gamma, P_0)$ if and only if $[\Gamma : \Gamma_0] < +\infty$.*

Proof. This is immediate from Theorem 1.1. □

1.5. Corollary. *Assume that Γ_0 is the subgroup of Γ generated by a subset Ω. If $[\Gamma : \Gamma_0] = +\infty$ and P is the projection onto $l^2(\Omega)$, then $\mathcal{I}_P \ne C_r^*(\Gamma, P)$.*

Proof. Let P_0 be the projection onto $l^2(\Gamma_0)$. If $\mathcal{I}_P = C_r^*(\Gamma, P)$, then $\mathcal{I}_{P_0} = C_r^*(\Gamma, P)$ by Theorem 1.1, since $\Omega \subset \Gamma_0$. □

2. Real rank of $\mathcal{T}_+$ and $\mathcal{T}'_+$

From now on, we will discuss some specific subsets Ω of the free group Γ on finitely many generators ($2 \leq n < +\infty$). In this and the next section, we take Ω to be $\Gamma'_+ = \Gamma_+ \cup \{e\}$, where

$$\Gamma_+ := \left\{ g_{i_1}^{n_1} g_{i_2}^{n_2} \dots g_{i_m}^{n_m} \in \Gamma : m, n_1, n_2, \dots, n_m \in \mathbb{N} \right\},$$

and we let P_+ and P'_+ be the projections onto the subspaces $l^2(\Gamma_+)$ and $l^2(\Gamma'_+)$, respectively.

We first consider $C_r^*(\Gamma, P'_+)$. For $h \in \Gamma'_+$ one observes immediately that

$$T_h T_h^* = P'_+ \text{ and } T_h^* T_h \text{ is the projection onto the subspace } l^2(h\Gamma'_+).$$

The C*-subalgebra $\mathcal{T}'_+$ of $\mathcal{L}(l^2(\Gamma'_+))$ generated by $\{T_g : g \in \Gamma'_+\}$ involves the following extension (see [7]):

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma'_+)) \longrightarrow \mathcal{T}'_+ \longrightarrow \mathcal{O}_n \longrightarrow 0,$$

where $\mathcal{K}(l^2(\Gamma'_+))$ is the algebra of all compact operators on $l^2(\Gamma'_+)$, and $\mathcal{O}_n$ is the Cuntz algebra generated by n isometries $\{S_i\}_{i=1}^n$ such that

$$S_1 S_1^* + S_2 S_2^* + \dots + S_n S_n^* = I.$$

The corner hereditary C*-subalgebra $\mathcal{A}_{P'_+}$, i.e., the corner $P'_+ C_r^*(\Gamma, P'_+) P'_+$, is generated by

$$\left\{ T_g := P'_+ U(g) P'_+ : g \in \Gamma \right\}.$$

It is obvious that $\mathcal{T}'_+$ is a *-subalgebra of $\mathcal{A}_{P'_+}$, $\mathcal{A}_{P'_+}$ is a C*-subalgebra of $\mathcal{I}_{P'_+}$, and $\mathcal{A}_{P'_+}$ generates $\mathcal{I}_{P'_+}$ as a closed ideal. We will clarify the relation between $\mathcal{T}'_+$ and $\mathcal{A}_{P'_+}$ by analyzing the elements of $\mathcal{A}_{P'_+}$, and then determine the (closed) ideal structure of $C_r^*(\Gamma, P'_+)$.

2.0. Proposition. $\mathcal{I}_{P'_+} \neq C_r^*(\Gamma, P'_+)$.

Proof. This is immediate from Theorem 1.1. □

The main result of this section is as follows:

2.1. Theorem.
(i) $\mathcal{T}'_+ = \mathcal{A}_{P'_+}$.
(ii) *The short sequence* $0 \longrightarrow \mathcal{K}(l^2(\Gamma'_+)) \longrightarrow \mathcal{T}'_+ \longrightarrow \mathcal{O}_n \longrightarrow 0$ *is exact.*
(iii) $RR(\mathcal{A}_{P'_+}) = RR(\mathcal{I}_{P'_+}) = 0$.

To prove this theorem, we need the following lemmas.

2.2. Lemma. *Suppose that $g \in \Gamma$ is represented by a reduced word $g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_n}^{\epsilon_n}$, where the ϵ_i's are nonzero integers. Then $T_g \neq 0$ if and only if there is $0 \leq m \leq n$ such that $\epsilon_i > 0$ for $1 \leq i \leq m$ and $\epsilon_i < 0$ for $m+1 \leq i \leq n$,* *where the cases $m = 0$ and $m = n$ are to be interpreted as $\epsilon_i < 0$ for $1 \le i \le n$ and $\epsilon_i > 0$ for $1 \le i \le n$, respectively.*

Proof. If $\epsilon_i < 0$ for all $1 \le i \le n$, then $T_g^* T_g = P'_+$ and $T_g T_g^*$ is the projection onto the subspace $l^2(g_{i_n}^{-\epsilon_n} g_{i_{n-1}}^{-\epsilon_{n-1}} \dots g_{i_2}^{-\epsilon_2} g_{i_1}^{-\epsilon_1} \Gamma'_+)$; note that $g_{i_n}^{-\epsilon_n} g_{i_{n-1}}^{-\epsilon_{n-1}} \dots g_{i_2}^{-\epsilon_2} g_{i_1}^{-\epsilon_1} \Gamma'_+$ is the set of all those elements of Γ'_+ that begin with $g_{i_n}^{-\epsilon_n} g_{i_{n-1}}^{-\epsilon_{n-1}} \dots g_{i_2}^{-\epsilon_2} g_{i_1}^{-\epsilon_1}$. If $\epsilon_i > 0$ for all $1 \le i \le n$, then $T_g T_g^* = P'_+$ and $T_g^* T_g$ is the projection onto the subspace associated with the subset

$$\left\{ h \in \Gamma'_+ : h \text{ starts with } \; g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_n}^{\epsilon_n} \right\}.$$

If there is m such that $1 < m < n$, $\epsilon_i > 0$ for $1 \le i \le m$, and $\epsilon_i < 0$ for $m < i \le n$, then $T_g T_g^*$ is the projection onto the subspace associated with

$$\left\{ h \in \Gamma'_+ : h \text{ starts with } \; g_{i_n}^{-\epsilon_n} g_{i_{n-1}}^{-\epsilon_{n-1}} \dots g_{i_{m+1}}^{-\epsilon_{m+1}} \right\};$$

and $T_g^* T_g$ is the projection onto the subspace associated with

$$\left\{ h \in \Gamma'_+ : h \text{ starts with } \; g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_m}^{\epsilon_m} \right\}.$$

This proves the direction "if" of the lemma.

