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ON THE REPRESENTATION OF DERIVATIVE ALGEBRAS IN CHARACTERISTIC $p > 0$

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Abstract. In this paper we show that neither the Weyl algebra $A_n(K)$ nor the derivative algebra $DA_n(K)$ has infinite irreducible representations in the case when the ground field K has characteristic $p > 0$. We also give a complete classification of irreducible representations of the first derivative algebra DA_1 when K is algebraically closed. Finally, we present an algorithm that determines, in finitely many steps, whether DA_1/L is a simple DA_1-module, where L is any left ideal of DA_1.

1. Introduction

Let K be a field with characteristic $\mathrm{ch}(K) = 0$ and $K[X] := K[x_1, \ldots, x_n]$ the polynomial ring in n variables. Then the Weyl algebra $A_n(K)$, the ring of differential operators $D(K[X])$, and the derivative algebra $\Delta(K[X])$ generated by $\{x_i, \partial_i : i = 1, \ldots, n\}$ in $\mathrm{End}_K\, K[X]$ are all isomorphic (see [8]). Because of this relation, the derivative algebra has defining relations as Weyl algebra, and hence has many applications. For example, symbolic computation in $\Delta(K[X])$ makes symbolic computation over $\mathcal{D}$-modules and automatic proving of function identities possible. However, if $\mathrm{ch}(K) = p > 0$, then $\Delta(K[X])$ is only a quotient of $A_n(K)$ (see [12]). Hence the study of $\Delta(K[X])$ and $D(K[X])$ becomes as difficult as any other problem in characteristic p, and only a few properties of $D(K[X])$ and $\Delta(K[X])$ are known in the case when $\mathrm{ch}(K) = p$ (see [9], [10], and [12]). Some elementary properties and the computing theory of $\Delta(K[X])$ were developed in the papers [12] and [11]. In this paper we consider the representation theory of $\Delta(K[X])$ in the case when $\mathrm{ch}(K) = p > 0$.

It is well known that, when $\mathrm{ch}(K) = 0$, the representation theory of the Weyl algebra $A_n(K)$ has important applications to several areas of mathematics and, in particular, to Lie algebras. Significant work has been done on the irreducible representations of the first Weyl algebra $A_1(K)$; see, e.g., [4], [5],

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[6], [1], [2]. The paper [13] gives a complete classification of finite dimensional simple A_1-modules when K is algebraically closed and $\mathrm{ch}(K) = p > 0$.

In this paper we use a simple fact from polynomial identity rings to show that, when $\mathrm{ch}(K) = p > 0$, both the Weyl algebra and the derivative algebra have only finite irreducible representations. Using this result, we give a complete classification of irreducible representations of the first derivative algebra $\Delta(K[x_1])$ for the case when K is algebraically closed with $\mathrm{ch}(K) = p > 0$. However, this classification does not provide the structure of simple modules. That is, given any left ideal L of $\Delta(K[x_1])$, the classification does not allow one to determine whether $\Delta(K[x_1])/L$ is a simple module. Using computational methods developed in recent years for commutative and noncommutative algebras (see, e.g., [7]), we will give an algorithm to determine, in finitely many steps, whether $\Delta(K[x_1])/L$ is simple, for any given left ideal L.

Throughout this paper, we suppose that K is a field with characteristic $p > 0$, and we set $K[X] = K[x_1, \dots, x_n]$. We denote by $Z_{\geq 0}$ the set of nonnegative integers. In order to stress the connection between A_n and $\Delta(K[X])$, we write DA_n for $\Delta(K[X])$. To make this paper self-contained, we state preliminary results in Section 2.

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2. Preliminaries

For convenience, we list some properties of the derivative algebra DA_n and the ring of differential operators $D(K[X])$.

2.1 Definition ([12]). Let $x_1, \dots, x_n$ be the left multiplication operators on $K[X]$ (that is, $x_i(f) = x_i{\cdot}f$ for any $f \in K[X]$), $\partial_1, \dots, \partial_n$ the partial differential operators on $K[X]$ (that is, $\partial_i(f) = \partial f/\partial x_i$ for any $f \in K[X]$). We denote by $DA_n(K)$ (or DA_n) the K-subalgebra of the endomorphism ring $\mathrm{End}_K(K[X])$ generated by $x_1, \dots, x_n, \partial_1, \dots, \partial_n$, and we call DA_n the derivative algebra of $K[X]$.

For n-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in Z^n_{\geq 0}$ set

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad \partial^\beta = \partial_1^{\beta_1} \dots \partial_n^{\beta_n}, \quad |\alpha| = \sum_{i=1}^n \alpha_i.$$

2.2 Proposition ([12]). *The set*

$$\big\{x^\alpha \partial^\beta : \alpha = (\alpha_1, \dots, \alpha_n),\ \beta = (\beta_1, \dots, \beta_n) \in Z^n_{\geq 0}, \\ \beta_i \leq p-1, \quad i = 1, \dots, n\big\}$$

is a K-basis of DA_n.

From now on we assume that, for any element $f \in DA_n$, f is expressed in terms of the above K-basis, and we call this representation the *standard form* of f.

2.3 Lemma ([12]). *For any $\alpha, \beta \in Z^n_{\geq 0}$, $|\alpha| \geq |\beta|$, we have*

$$\partial^\alpha(x^\beta) = \begin{cases} \alpha! & \text{if } \alpha = \beta,\ \alpha_i \leq p-1,\ 1 \leq i \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha! = \alpha_1!\cdot\alpha_2! \ldots \alpha_n!$, $0! = 1$.

2.4 Lemma ([12]). *We have*

$$\partial_i\cdot x_j = x_j\cdot\partial_i \quad (i \neq j), \quad \partial_i\cdot x_i = x_i\cdot\partial_i + 1,$$

$$\partial_i^m x_i^s = x_i^s\partial_i^m + s\cdot m\cdot x_i^{s-1}\partial_i^{m-1} + s(s-1)\cdot m(m-1)x_i^{s-2}\partial_i^{m-2} + s(s-1)(s-2)\cdot m(m-1)(m-2)\cdot x_i^{s-3}\partial_i^{m-3} + \ldots.$$

2.5 Theorem ([12]). *Let A_n be the nth Weyl algebra over K, that is, the associative K-algebra generated by $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and the relations $y_i x_j - x_j y_i = \delta_{ij}$, $x_i x_j - x_j x_i = y_i y_j - y_j y_i = 0$, $i, j = 1, \ldots, n$. Then there exists a K-algebra isomorphism $DA_n \cong A_n/\langle y_1^p, \ldots, y_n^p\rangle$. Furthermore, the center of A_n (resp. DA_n) is $K[x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p]$, (resp. $K[x_1^p, \ldots, x_n^p]$), and A_n (resp. DA_n) is a finitely generated free module over its center.*

By Theorem 2.5, we have $DA_n \cong A_n/\langle y_1^p, \ldots, y_n^p\rangle$. Hence there is a one-to-one correspondence between the set of left ideals of DA_n and the set of left ideals of A_n which contain $\langle y_1^p, \ldots, y_n^p\rangle$. This correspondence can be obtained as follows:

By Proposition 2.2, any element $f \in DA_n$, can be written in the standard form

$$f = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in Z^n_{\geq 0}$, $\beta_i \leq p-1$, $i = 1, \ldots, n$, $c_{\alpha\beta} \in K$. Let $f' \in A_n$, $f' = \sum c_{\alpha\beta} x^\alpha y^\beta$, with the same tuples α and β as in the representation of f. Suppose $L = \langle f_1, \ldots, f_s\rangle$ is any left ideal of DA_n. Then it is easy to show that

$$L' = \langle f_1', \ldots, f_s', y_1^p, \ldots, y_n^p\rangle$$

is the left ideal of A_n corresponding to L. Thus we have the following isomorphisms of K-vector spaces:

$$\begin{aligned} DA_n/L &\cong (A_n/\langle y_1^p, \ldots, y_n^p\rangle)/(\langle f_1', \ldots, f_s', y_1^p, \ldots, y_n^p\rangle/\langle y_1^p, \ldots, y_n^p\rangle) \\ &\cong A_n/L'. \end{aligned} \tag{2.1}$$

This isomorphism is especially important in the computing theory of DA_n, since DA_n is not a domain and therefore does not have a Groebner basis. We can use the above isomorphism and Groebner bases in A_n to perform efficient computations in DA_n; see [11].