We now verify the direction "only if". Assume that $T_g \neq 0$ and that ϵ_m is the last positive power occurring in the reduced word $g = g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_n}^{\epsilon_n}$. It suffices to show that $\epsilon_i > 0$ for any $1 \le i \le m$. If $\epsilon_i < 0$ for some $1 \le i \le m-1$, then $g\Gamma'_+ \cap \Gamma'_+ = \emptyset$; but this would imply $T_g = 0$. □

2.3. Lemma. *Let $g = g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_m}^{\epsilon_m} \in \Gamma'_+$, where $\epsilon_i > 0$ for $1 \le i \le m$. Then:*

(i) *The unitary operator $U(g)$ can be written, with respect to the decomposition $P'_+ \oplus P'^{\perp}_+ = I$, in the matrix form*

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix},$$

where $A = P'_+ U(g) P'_+$, $C = P'^{\perp}_+ U(g) P'_+$, and $D = P'^{\perp}_+ U(g) P'^{\perp}_+$.

(ii) $T_{g_1^{\epsilon_1} g_2^{\epsilon_2} \dots g_m^{\epsilon_m}} = T_{g_m^{\epsilon_m}} T_{g_{m-1}^{\epsilon_{m-1}}} \dots T_{g_2^{\epsilon_2}} T_{g_1^{\epsilon_1}}$.

Proof. (i) Since $U(g)^* P'_+ U(g)$ is the projection onto the subspace $l^2(g\Gamma'_+)$ and $g\Gamma'_+ \subset \Gamma'_+$, it follows that $P'^{\perp}_+ U(g)^* P'_+ U(g) P'^{\perp}_+ = 0$. Thus $P'_+ U(g) P'^{\perp}_+ = 0$.

(ii) It is easily checked with a simple matrix multiplication that $P'_+ U(g_1^{\epsilon_1} g_2^{\epsilon_2}) P'_+ = P'_+ U(g_2^{\epsilon_2}) P'_+ U(g_1^{\epsilon_1}) P'_+$ whenever $\epsilon_i > 0$. The general situation follows by induction. □

2.4. Lemma. *Assume that $T_g \neq 0$, where $g = g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_n}^{\epsilon_n}$, $\epsilon_i > 0$ for $1 \le i \le m$, and $\epsilon_i < 0$ for $m+1 \le i \le n$ $(0 \le m \le n)$. Then*

$$T_g = T_{g_{i_n}}^{*-\epsilon_n} T_{g_{i_{n-1}}}^{*-\epsilon_{n-1}} \dots T_{g_{i_{m+1}}}^{*-\epsilon_{m+1}} T_{g_{i_m}}^{\epsilon_m} T_{g_{i_{m-1}}}^{\epsilon_{m-1}} \dots T_{g_{i_1}}^{\epsilon_1}.$$

Proof. By the definition we have $U(g) = U(g_{i_{m+1}}^{\epsilon_{m+1}} \dots g_{i_n}^{\epsilon_n})U(g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m})$. We claim that

$$P'_+U(g_{i_{m+1}}^{\epsilon_{m+1}} \dots g_{i_n}^{\epsilon_n})P'^{\perp}_+U(g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m})P'_+ = 0.$$

For the cases $m = 0$ and $m = n$ this equality is trivial. Assume $1 < m < n$. First, observe that the range projection of $P'^{\perp}_+U(g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m})P'_+$ is a subprojection of the projection onto the subspace associated with the set of all reduced words starting with $g_{i_m}^{-\epsilon_m} \dots g_{i_k}^{-\epsilon_k}$ for some $1 \le k \le m-1$; secondly, notice that the range projection of $U(g_{i_{m+1}}^{\epsilon_{m+1}} \dots g_{i_n}^{\epsilon_n})P'^{\perp}_+U(g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m})P'_+$ is a subprojection of the projection onto the subspace associated with the set of all reduced words starting with

$$g_{i_n}^{-\epsilon_n} \dots g_{i_{m+1}}^{-\epsilon_{m+1}} g_{i_m}^{-\epsilon_m} \dots g_{i_k}^{-\epsilon_k}$$

for some $1 \le k \le m-1$; thirdly, the set of all reduced words starting with

$$g_{i_n}^{-\epsilon_n} \dots g_{i_{m+1}}^{-\epsilon_{m+1}} g_{i_m}^{-\epsilon_m} \dots g_{i_k}^{-\epsilon_k}$$

is disjoint from Γ'_+. Thus, the above equality is proved.

Using this equality, we have

$$\begin{aligned} P'_+U(g_{i_1}^{\epsilon_1} \dots g_{i_n}^{\epsilon_n})P'_+ &= P'_+U(g_{i_{m+1}}^{\epsilon_{m+1}} \dots g_{i_n}^{\epsilon_n})P'_+U(g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m})P'_+ \\ &\quad + P'_+U(g_{i_{m+1}}^{\epsilon_{m+1}} \dots g_{i_n}^{\epsilon_n})P'^{\perp}_+U(g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m})P'_+ \\ &= P'_+U(g_{i_{m+1}}^{\epsilon_{m+1}} \dots g_{i_n}^{\epsilon_n})P'_+U(g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m})P'_+. \end{aligned}$$

Since $\epsilon_i > 0$ for $1 \le i \le m$, from the definition and the matrix form of $U(g_i)$ as given in Lemma 2.3(i) one sees that

$$\begin{aligned} T_{g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m}} &= P'_+U(g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_m}^{\epsilon_m})P'_+ \\ &= (P'_+U(g_{i_m})P'_+)^{\epsilon_m}(P'_+U(g_{i_{m-1}})P'_+)^{\epsilon_{m-1}} \dots (P'_+U(g_{i_1})P'_+)^{\epsilon_1} \\ &= T_{g_{i_m}}^{\epsilon_m} T_{g_{i_{m-1}}}^{\epsilon_{m-1}} \dots T_{g_{i_1}}^{\epsilon_1}. \end{aligned}$$

Since $U(g^{-1}) = U(g)^*$ for any $g \in \Gamma$ and $\epsilon_i < 0$ for $m+1 \le i \le n$, it is again easily seen from Lemma 2.3(i) that

$$\begin{aligned} T_{g_{i_{m+1}}^{\epsilon_{m+1}} \dots g_{i_n}^{\epsilon_n}} &= P'_+U(g_{i_{m+1}}^{\epsilon_{m+1}} \dots g_{i_n}^{\epsilon_n})P'_+ \\ &= (P'_+U(g_{i_n}^{-1})P'_+)^{-\epsilon_n} \dots (P'_+U(g_{i_{m+1}}^{-1})P'_+)^{-\epsilon_{m+1}} \\ &= T_{g_{i_n}^{-1}}^{-\epsilon_n} \dots T_{g_{i_{m+1}}^{-1}}^{-\epsilon_{m+1}} \\ &= T_{g_{i_n}}^{*-\epsilon_n} T_{g_{n-1}}^{*-\epsilon_{n-1}} \dots T_{g_{i_{m+1}}}^{*-\epsilon_{m+1}}. \end{aligned}$$

Therefore, we have the equality

$$T_g = T_{g_{i_n}}^{*-\epsilon_n} T_{g_{i_{n-1}}}^{*-\epsilon_{n-1}} \dots T_{g_{i_{m+1}}}^{*-\epsilon_{m+1}} T_{g_{i_m}}^{\epsilon_m} T_{g_{i_{m-1}}}^{\epsilon_{m-1}} \dots T_{g_{i_1}}^{\epsilon_1}. \qquad \square$$

2.5. Proof of Theorem 2.1. (i) By definition the C*-algebra $\mathcal{T}'_+$ is generated by $\{T_g : g \in \Gamma'_+\}$, i.e., $\mathcal{T}'_+$ is the norm closure of the linear span of all possible words of elements in $\{T_h^*; h \in \Gamma'_+\} \cup \{T_{h'} : h, h' \in \Gamma'_+\}$. From Lemmas 2.2 and 2.4 one sees that $\mathcal{T}'_+$ coincides with the corner algebra $\mathcal{A}_{P'_+}$ that is generated by the apparently larger set $\{T_g : g \in \Gamma\}$.