3. Representation theory of DA_n

3.1 Theorem. *If K is a field with characteristic $p > 0$, then neither $DA_n(K)$ nor $A_n(K)$ has an infinite dimensional irreducible representation,*

Proof. By Theorem 2.5, $A_n(K)$ and $DA_n(K)$ are finitely generated over their center, Hence, by [8, Corollary 13.1.13], they are polynomial identity rings. Moreover, it is obvious that both A_n and DA_n are affine K-algebras.

By [8, Theorem 13.10.3], in a affine polynomial identity K-algebra R, any simple left R-module is a finite dimensional K-vector space. Thus every simple module over A_n or DA_n is finite over K, so A_n and DA_n do not have infinite irreducible representations. This completes the proof. □

It is easy to derive the following corollaries.

3.2 Corollary. *$K[X]$ is not a simple DA_n-module.*

Notice that if $\mathrm{ch}(K) = 0$, then $DA_n \cong A_n$, and $K[X]$ is a simple DA_n-module (see [3]). By the corollary this result does not hold in the case when $\mathrm{ch}(K) = p > 0$.

3.3 Corollary. *Let M be any left DA_n-module (or left A_n-module). If* $\text{GK-dim}\, M > 0$*, then M is not simple.*

Proof. By the definition of the Gelfand-Kirillov dimension (see [8]), M is not finite dimensional over K when $\text{GK-dim}\, M > 0$. Thus M is not a simple module. □

Since for general n the classification of simple DA_n-modules is quite complex, we consider here only the case $n = 1$. We shall use the concept of Harish-Chandra modules to obtain a classification for finite irreducible representations of DA_1 when K is also algebraically closed. Since, by Theorem 3.1, DA_1 has only finite irreducible representations, this classification provides a complete classification of all simple DA_1-modules.

3.4 Definition. Let $h_1 = \partial_1 x_1$, and let V be any DA_1-module. If V satisfies

(i) $V = \bigoplus_{\lambda \in K} V_\lambda$, where $V_\lambda = \{v \in V : h_1 v = \lambda v\}$,
(ii) $\dim_K V_\lambda < \infty$, for all $\lambda \in K$,

then V is called a Harish-Chandra module over (DA_1, h_1).

We retain the notation V_λ through the rest of this paper.

3.5 THEOREM. *Let $\Lambda = \{0, 1, \ldots, p-1\} \subseteq K$, $\lambda, \mu \in K$.*

(1) *If $\mu \neq 0$, let $V(\lambda, \mu) = \bigoplus_{i \in \Lambda} K v_i$, where $\{v_i : i \in \Lambda\}$ is an arbitrary set of p elements. Define the action of DA_1 on $V(\lambda, \mu)$ as follows:*

$$\begin{aligned} x_1 v_i &= v_{i+1}, \quad 0 \leq i < p-1, \\ x_1 v_{p-1} &= \mu v_0, \\ \partial_1 v_0 &= \frac{\lambda - 1}{\mu} v_{p-1}, \\ \partial_1 v_j &= (\lambda + j - 1) v_{j-1}, \quad 0 < j \leq p-1, \end{aligned}$$

where $v_{-1} = v_{p-1}$, $v_p = v_0$. Then $V(\lambda, \mu)$ is a finite dimensional irreducible DA_1-module and also a Harish-Chandra module.

(2) *Let $\overline{V} = \bigoplus_{i \in \Lambda} K v_i$, where $\{v_i : i \in \Lambda\}$ is an arbitrary set of p elements. Define the action of DA_1 on $\overline{V}$ as follows:*

$$\begin{aligned} x_1 v_i &= -i v_{i-1}, \quad 0 \leq i \leq p-1, \\ \partial_1 v_i &= v_{i+1}, \quad 0 \leq i < p-1, \\ \partial_1 v_{p-1} &= 0, \end{aligned}$$

where $v_{-1} = v_{p-1}$, $v_p = v_0$. Then $\overline{V}$ is a finite dimensional irreducible DA_1-module and also a Harish-Chandra module.