(ii) By a result from [7] one has the *-isomorphism

$$\mathcal{T}'_+/\mathcal{K}(l^2(\Gamma'_+)) \cong \mathcal{O}_n$$

via by the exact sequence $0 \longrightarrow \mathcal{K}(l^2(\Gamma'_+)) \longrightarrow \mathcal{T}'_+ \longrightarrow \mathcal{O}_n \longrightarrow 0$. Since $\mathcal{A}_{P'_+} = \mathcal{T}'_+$, we have therefore $\mathcal{A}_{P'_+}/\mathcal{K}(l^2(\Gamma'_+)) \cong \mathcal{O}_n$, and the following sequence is also exact:

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma'_+)) \longrightarrow \mathcal{A}_{P'_+} \longrightarrow \mathcal{O}_n \longrightarrow 0.$$

(iii) Since $K_1(\mathcal{K}(l^2(\Gamma'_+)) = 0$, $RR(\mathcal{K}(l^2(\Gamma'_+))) = 0$, and $RR(\mathcal{O}_n) = 0$ (see [13] or [15]), it follows from [2, 3.14] or [15, 2.4] that $RR(\mathcal{A}_{P'_+}) = 0$. Thus, $RR(\mathcal{A}_{P'_+} \otimes \mathcal{K}) = 0$ (see [2, 2.5]). Since $\mathcal{A}_{P'_+}$ is a full corner of $\mathcal{I}_{P'_+}$ (i.e., $\mathcal{A}_{P'_+}$ generates $\mathcal{I}_{P'_+}$ as a closed ideal), by [1, 2.8] one has

$$\mathcal{I}_{P'_+} \otimes \mathcal{K} \cong \mathcal{A}_{P'_+} \otimes \mathcal{K}.$$

Therefore, $RR(\mathcal{I}_{P'_+} \otimes \mathcal{K}) = 0$, and hence $RR(\mathcal{I}_{P'_+}) = 0$.

The following is a necessary condition for the product $T_{h_1}T_{h_2}\dots T_{h_k}$ to be a nonzero operator.

2.6. Proposition. *Assume that $h_1, h_2, \dots, h_k \in \Gamma$ and $T_{h_1}, T_{h_2}, \dots, T_{h_k}$ satisfy*

$$T_{h_1}T_{h_2}\dots T_{h_k} \neq 0.$$

Then, after canceling all factors of the forms gg^{-1} or $g^{-1}g$, the element $h_k h_{k-1} \dots h_2 h_1$ can be simplified to either e or to the form $g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} \dots g_{i_l}^{\epsilon_l}$, where $\epsilon_i > 0$ for $0 \le i \le m$ and $\epsilon_i < 0$ for $m+1 \le i \le l$ (for some $m \le l$ as in Lemma 2.2).

Proof. By induction we only need to prove the lemma for $k = 2$. Because $T_e = P'_+$ is the identity of $\mathcal{T}_+$, we can assume that $h_i \neq e$ for $1 \le i \le k$.

Since $T_{h_1} \neq 0$ and $T_{h_2} \neq 0$, by Lemma 2.2 one can write

$$h_1 = g_{j_1}^{n_1} g_{j_2}^{n_2} \dots g_{j_t}^{n_t} g_{j_{t+1}}^{-n_{t+1}} g_{j_{t+2}}^{-n_{t+2}} \dots g_{j_{t_0}}^{-n_{t_0}},$$
$$h_2 = g_{k_1}^{m_1} g_{k_2}^{m_2} \dots g_{k_s}^{m_s} g_{k_{s+1}}^{-m_{s+1}} g_{k_{s+2}}^{-m_{s+2}} \dots g_{k_{s_0}}^{-m_{s_0}},$$

where $n_1, n_2, \dots, n_{t_0}$, $m_1, m_2, \dots, m_{s_0}$ are all positive integers. By definition, for the range projection $R_2 := T_{h_2}T_{h_2}^*$ of T_{h_2} there are three possibilities:

(1) When $s = 0$, R_2 is the projection onto the subspace associated with the subset

$$\left\{h \in \Gamma'_+ : h \text{ starts with } \; g_{k_{s_0}}^{m_{s_0}} \dots g_{k_2}^{m_2} g_{k_1}^{m_1}\right\};$$

(2) When $s = s_0$, $R_2 = P'_+$.

(3) When $1 \le s < s_0$, R_2 is the projection onto the subspace associated with the subset

$$\left\{h \in \Gamma'_+ : h \text{ starts with } \ g_{k_{s_0}}^{m_{s_0}} \dots g_{k_{s+1}}^{m_{s+1}}\right\}.$$

Notice that $T_{h_1}T_{h_2} = P'_+U(h_1)T_{h_2}$. In case (1) the range projection of $U(h_1)T_{h_2}$ is onto the subspace associated with the subset

$$\left\{h \in \Gamma'_+ : h \text{ starts with the reduced form of } \ h_1^{-1}h_2^{-1}\right\}.$$

If $P'_+U(h_1)T_{h_2} \neq 0$, then, by Lemma 2.2, $h_1^{-1}h_2^{-1}$ can be simplified to the required form (after canceling all factors of the form $g_jg_j^{-1}$ or $g_j^{-1}g_j$); equivalently, h_2h_1 can be simplified to the required form. In case (2) we always have $T_{h_1}T_{h_2} \neq 0$, since h_2h_1 is of the required form for any $T_{h_1} \neq 0$. In case (3) the range projection of $U(h_1)T_{h_2}$ is onto the subspace associated with the subset

$$\left\{h \in \Gamma'_+ : h \text{ starts with the reduced form of } \ h_1^{-1}g_{k_{s_0}}^{m_{s_0}} \dots g_{k_{s+1}}^{m_{s+1}}\right\}.$$

If $T_{h_1}T_{h_2} \neq 0$, then, by applying Lemma 2.2 again, $h_1^{-1}g_{k_{s_0}}^{m_{s_0}} \dots g_{k_{s+1}}^{m_{s+1}}$ can be simplified to the required form; this happens if and only if $h_1^{-1}h_2^{-1}$ can be simplified to the required form, which in turn holds if and only if h_2h_1 can be reduced to the required form. □