Proof. (1) It is obvious from the definition that $V(\lambda, \mu)$ is a finite dimensional left DA_1-module, and that

$$\begin{aligned} h_1 v_i &= \partial_1 x_1 v_i = \partial_1 v_{i+1} = (\lambda + i) v_i, \quad 0 \leq i < p-1, \\ h_1 v_{p-1} &= \partial_1 x_1 v_{p-1} = \mu \partial_1 v_0 = (\lambda - 1) v_{p-1} = (\lambda + p - 1) v_{p-1}. \end{aligned}$$

Let $V_{\lambda+i} = K v_i$ for $0 \leq i \leq p-1$, and set $V_\delta = 0$ for $\delta \in K$ and $\delta \neq \lambda + i$, $i = 0, 1, \ldots, p-1$. Clearly, $V(\lambda, \mu) = \bigoplus_{\delta \in K} V_\delta$ is a Harish-Chandra module. We now prove that $V(\lambda, \mu)$ is an irreducible DA_1-module.

Suppose this is not true. Then $V(\lambda, \mu)$ has some nonzero proper submodule N. If there exists an element $v_i \in N$, $i \in \Lambda$, then $\{v_j; j \in \Lambda\} \subseteq N$ by definition. Hence $N = V(\lambda, \mu)$, contradicting our hypothesis. Therefore $v_i \notin N$ for all $i \in \Lambda$. Take any nonzero element f in N. Then f has the form

$$f = a_1 v_{i_1} + a_2 v_{i_2} + \cdots + a_s v_{i_s},$$

where $s > 1$, $a_j \in K$, $a_j \neq 0$, $j = 1, \ldots, s$, and $p - 1 \geq i_1 > i_2 > \cdots > i_s \geq 0$. (We have $s > 1$ since for any $i \in \Lambda$, $v_i \notin N$.)

Note that $h_1 v_i = (\lambda + i) v_i$ for $i = 0, \dots, p-1$. Hence

$$\begin{aligned} f &= a_1 v_{i_1} + a_2 v_{i_2} + \dots + a_s v_{i_s} \in N, \\ h_1 f &= (\lambda + i_1) a_1 v_{i_1} + (\lambda + i_2) a_2 v_{i_2} + \dots + (\lambda + i_s) a_s v_{i_s} \in N, \\ h_1^2 f &= (\lambda + i_1)^2 a_1 v_{i_1} + (\lambda + i_2)^2 a_2 v_{i_2} + \dots + (\lambda + i_s)^2 a_s v_{i_s} \in N, \\ &\vdots \\ h_1^{s-1} f &= (\lambda + i_1)^{s-1} a_1 v_{i_1} + (\lambda + i_2)^{s-1} a_2 v_{i_2} + \dots \\ &\qquad + (\lambda + i_s)^{s-1} a_s v_{i_s} \in N. \end{aligned}$$

We put this system of equations in matrix form:

$$\begin{pmatrix} f \\ h_1 f \\ \vdots \\ h_1^{s-1} f \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_s \\ (\lambda + i_1) a_1 & (\lambda + i_2) a_2 & \dots & (\lambda + i_s) a_s \\ \vdots & \vdots & & \vdots \\ (\lambda + i_1)^{s-1} a_1 & (\lambda + i_2)^{s-1} a_2 & \dots & (\lambda + i_s)^{s-1} a_s \end{pmatrix} \begin{pmatrix} v_{i_1} \\ v_{i_2} \\ \vdots \\ v_{i_s} \end{pmatrix}. \tag{3.1}$$

The determinant of the above matrix equals

$$a_1 a_2 \dots a_s \begin{vmatrix} 1 & 1 & \dots & 1 \\ (\lambda + i_1) & (\lambda + i_2) & \dots & (\lambda + i_s) \\ \vdots & \vdots & & \vdots \\ (\lambda + i_1)^{s-1} & (\lambda + i_2)^{s-1} & \dots & (\lambda + i_s)^{s-1} \end{vmatrix}. \tag{3.2}$$

Since $a_1 a_2 \dots a_s \neq 0$ and the determinant in (3.2) is a Vandermonde determinant with pairwise distinct entries $\lambda + i_j$, $j = 1, \dots, s$, the determinant of the system (3.1) is non-zero. Hence this system has a unique solution. Thus $v_{i_1}, \dots, v_{i_s}$ can be expressed as K-combinations of $f, h_1 f, \dots, h_1^{s-1} f$, and hence of $v_{i_1}, \dots, v_{i_s} \in N$. This contradicts the fact that $v_i \notin N$ for all i. Hence $V(\lambda, \mu)$ is an irreducible left DA_1-module.