2.7. Corollary. *Assume that the final projection of T_{h_i} is a subprojection of the initial projection of $T_{h_{i-1}}$ for $2 \le i \le k$, and that $h_kh_{k-1}\dots h_2h_1$ can be simplified to a reduced word $g_{j_1}^{n_1}g_{j_2}^{n_2}\dots g_{j_m}^{n_m}g_{j_{m+1}}^{-n_{m+1}}\dots g_{j_l}^{-n_l}$, where $0 \le m \le l$ and $n_1, n_2, \dots, n_l$ are all non-negative integers. Then*

$$\begin{aligned} T_{h_1}T_{h_2}\dots T_{h_k} &= T_{g_{j_l}}^{*n_l}T_{g_{j_{l-1}}}^{*n_{l-1}}\dots T_{g_{j_{m+1}}}^{*n_{m+1}}T_{g_{j_m}}^{n_m}T_{g_{j_{m-1}}}^{n_{m-1}}\dots T_{g_{j_1}}^{n_1} \\ &= T^*_{g_{j_{m+1}}^{n_{m+1}}g_{j_{m+2}}^{n_{m+2}}\dots g_{j_l}^{n_l}}T_{g_{j_1}^{n_1}g_{j_2}^{n_2}\dots g_{j_m}^{n_m}}. \end{aligned}$$

Proof. This follows by combining Lemma 2.4 and Proposition 2.6. □

2.8. Remark. Assume that $h_1, h_2, \dots, h_k \in \Gamma$ are such that $T_{h_1}T_{h_2}\dots T_{h_k} \neq 0$. The reader is reminded that the relation $T_{h_1}T_{h_2}\dots T_{h_k} = T_{h_kh_{k-1}\dots h_2h_1}$ is not valid in general; thus, the condition in Proposition 2.6 is necessary, but not sufficient.

An immediate counterexample is given by $h_1 = g_1^{-3}$, $h_2 = g_1^3$, and $h_3 = g_1$; in this case,

$$T_{h_1}T_{h_2}T_{h_3} = T^*_{g_1^3}T_{g_1^3}T_{g_1} = P_{g_1^3}T_{g_1} \neq T_{g_1},$$

where $P_{g_1^3}$ is the projection onto the subspace $l^2(g_1^3\Gamma'_+)$. In fact, the initial projection of T_{g_1} is the projection P_{g_1} onto the subspace $l^2(g_1\Gamma'_+)$, while the initial projection of $P_{g_1^3}T_{g_1}$ is $P_{g_1^4}$, the projection onto $l^2(g_1^4\Gamma'_+)$.

3. Ideal structure of $\mathbf{C^*_r(\Gamma, P'_+)}$

In this section we will determine all non-trivial closed ideals of $C^*_r(\Gamma, P'_+)$ and the structure of $\mathcal{I}_{P'_+}$. The main result is the following theorem.

3.1. Theorem.

(i) *The only nontrivial closed ideals of* $C^*_r(\Gamma, P'_+)$ *are* $\mathcal{K}(l^2(\Gamma))$ *and* $\mathcal{I}_{P'_+}$.

(ii) $\mathcal{I}_{P'_+} \cong \mathcal{I}_{P'_+} \otimes \mathcal{K}$ *and* $\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_n \otimes \mathcal{K}$.

Clearly, $C^*_r(\Gamma, P'_+)$ contains $\mathcal{K}(l^2(\Gamma))$ as a closed ideal, since $\mathcal{T}'_+$ contains a rank one projection onto the subspace spanned by f_e. We prove the remaining assertions with the following lemmas.

3.2. Lemma. *The following short sequence is exact:*

$$0 \longrightarrow \mathcal{I}_{P'_+} \longrightarrow C^*_r(\Gamma, P'_+) \longrightarrow C^*_r\Gamma \longrightarrow 0.$$

Proof. To prove the exactness of the above short sequence, one only needs to show that the canonical map from $C^*_r(\Gamma, P'_+)$ to the quotient

$$C^*_r(\Gamma, P'_+)/\mathcal{I}_{P'_+}$$

is injective. In fact, since $C^*_r\Gamma$ is simple [10], each nonzero element of $C^*_r\Gamma$ generates $C^*_r\Gamma$ as a closed ideal. If a nonzero element Y of $C^*_r\Gamma$ is in $\mathcal{I}_{P'_+}$, then the closed ideal generated by Y, that is, $C^*_r\Gamma$, would be in $\mathcal{I}_{P'_+}$. This contradicts the fact that $\mathcal{I}_{P'_+}$ is a non-trivial closed ideal of $C^*_r\Gamma$. □

3.3. Lemma. $\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_n \otimes \mathcal{K}$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma)) \longrightarrow \mathcal{I}_{P'_+} \longrightarrow \mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma)) \longrightarrow 0.$$

Since $P'_+\mathcal{I}_{P'_+}P'_+ = \mathcal{A}_{P'_+}$, by [1, 2.8] it follows that

$$\mathcal{A}_{P'_+} \otimes \mathcal{K} \cong \mathcal{I}_{P'_+} \otimes \mathcal{K}.$$

By Theorem 2.1, $\mathcal{A}_{P'_+}/\mathcal{K}(l^2(\Gamma'_+)) \cong \mathcal{O}_n$. Since $\mathcal{O}_n$ is simple [4], it is clear that $\mathcal{K}(l^2(\Gamma))$ is the only non-trivial closed ideal of $\mathcal{I}_{P'_+}$. Let π be the Calkin map from $\mathcal{L}(l^2(\Gamma))$ to $\mathcal{L}(l^2(\Gamma))/\mathcal{K}(l^2(\Gamma))$. Then it is obvious that

$$\pi(P'_+)\left\{\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma))\right\}\pi(P'_+) = \mathcal{A}_{P'_+}/\mathcal{K}(l^2(\Gamma'_+)).$$

It follows from [1, 2.8] again (or by a direct proof) that

$$\left\{\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma))\right\} \otimes \mathcal{K} \cong \mathcal{O}_n \otimes \mathcal{K}.$$

Thus, $\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma))$ is a purely infinite, simple C*-algebra (see [4] and [15, 1.4]). By using a structural result in [15, 1.2] stating that a σ-unital (in particular, separable), purely infinite, simple C*-algebra is either unital or stable, we see that $\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma))$ is stable; i.e., we have

$$\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma)) \cong \left\{\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma))\right\} \otimes \mathcal{K},$$

since $\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma))$ is non-unital (by Proposition 2.0) and separable. Finally,

$$\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_n \otimes \mathcal{K}. \qquad \square$$

3.4. Corollary. *The *-isomorphism of Lemma 3.3 between* $\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma))$ *and* $\mathcal{O}_n \otimes \mathcal{K}$ *induces the following exact sequence:*

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma)) \longrightarrow \mathcal{I}_{P'_+} \longrightarrow \mathcal{O}_n \otimes \mathcal{K} \longrightarrow 0.$$

Proof. This is obvious. $\square$

3.5. Lemma. $\mathcal{I}_{P'_+}$ *and* $\mathcal{K}(l^2(\Gamma))$ *are the only non-trivial closed ideals of* $C_r^*(\Gamma, P'_+)$.

Proof. Using the fact that $C_r^*\Gamma$ is simple [10] and Lemma 3.2, one concludes that there is no closed ideal between $\mathcal{I}_{P'_+}$ and $C_r^*(\Gamma, P'_+)$. There is also no closed ideal between $\mathcal{K}(l^2(\Gamma))$ and $\mathcal{I}_{P'_+}$, since $\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_n \otimes \mathcal{K}$ is simple. There is obviously no other closed ideal in $C_r^*(\Gamma, P'_+)$. $\square$

To finish the proof of Theorem 3.1, it remains to show that $\mathcal{I}_{P'_+}$ is a stable C*-algebra. The following is an auxiliary lemma with a standard proof (see [14, 2.5] for similar results).