(2) It is obvious that $\overline{V}$ is a finite dimensional DA_1-module, and that

$$\begin{aligned} h_1 v_i &= \partial_1 x_1 v_i = -i \partial_1 v_{i-1} = -i v_i, \quad i = 1, \dots, p-1, \\ h_1 v_0 &= \partial_1 x_1 v_0 = 0. \end{aligned}$$

Let

$$\begin{aligned} V_{-i} &= K v_i, & & i = 0, 1, \dots, p-1, \\ V_\delta &= 0, & & \delta \in K, \quad \delta \neq -i, \quad i = 0, 1, \dots, p-1. \end{aligned}$$

Then $\overline{V} = \bigoplus_{\delta \in K} V_\delta$ is a Harish-Chandra module. We now show that $\overline{V}$ is an irreducible DA_1-module.

Suppose this is not true. Then there exists a nonzero proper submodule N of $\overline{V}$. As in the proof of part (1) we see that if $v_i \in N$ then $N = \overline{V}$. Thus

$v_i \notin N$ for all $i \in \Lambda$. Take any nonzero element f in N. Then f has the form

$$f = a_1 v_{i_1} + a_2 v_{i_2} + \cdots + a_s v_{i_s},$$

where $s > 1$, $a_j \in K$, $a_j \neq 0$, $j = 1, \ldots, s$, $p - 1 \geq i_1 > i_2 > \cdots > i_s \geq 0$. Since $h_1 v_i = -i v_i$ for $i = 0, \ldots, p-1$, we have

$$\begin{aligned} f &= a_1 v_{i_1} + a_2 v_{i_2} + \cdots + a_s v_{i_s} \in N, \\ h_1 f &= (-i_1) a_1 v_{i_1} + (-i_2) a_2 v_{i_2} + \cdots + (-i_s) a_s v_{i_s} \in N, \\ &\vdots \\ h_1^{s-1} f &= (-i_1)^{s-1} a_1 v_{i_1} + (-i_2)^{s-1} a_2 v_{i_2} + \cdots + (-i_s)^{s-1} a_s v_{i_s} \in N. \end{aligned}$$

Thus

$$\begin{pmatrix} f \\ h_1 f \\ \vdots \\ h_1^{s-1} f \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \ldots & a_s \\ (-i_1) a_1 & (-i_2) a_2 & \ldots & (-i_s) a_s \\ \vdots & \vdots & & \vdots \\ (-i_1)^{s-1} a_1 & (-i_2)^{s-1} a_2 & \ldots & (-i_s)^{s-1} a_s \end{pmatrix} \begin{pmatrix} v_{i_1} \\ v_{i_2} \\ \vdots \\ v_{i_s} \end{pmatrix}.$$

As in the proof of (1), we see that the above matrix is invertible, and the above system equations therefore has a unique solution. Thus $v_{i_1}, \ldots, v_{i_s}$ can be expressed as K-combinations of $f, h_1 f, \ldots, h_1^{s-1} f$. Hence $v_{i_j} \in N, j = 1, \ldots, s$, contradicting the fact that $v_j \notin N$ for all j. Hence $\overline{V}$ is an irreducible DA_1-module. This completes the proof.

After these preliminaries, we can now state the classification announced in the Introduction.

3.6 THEOREM. *Let K be an algebraically closed field with characteristic $p > 0$, and let V be any irreducible left $DA_1(K)$-module. Then either V is isomorphic to $\overline{V}$, or there exist $\lambda, \mu \in K, \mu \neq 0$, such that V is isomorphic to $V(\lambda, \mu)$.*

Proof. By Theorem 3.1, V must be a finite dimensional K-vector space. Since $h_1 V \subseteq V$ and K is algebraically closed, by the eigenvalue theory of linear operators there exist $\lambda \in K$ and $0 \neq u_1 \in V$ such that $h_1 u_1 = \lambda u_1$.

We now show that $\sum_{i \geq 0} K x_1^i u_1 + \sum_{j \geq 0} K \partial_1^j u_1$ is a nonzero submodule of V. Indeed, for $i \geq 1$ we have, by Lemma 2.4 and the relation $h_1 u_1 = \lambda u_1$,

$$\begin{aligned} x_1 \cdot \partial_1^i u_1 &= \left(\partial_1^i x_1 - i \partial_1^{i-1} \right) u_1 \\ &= \partial_1^{i-1} h_1 u_1 - i \partial_1^{i-1} u_1 \\ &= (\lambda - i) \partial_1^{i-1} u_1 \in K \partial_1^{i-1} u_1. \end{aligned}$$

If $j \geq 2$, then

$$\begin{aligned}\partial_1 \cdot x_1^j u_1 &= \left(\partial_1 x_1^{j-1}\right) x_1 u_1 \\ &= x_1^{j-1} h_1 u_1 + (j-1) x_1^{j-1} u_1 \\ &= (\lambda + j - 1) x_1^{j-1} u_1 \in K x_1^{j-1} u_1,\end{aligned}$$

while for $j = 1$ we have

$$\partial_1 \cdot x_1 u_1 = h_1 u_1 = \lambda u_1 \in K u_1.$$

Thus $\sum_{i \geq 0} K x_1^i u_1 + \sum_{j \geq 0} K \partial_1^j u_1$ is a nonzero submodule of V, and since V is irreducible, we have

$$V = \sum_{i \geq 0} K x_1^i u_1 + \sum_{j \geq 0} K \partial_1^j u_1.$$

Let

$$V_{\lambda+i} = \{v \in V : h_1 v = (\lambda + i) v\}.$$

Then $V_{\lambda+i}$ is a K-subspace of V, and for $i \geq 1$ we have

$$\begin{aligned}h_1 \cdot x_1^i u_1 &= \partial_1 x_1 x_1^i u_1 \\ &= x_1^i h_1 u_1 + i x_1^i u_1 \\ &= (\lambda + i) x_1^i u_1, \\ h_1 \cdot \partial_1^i u_1 &= \partial_1 (x_1 \partial_1^i) u_1 \\ &= \partial_1^i h_1 u_1 - i \partial_1^i u_1 \\ &= (\lambda - i) \partial_1^i u_1.\end{aligned}$$

Thus $x_1^i u_1 \in V_{\lambda+i}$ and $\partial_1^i u_1 \in V_{\lambda-i} = V_{\lambda+p-i}$. Notice that $u_1 \in V_\lambda$. Therefore $V \subseteq \sum_{i \in \Lambda} V_{\lambda+i}$, and $V = \sum_{i \in \Lambda} V_{\lambda+i}$. We now show that $\sum_{i \in \Lambda} V_{\lambda+i}$ is, in fact, a direct sum, i.e.,

$$\sum_{i \in \Lambda} V_{\lambda+i} = \bigoplus_{i \in \Lambda} V_{\lambda+i}. \tag{3.3}$$

To prove this, let $v_i \in V_{\lambda+i}$, $i = 0, 1, \ldots, p-1$, and suppose there exist $a_i \in K$, such that

$$a_0 v_0 + a_1 v_1 + \cdots + a_{p-1} v_{p-1} = 0.$$

By considering the action of $h_1^0 (= 1), h_1^1, \ldots, h_1^{p-1}$ on the above equation and using the relation $h_1 v_i = (\lambda + i) v_i$, we obtain the system

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda & (\lambda+1) & \ldots & (\lambda+p-1) \\ \vdots & \vdots & \vdots & \vdots \\ \lambda^{p-1} & (\lambda+1)^{p-1} & \ldots & (\lambda+p-1)^{p-1} \end{pmatrix} \begin{pmatrix} a_0 v_0 \\ a_1 v_1 \\ \vdots \\ a_{p-1} v_{p-1} \end{pmatrix} = 0.$$

Since the above matrix is a Vandermonde matrix and the numbers $\lambda, \lambda+1, \ldots,$ $\lambda + p - 1$ are pairwise distinct, this matrix is invertible. Hence the above homogeneous system has only the trivial solution, i.e., we have

$$a_0 v_0 = a_1 v_1 = \cdots = a_{p-1} v_{p-1} = 0.$$

It follows that for any i with $0 \leq i \leq p-1$ we have $a_i = 0$ whenever $v_i \neq 0$. Hence $\sum_{i \in \Lambda} V_{\lambda+i}$ is a direct sum, proving (3.3).