3.6. Lemma (cf. [14, 2.5]). *Assume that* $\mathcal{I}$ *is a stable closed ideal of a C*-algebra* $\mathcal{A}$ *with* $RR(\mathcal{I}) = 0$, *and assume that every projection in* $\mathcal{A}/\mathcal{I}$ *lifts to a projection in* $\mathcal{A}$. *If a projection* $\bar{R}_1 \in \mathcal{A}/\mathcal{I}$ *lifts to a projection* $R_1 \in \mathcal{A}$ *and a projection* $\bar{R}_2 \in (\bar{I} - \bar{R}_1)\mathcal{A}/\mathcal{I}(\bar{I} - \bar{R}_1)$ *is equivalent to* $\bar{R}_1$, *then* $\bar{R}_2$ *lifts to a projection* $R_2 \in (I - R_1)\mathcal{A}(I - R_1)$ *such that* $R_1 \sim R_2$.

Proof. Let $\bar{V}$ be a partial isometry in $\mathcal{A}/\mathcal{I}$ such that $\bar{V}^*\bar{V} = \bar{R}_1$ and $\bar{V}\bar{V}^* = \bar{R}_2$. Let $V \in \mathcal{A}$ be such that $\bar{V}$ is the image of V in $\mathcal{A}/\mathcal{I}$. Set $W = (I - R_1)VR_1$. Then $W - V \in \mathcal{I}$, since $\bar{R}_1\bar{R}_2 = \bar{0}$. Since the real rank of $R_1\mathcal{I}R_1$ is again zero, one can take a projection $R \in R_1\mathcal{I}R_1$ such that

$$\|(R_1 - R)(R_1 - W^*W)(R_1 - R)\| < 1.$$

Set

$$U = \{(R_1 - R)W^*W(R_1 - R)\}^{-1/2}\, W^*.$$

Then $UU^* = R_1 - R$ and $U^*U \le I - R_1$. Furthermore, from the construction it is easy to see that the image of U^*U in $\mathcal{A}/\mathcal{I}$ is $\bar{R}_2$. Since $\mathcal{I}$ is stable, one can find a projection $R' \in (I - R_1 - U^*U)\mathcal{I}(I - R_1 - U^*U)$ such that $R \sim R'$. Let V_1 be a partial isometry in $\mathcal{I}$ such that $V_1^*V_1 = R'$ and $V_1V_1^* = R$.

Set $W_0 = U + V_1$. Then $W_0 W_0^* = R_1$ and $W_0^* W_0 = U^*U \oplus R' := R_2$, as desired. $\square$

3.7. LEMMA. $\mathcal{I}_{P'_+} \cong \mathcal{I}_{P'_+} \otimes \mathcal{K}$.

Proof. Consider $\mathcal{I}_{P'_+}/\mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_n \otimes \mathcal{K}$. Let 1 denote the identity of $\mathcal{O}_n$, and let $\{e_{ij}\}$ be the set of matrix units of $\mathcal{K}$. By repeatedly applying Lemma 3.6, one can lift the projections $1 \otimes e_{ii}$ of $\mathcal{O}_n \otimes \mathcal{K}$ to mutually orthogonal projections $P_1, P_2, \dots P_n, \dots$ of $\mathcal{I}_{P'_+}$ that are all equivalent in $\mathcal{I}_{P'_+}$. Then $(I - \sum_{i=1}^{\infty} P_i)\mathcal{I}_{P'_+}(I - \sum_{i=1}^{\infty} P_i) \subset \mathcal{K}(l^2(\Gamma))$, where the reader is reminded that the infinite sums above and below are taken in the corresponding multiplier algebras instead of the underlying C*-algebras. Take mutually orthogonal, one-dimensional projections $\{Q_k\}$ in $\mathcal{I}_{P'_+}$ such that

$$I - \sum_{i=1}^{\infty} P_i = \sum_k Q_k$$

(where the sum $\sum_k Q_k$ may contain a finite or infinite number of terms). Take a one-dimensional subprojection R_i of P_i for each $i \geq 1$ such that all $P_i - R_i$ $(i \geq 1)$ are still equivalent in $\mathcal{I}_{P'_+}$; this can be done by taking a one-dimensional subprojection R_1 of P_1, and letting R_i $(i \geq 2)$ be the one-dimensional subprojection of P_i under the equivalence of P_1 and P_i. Write $I - \sum_{i=1}^{\infty}(P_i - R_i) = \sum_{j=1}^{\infty} R'_j$, where

$$\left\{R'_j : j \in \mathbb{N}\right\} = \{Q_k : k\} \cup \{R_i : i \in \mathbb{N}\}.$$

Set $P'_i = (P_i - R_i) \oplus R'_i$ for $i \geq 1$. Then all P'_i are mutually orthogonal projections in $\mathcal{I}_{P'_+}$ and they are still mutually equivalent in $\mathcal{I}_{P'_+}$. Also,

$$\sum_{i=1}^{\infty} P'_i = I.$$

Then it is clear that

$$\mathcal{I}_{P'_+} \cong (P'_1 \mathcal{I}_{P'_+} P'_1) \otimes \mathcal{K}.$$

Since P_1 generates $\mathcal{I}_{P'_+}$ as a closed ideal, one sees from [1, 2.8] that

$$\mathcal{I}_{P'_+} \cong \mathcal{I}_{P'_+} \otimes \mathcal{K}. \qquad \square$$

We have completed the proof of Theorem 3.1.

3.8 REMARK-PROPOSITION. Using exactly the same arguments in the proofs of Lemma 3.6 and Lemma 3.7, one reaches the following general conclusion:

PROPOSITION. *Assume that $\mathcal{H}$ is any separable infinite-dimensional Hilbert space. If $\mathcal{A}$ is a separable C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ such that $\mathcal{K}(\mathcal{H}) \subset \mathcal{A}$ and*

$\mathcal{A}/\mathcal{K}(\mathcal{H})$ is a non-unital, purely infinite, simple C^*-algebra, then $\mathcal{A}$ is a stable C^*-algebra.

4. The C*-algebra $\mathbf{C_r^*(\Gamma, P_+)}$

After investigating the structure of $C_r^*(\Gamma, P'_+)$ in the last two sections, we now consider the relation between $C_r^*(\Gamma, P'_+)$ and $C_r^*(\Gamma, P_+)$, where P_+ is the projection onto the subspace $l^2(\Gamma_+)$. The following are the conclusions:

4.1. Theorem.

(i) *$C_r^*(\Gamma, P'_+) = C_r^*(\Gamma, P_+)$, and $\mathcal{I}_{P'_+} = \mathcal{I}_{P_+}$.*

(ii) *The Toeplitz algebra $\mathcal{T}_+$ associated with P_+ coincides with the corner $\mathcal{A}_{P_+} := P_+C_r^*(\Gamma, P_+)P_+$; and $\mathcal{A}_{P_+}/\mathcal{K}(l^2(\Gamma_+)) \cong \mathcal{O}_n$.*

Proof. (i) The projection $P_1 = P_+U(g_1g_2^{-1})^*P_+U(g_1g_2^{-1})P_+$ is onto the subspace $l^2(g_1g_2^{-1}\Gamma_+ \cap \Gamma_+)$. Clearly, P_1 is the projection onto the subspace $l^2(\Gamma_+(g_1))$, since

$$g_1g_2^{-1}\Gamma_+ \cap \Gamma_+ = g_1g_2^{-1}\Gamma_+(g_2) = \Gamma_+(g_1),$$

where $\Gamma_+(g_i)$ is the set of all reduced words in Γ_+ with initial word g_i. The projection $P_2 := P_+U(g_1)^*P_+U(g_1)P_+$ is onto the subspace $l^2(g_1\Gamma_+ \cap \Gamma_+)$. Since $g_1\Gamma_+ \cap \Gamma_+ = g_1\Gamma_+ = \Gamma_+(g_1) \setminus \{g_1\}$, $P_1 - P_2$ is the one-dimensional projection P_1 onto the subspace spanned by f_{g_1}. Consequently, $\mathcal{K}(l^2(\Gamma))$ is a subalgebra of $C_r^*(\Gamma, P_+)$.

Let P_e be the one-dimensional projection onto the subspace spanned by f_e. The relation $P'_+ = P_+ + P_e$ implies that

$$P_+ \in C_r^*(\Gamma, P'_+) \quad \text{and} \quad P'_+ \in C_r^*(\Gamma, P_+).$$

Therefore, $C_r^*(\Gamma, P_+) = C_r^*(\Gamma, P'_+)$.

(ii) First, all conclusions of Lemmas 2.2, 2.3 and 2.4 remain valid if Γ'_+ is replaced by Γ_+; the details are left to the reader. Then $\mathcal{A}_{P_+} = \mathcal{T}_+$, the algebra generated by all Toeplitz operators

$$\left\{T_h := P_+U(h)P_+ : h \in \Gamma'_+\right\}.$$

Obviously, $\sum_{i=1}^n T_{g_i}^*T_{g_i} = P_+$, since Γ_+ is the disjoint union

$$\bigcup_{i=1}^n \{h \in \Gamma_+ : h \text{ starts with } g_i\}.$$

Thus, the same arguments as in [7] show that the following sequence is exact:

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma_+)) \longrightarrow \mathcal{T}_+ \longrightarrow \mathcal{O}_n \longrightarrow 0.$$

Therefore, $\mathcal{T}_+/\mathcal{K}(l^2(\Gamma_+)) \cong \mathcal{O}_n$. □

5. $\mathbf{C_r^*(\Gamma, R)}$ and the Cuntz-Krieger algebras

Assume from now on that A is not a permutation matrix. An $n \times n$ matrix $A = [t_{ij}]$ with all entries either 0 or 1 is said to be *irreducible* if for any pair (i, j) there is $k_{ij} \in \mathbb{N}$ such that the (i, j)-entry of $A^{k_{ij}}$ is nonzero. The Cuntz-Krieger algebra $\mathcal{O}_A$ is generated by nonzero partial isometries $\{S_i : i = 1, 2, \ldots, n\}$ on a separable Hilbert space such that

$$S_i^* S_i = \sum_{j=1}^{n} t_{i,j} S_j S_j^*, \quad S_l^* S_k = 0 \quad (l \neq k).$$

In particular, if A is an $n \times n$ matrix with all entries 1, then $\mathcal{O}_A = \mathcal{O}_n$. The reader can find more information about $\mathcal{O}_A$ in [6], [4], and some of the subsequent references.

Let Ω_A be the subset of Γ_+ consisting of the generators $\{g_1, g_2, \ldots, g_n\}$, the identity e, and all *admissible* reduced words with respect to A, where a reduced word $g_{i_1} g_{i_2} \ldots g_{i_m}$ ($i_j = i_k$ for $j \neq k$ is allowed) is said to be admissible with respect to A if $\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$ and $t_{i_1,i_2} = t_{i_2,i_3} = \cdots = t_{i_{m-1},i_m} = 1$ ([8]).

Let R be the projection onto the subspace $l^2(\Omega_A)$. The Toeplitz algebra $\mathcal{T}_A$ generated by $\{T_h := RU(h)R : h \in \Omega_A\}$ has been studied in [8]. The following short sequence is exact (see [8]):

$$0 \longrightarrow \mathcal{K}(l^2(\Omega_A)) \longrightarrow \mathcal{T}_A \longrightarrow \mathcal{O}_A \longrightarrow 0.$$

Here we are interested in studying the structure of $C_r^*(\Gamma, R)$ and the corner algebra $\mathcal{A}_R := RC_r^*(\Gamma, R)R$. As in [8] we assume that the number of generators of Γ is precisely the matrix size n.

5.1. Theorem. *Assume that $\mathcal{A}$ is an irreducible $n \times n$ matrix with entries in $\{0, 1\}$ and Γ is the free group on n generators. Then:*

(i) $\mathcal{A}_R = \mathcal{T}_A$.
(ii) $C_r^*(\Gamma, R)$ *has two nontrivial closed ideals,* $\mathcal{K}(l^2(\Gamma))$ *and the closed ideal* $\mathcal{I}_R$ *generated by* R.
(iii) $\mathcal{I}_R \cong \mathcal{I}_R \otimes \mathcal{K}$, $\mathcal{I}_R / \mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_A \otimes \mathcal{K}$.
(iv) $RR(\mathcal{A}_R) = RR(\mathcal{I}_R) = 0$.

To prove this result, we again proceed in several steps as follows.

5.2. Lemma. *We have $T_h \neq 0$ if and only if $h = g_{i_1} g_{i_2} \ldots g_{i_l}$, or $h = g_{j_k}^{-1} g_{j_{k-1}}^{-1} \ldots g_{j_2}^{-1} g_{j_1}^{-1}$, or $h = g_{i_1} g_{i_2} \ldots g_{i_l} g_{j_k}^{-1} g_{j_{k-1}}^{-1} \ldots g_{j_2}^{-1} g_{j_1}^{-1}$ such that $i_l > 0$ and $t_{i_l,j} = t_{j_k,j} = 1$ for some $j \in \{1, 2, \ldots, n\}$, where all of the above words are reduced words such that $g_{i_1} g_{i_2} \ldots g_{i_l}$ and $g_{j_1} g_{j_2} \ldots g_{j_k}$ are elements in Ω_A.*

Proof. Since, for any $h \in \Gamma$, $T_h T_h^*$ is the projection onto $l^2(h^{-1}\Omega_A \cap \Omega_A)$ and $T_h^* T_h$ is the projection onto $l^2(h\Omega_A \cap \Omega_A)$, one sees that $T_h \neq 0$ iff $h\Omega_A \cap \Omega_A$

is not empty. Thus, in order for $T_h \neq 0$, it is necessary that $h = h_1 h_2^{-1}$ for some $h_1, h_2 \in \Omega_A$, that is,

$$h = g_{i_1} g_{i_2} \dots g_{i_l} g_{j_k}^{-1} g_{j_{k-1}}^{-1} \dots g_{j_2}^{-1} g_{j_1}^{-1},$$

where $t_{i_1,i_2} = t_{i_2,i_3} = \cdots = t_{i_{l-1},i_l} = 1$, and $t_{j_1,j_2} = t_{j_2,j_3} = \cdots = t_{j_{k-1},j_k} = 1$. Two extreme cases here are when either $h_1 = e$ or $h_2 = e$.