We now distinguish between two cases, $\lambda \notin \Lambda$ and $\lambda \in \Lambda$.

Case 1. $\lambda \notin \Lambda$.

If there exists $i \in \Lambda$ and a nonzero element $v \in V_{\lambda+i}$ such that $x_1 v = 0$, then $h_1 v = \partial_1 x_1 v = 0$, and $\lambda + i = 0$, $\lambda = -i = p - i$, contradicting the assumption $\lambda \notin \Lambda$. Thus, for any $i \in \Lambda$, there is no nonzero element $v \in V_{\lambda+i}$ such that $x_1 v = 0$. Therefore the action of x_1 on $V_{\lambda+i}$ is faithful.

Given $v \in V_\lambda$, it is easy to see that $h_1 x_1^p v = \lambda x_1^p v$. Thus $x_1^p v \in V_\lambda$, and x_1^p may be viewed as a K-linear endomorphism of V_λ. Therefore there exists an eigenvalue $\mu \in K$ and a nonzero eigenvector $v_0 \in V_\lambda$, such that $x_1^p v_0 = \mu v_0$. We now show that $\mu \neq 0$.

Suppose $\mu = 0$. Then $x_1^p v_0 = 0$. Since the action of x_1 on V_λ is faithful, there exists i, $1 \leq i \leq p-1$, such that $x_1^i v_0 \neq 0$, $x_1^{i+1} v_0 = 0$. But

$$\begin{aligned} h_1 x_1^i v_0 &= \partial_1 x_1^i x_1 v_0 \\ &= x_1^i h_1 v_0 + i x_1^i v_0 \\ &= (\lambda + i) x_1^i v_0, \end{aligned}$$

so $x_1^i v_0 \in V_{\lambda+i}$. Since $x_1 \cdot x_1^i v_0 = x_1^{i+1} v_0 = 0$, this contradicts the fact that x_1 is faithful on $V_{\lambda+i}$. Hence we have $\mu \neq 0$.

Next, we show that $\sum_{i \in \Lambda} K x_1^i v_0$ is a submodule of V. Indeed, if $i = 2$, $3, \ldots, p-1$, then

$$\begin{aligned} \partial_1 \cdot x_1^i v_0 &= \partial_1 x_1^{i-1} \cdot x_1 v_0 \\ &= x_1^{i-1} h_1 v_0 + (i-1) x_1^{i-1} v_0 \\ &= (\lambda + i - 1) x_1^{i-1} v_0 \in K x_1^{i-1} v_0. \end{aligned} \tag{3.4}$$

If $i = 0$, then

$$\begin{aligned} \partial_1 \cdot v_0 &= \partial_1 \left(\frac{1}{\mu} x_1^p v_0 \right) \\ &= \frac{1}{\mu} \partial_1 x_1^{p-1} x_1 v_0 \\ &= \frac{1}{\mu} x_1^{p-1} h_1 v_0 + \frac{p-1}{\mu} x_1^{p-1} v_0 \\ &= \frac{\lambda + p - 1}{\mu} x_1^{p-1} v_0 \in K x_1^{p-1} v_0, \end{aligned} \tag{3.5}$$

and if $i = 1$, then

$$\partial_1 \cdot x_1 v_0 = h_1 v_0 = \lambda v_0 \in K v_0. \tag{3.6}$$

Moreover, if $j = 0, 1, \ldots, p-2$, then

$$x_1 \cdot x_1^j v_0 = x_1^{j+1} v_0 \in K x_1^{j+1} v_0,$$

and for $j = p-1$ we have

$$x_1 \cdot x_1^{p-1} v_0 = x_1^p v_0 = \mu v_0 \in K v_0.$$

Thus $\sum_{i \in \Lambda} K x_1^i v_0$ is a nonzero submodule of V.