The above condition is also sufficient. In fact, if $h = g_{j_k}^{-1} g_{j_{k-1}}^{-1} \dots g_{j_2}^{-1} g_{j_1}^{-1}$ (when $h_1 = e$), then $h\Omega_A \cap \Omega_A$ is the set of those reduced words in Ω_A that start with g_j and satisfy $t_{j_k,j} = 1$ and e. Thus $T_h \neq 0$. If $h = g_{i_1} g_{i_2} \dots g_{i_l}$ (when $h_2 = e$), then

$$T_h = T^*_{g_{i_l}^{-1} g_{i_{l-1}}^{-1} \dots g_{i_2}^{-1} g_{i_1}^{-1}}.$$

Thus $T_h \neq 0$.

If $h = g_{i_1} g_{i_2} \dots g_{i_l} g_{j_k}^{-1} g_{j_{k-1}}^{-1} \dots g_{j_2}^{-1} g_{j_1}^{-1}$ with $i_l > 0$, then $h\Omega_A \cap \Omega_A$ is the set of those reduced words in Ω_A that start with $g_{i_1} g_{i_2} \dots g_{i_l} g_j$, where $t_{j_k,j} = 1$; furthermore, $t_{i_l,j} = 1$ is also necessary for $g_{i_1} g_{i_2} \dots g_{i_l} g_j \in \Omega_A$. Therefore, the existence of such a j with $t_{i_l,j} = t_{j_k,j} = 1$ is a necessary and sufficient condition in order for $h\Omega_A \cap \Omega_A \neq \emptyset$. $\square$

5.3. Lemma. *Assume that* $h = g_{i_1} g_{i_2} \dots g_{i_l} g_{j_k}^{-1} g_{j_{k-1}}^{-1} \dots g_{j_2}^{-1} g_{j_1}^{-1}$ *with* $i_l \geq 1$ *and that* $t_{i_l,j} = t_{j_k,j} = 1$ *for some* $j \in \{1, 2, \dots, n\}$. *Then* $T_h = T^*_{g_{j_1} g_{j_2} \dots g_{j_k}} T_{g_{i_1} g_{i_2} \dots g_{i_l}}$.

Proof. Let $h_1 = g_{i_1} g_{i_2} \dots g_{i_l}$ and $h_2 = g_{j_1} g_{j_2} \dots g_{j_k}$. Then

$$T_h = T_{h_1 h_2^{-1}} = RU(h_1 h_2^{-1})R = RU(h_2)^* U(h_1)R.$$

To show $T_{h_1 h_2^{-1}} = T^*_{h_2} T_{h_1}$, it suffices to show that

$$RU(h_2)^*(I - R)U(h_1)R = 0.$$

The range of $(I - R)U(h_1)R$ is the subspace

$$l^2(g_{i_l}^{-1} \dots g_{i_2}^{-1} g_{i_1}^{-1} \Omega_A \cap (\Gamma \setminus \Omega_A)).$$

Clearly, each reduced word in $g_{i_l}^{-1} \dots g_{i_2}^{-1} g_{i_1}^{-1} \Omega_A \cap (\Gamma \setminus \Omega_A)$ starts with $g_{i_l}^{-1} \dots g_{i_k}^{-1}$ for some $1 \leq k \leq l - 1$. It follows that the range of $U(h_2)^*(I - R)U(h_1)R$ is associated with the subset of Γ in which each reduced word starts with the reduced word $h_2 g_{i_l}^{-1} \dots g_{i_k}^{-1}$ for some $1 \leq k \leq l - 1$. Thus the range projection of $U(h_2)^*(I - R)U(h_1)R$ is a subprojection of $I - R$. Therefore, $RU(h_2)^*(I - R)U(h_1)R = 0$. $\square$

5.4 Proof of Theorem 5.1. From Lemma 5.2 and Lemma 5.3 one sees that the two sets $\{T_h : h \in \Omega_A\}$ and $\{T_h : h \in \Gamma\}$ generate the same C*-algebra. Equivalently, the Toeplitz algebra $\mathcal{T}_A$ is exactly the corner $\mathcal{A}_R := RC^*_r(\Gamma, R)R$.

It follows from Theorem 1.1 that the projection R generates a nontrivial closed ideal $\mathcal{I}_R$ of $C_r^*(\Gamma, R)$. The same argument as for Lemma 3.2 shows that the natural short sequence

$$0 \longrightarrow \mathcal{I}_R \longrightarrow C_r^*(\Gamma, R) \longrightarrow C_r^*\Gamma \longrightarrow 0$$

is exact. Using the short exact sequence (see [8])

$$0 \longrightarrow \mathcal{K}(l^2(\Omega_A)) \longrightarrow \mathcal{T}_A \longrightarrow \mathcal{O}_A \longrightarrow 0$$

and the fact that $\mathcal{T}_A = \mathcal{A}_R$, one obtains the exact sequence

$$0 \longrightarrow \mathcal{K}(l^2(\Omega_A)) \longrightarrow \mathcal{A}_R \longrightarrow \mathcal{O}_A \longrightarrow 0.$$

Since A is irreducible, the C*-algebra $\mathcal{O}_A$ is purely infinite and simple [4]. Since $\mathcal{A}_R$ generates $\mathcal{I}_R$ as a closed ideal, $\mathcal{I}_R/\mathcal{K}(l^2(\Gamma))$ is stably isomorphic to $\mathcal{A}_R/\mathcal{K}(l^2(\Omega_A))$, and hence $\mathcal{I}_R/\mathcal{K}(l^2(\Gamma))$ is stably isomorphic to $\mathcal{O}_A$. Thus, $\mathcal{I}_R/\mathcal{K}(l^2(\Gamma))$ is purely infinite and simple; furthermore, $\mathcal{I}_R/\mathcal{K}(l^2(\Gamma))$ is non-unital and separable. From the general result [15, 1.2] that every non-unital, σ-unital, purely infinite simple C*-algebra is stable it follows that

$$\mathcal{I}_R/\mathcal{K}(l^2(\Gamma)) \cong \mathcal{O}_A \otimes \mathcal{K}.$$

Hence the following short sequence is exact:

$$0 \longrightarrow \mathcal{K}(l^2(\Gamma)) \longrightarrow \mathcal{I}_R \longrightarrow \mathcal{O}_A \otimes \mathcal{K} \longrightarrow 0.$$

Using exactly the same proof as for Lemma 3.7, one shows that $\mathcal{I}_R$ is a stable C*-algebra, that is, $\mathcal{I}_R \cong \mathcal{I}_R \otimes \mathcal{K}$.

The above two exact sequences imply that $C_r^*(\Gamma, R)$ has only two nontrivial closed ideals, $\mathcal{K}(l^2(\Gamma))$ and $\mathcal{I}_R$.

Finally, we obtain $RR(\mathcal{A}_R) = 0$ by combining the results of [4] and [13] provided A is irreducible. Also, $RR(\mathcal{I}_R) = 0$, since $\mathcal{I}_R$ and $\mathcal{A}_R$ are stably *-isomorphic (by [1, 2.8]).

5.5. Remark. In contrast to the cases Γ'_+ and Γ_+ we discussed earlier, it seems that $\mathcal{T}_A$ is not the closed linear span of Toeplitz operators associated with R.

6. $\mathbf{C_r^*(\Gamma, P)}$ associated with subgroups of $\mathbf{\Gamma}$

Let Γ_0 be a non-trivial subgroup of Γ (i.e., $\Gamma_0 \neq \Gamma, \{e\}$). Then Γ_0 is also a free group, and, of course, Γ_0 as well as $\Gamma \setminus \Gamma_0$ are infinite sets. Let P_0 be the projection onto the subspace $l^2(\Gamma_0)$. Consider the C*-algebra $C_r^*(\Gamma, P_0)$ generated by $C_r^*\Gamma$ and P_0. In this situation, $T_h := P_0U(h)P_0$ for each $h \in \Gamma$, and the Toeplitz algebra is denoted by $\mathcal{T}_0$.

6.1. Theorem.

(i) $P_0C_r^*(\Gamma, P_0)P_0 = \mathcal{T}_0 \cong C_r^*\Gamma_0$.

(ii) *The closed ideal* $\mathcal{I}_0$ *of* $C_r^*(\Gamma, P_0)$ *generated by the projection* P_0 *is isomorphic to* $C_r^*\Gamma_0 \otimes \mathcal{K}$ *(of course,* $\mathcal{I}_0 \neq C_r^*(\Gamma, P_0)$ *).*

(iii) *The following short sequence is exact:*

$$0 \longrightarrow C_r^*\Gamma_0 \otimes \mathcal{K} \longrightarrow C_r^*(\Gamma, P_0) \longrightarrow C_r^*\Gamma \longrightarrow 0.$$

Proof. The result follows from the following claims.

Claim 1. $T_h \neq 0$ *if and only if* $h \in \Gamma_0$.

In fact, $T_h \neq 0$ iff $h\Gamma_0 \cap \Gamma_0 \neq \emptyset$. Since Γ_0 is a subgroup of Γ, we have $h\Gamma_0 \cap \Gamma_0 \neq \emptyset$ iff $h \in \Gamma_0$.

Claim 2. $P_0 C_r^*(\Gamma, P_0) P_0 = C_r^*\Gamma_0$.

In fact, one needs only to observe that $T_h = U(h)P_0$ is a unitary operator onto $l^2(\Gamma_0)$ for each $h \in \Gamma_0$, and that $P_0 U(h) P_0 = 0$ for all $h \notin \Gamma_0$.

Claim 3. *The closed ideal* $\mathcal{I}_{P_0}$ *of* $C_r^*(\Gamma, P_0)$ *generated by* P_0 *is nontrivial (cf. Corollary 1.4).*

Claim 4. $\mathcal{I}_{P_0} \cong C_r^*\Gamma_0 \otimes \mathcal{K}$.

In fact, since Γ_0 is a subgroup of Γ, it is clear that $h_1\Gamma_0 \cap h_2\Gamma_0 \neq \emptyset$ if and only if $h_1\Gamma_0 = h_2\Gamma_0$. One can choose recursively a sequence $h_1 = e, h_2, \ldots, h_n, \ldots$ in Γ such that

$$h_i\Gamma_0 \cap h_j\Gamma_0 = \emptyset \quad (i \neq j) \quad \text{and} \quad \Gamma = \bigcup_{i=1}^{\infty} h_i\Gamma_0.$$

On the other hand, $U(h_i)^* P_0 U(h_i)$ is the projection onto the subspace $l^2(h_i\Gamma_0)$ for $i \geq 1$. Since $P_0 \sim U(h_1)^* P_0 U(h_i)$ for $i \geq 1$ and $\sum_{i=1}^{\infty} U(h_i)^* P_0 U(h_i) = I$, it is clear that

$$P_0 C_r^*(\Gamma, P_0) P_0 \otimes \mathcal{K} = \mathcal{I}_0.$$

Hence it follows from Claim 2 above that $\mathcal{I}_0 \cong C_r^*\Gamma_0 \otimes \mathcal{K}$.

Claim 5. *The natural short sequence* $0 \longrightarrow C_r^*\Gamma_0 \otimes \mathcal{K} \longrightarrow C^*(\Gamma, P_0) \longrightarrow C_r^*\Gamma \longrightarrow 0$ *is exact.*

In fact, the same argument as in the proof of Lemma 3.2 applies.

Combining the claims yields a complete proof of Theorem 6.1. □

6.2 Example. Let us look at the following particular case. First, take any element $g \in \Gamma \backslash \{e\}$ and let $\Gamma_1 := \{g^n : n \in \mathbb{Z}\}$. Then Γ_1 is an infinite subgroup of Γ such that $\Gamma \setminus \Gamma_1$ is an infinite subset of Γ. Let P_1 be the projection onto the subspace $l^2(\Gamma_1)$. Consider the C*-algebra $C_r^*(\Gamma, P_1)$ generated by $C_r^*\Gamma$ and P_1. Note that a Toeplitz operator with respect to P_1 is of the form

$T_h := P_1 U(h) P_1$ for any $h \in \Gamma$. We will denote the corresponding Toeplitz algebra by $\mathcal{T}_{\mathbb{Z}}$. Then we have the following corollary.

Corollary.
(i) $P_1 C_r^*(\Gamma, P_1) P_1 = \mathcal{T}_{\mathbb{Z}} \cong C(S^1)$.
(ii) *The closed ideal* $\mathcal{I}_0$ *of* $C_r^*(\Gamma, P_1)$ *generated by* P_1 *is *-isomorphic to* $C(S^1) \otimes \mathcal{K}$ *(of course,* $\mathcal{I}_0 \neq C_r^*(\Gamma, P_1)$ *).*
(iii) *The following short sequence is exact:*

$$0 \longrightarrow C(S^1) \otimes \mathcal{K} \longrightarrow C_r^*(\Gamma, P_1) \longrightarrow C_r^* \Gamma \longrightarrow 0.$$

Proof. (i) The corner algebra $P_1 C_r^*(\Gamma, P_1) P_1$ is generated by $\{T_h : h \in \Gamma\}$. On the other hand, Claim 1 in the proof of Theorem 6.1 implies that

$$\{T_h : h \in \Gamma\} = \left\{T_{g_1^n} : n \in \mathbb{Z}\right\}.$$

Hence $\mathcal{T}_{\mathbb{Z}}$ is the abelian C*-algebra generated by the bilateral shift $U(g_1)$. It is well known that $\mathcal{T}_{\mathbb{Z}} \cong C(S^1)$.

(ii) and (iii) follow from Theorem 6.1 and the trivial fact that $C_r^* \Gamma_1 = C(S^1)$. □

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