Similarly to (3.4), (3.5), and (3.6), we see that

$$h_1 x_1^i v_0 = \partial_1 x_1^{i+1} v_0 = (\lambda + i) x_1^i v_0, \quad i = 0, 1, \ldots, p-1.$$

Hence $x_1^i v_0 \in V_{\lambda+i}$. By (3.3), $\sum_{i \in \Lambda} K x_1^i v_0$ is a direct sum, i.e.,

$$\sum_{i \in \Lambda} K x_1^i v_0 = \bigoplus_{i \in \Lambda} K x_1^i v_0.$$

Since V is an irreducible left DA_1-module, we have $V = \bigoplus_{i \in \Lambda} K x_1^i v_0$. Let $v_i = x_1^i v_0$, $i = 1, \ldots, p-1$, $v_p = v_0$, $v_{-1} = v_{p-1}$. Then

$$\begin{aligned}
x_1 v_i &= v_{i+1}, i = 0, 1, \ldots, p-2, \\
x_1 v_{p-1} &= x_1^p v_0 = \mu v_0, \\
\partial_1 v_0 &= \frac{\lambda + p - 1}{\mu} x_1^{p-1} v_0 = \frac{\lambda - 1}{\mu} v_{p-1} \quad \text{(by (3.5))}, \\
\partial_1 v_j &= (\lambda + j - 1) v_{j-1}, \quad j = 1, \ldots, p-1 \quad \text{(by (3.4) and (3.6))}.
\end{aligned}$$

Thus we have $V \cong V(\lambda, \mu)$ when $\mu \neq 0$.

Case 2. $\lambda \in \Lambda$.

In this case it is easy to see that $V = \bigoplus_{i \in \Lambda} V_i$. We now distinguish two subcases.

Subcase 2.1. There exist $s \in \Lambda$ and nonzero element $u \in V_s$, such that $x_1 u = 0$.

In this case, we have

$$su = h_1 u = \partial_1 x_1 u = \partial_1 0 = 0,$$

and thus $s = 0$ and $u \in V_0$. Let $U = \sum_{i \in \Lambda} K \partial_1^i u$. Then for $1 \leq i \leq p-1$ we have

$$x_1 \cdot \partial_1^i u = \left(\partial_1^i x_1 - i \partial_1^{i-1}\right) u = -i \partial_1^{i-1} u \in U.$$

Since $\partial_1^p = 0$, we also have

$$\begin{aligned}
\partial_1 \cdot \partial_1^{p-1} u &= 0 \in U, \\
\partial_1 \cdot \partial_1^j u &= \partial_1^{j+1} u \in U, \quad j = 0, 1, \ldots, p-2.
\end{aligned}$$

Hence U is a nonzero submodule of V, and since V is irreducible, we have $U = V$.

For $1 \leq i \leq p-1$ we have $h_1\partial_1^i u = -i\partial^i u$, and $h_1 u = 0$. Thus $\partial_1^j u \in V_{-j}$ for $j = 0, 1, \ldots, p-1$, Combined with (3.3), this shows that $\sum_{i\in\Lambda} K\partial_1^i u$ is a direct sum, i.e.,

$$\sum_{i\in\Lambda} K\partial_1^i u = \bigoplus_{i\in\Lambda} K\partial_1^i u.$$

Hence $V = \bigoplus_{i\in\Lambda} K\partial_1^i u$.

Let $v_0 = u$, $v_i = \partial_1^i v_0$, $i = 1, \ldots, p-1$. Then

$$\begin{aligned} x_1 v_0 &= 0, \\ x_1 v_i &= x_1\partial_1^i v_0 \left(\partial_1^i x_1 - i\partial_1^{i-1}\right) v_0 \\ &= -i\partial_1^{i-1} v_0 = -iv_{i-1},\ i = 1, \ldots, p-1, \\ \partial_1 v_j &= v_{j+1},\ j = 0, 1, \ldots, p-2, \\ \partial_1 v_{p-1} &= \partial_1^p v_0 = 0. \end{aligned}$$

Thus $V \cong \overline{V}$.

Subcase 2.2. For any $s \in \Lambda$, there is no nonzero element $u \in V_s$ such that $x_1 u = 0$.

In this case, for any $s \in \Lambda$, the action of x_1 on V_s is faithful. It is easy to see that $h_1 x_1^p v = x_1^p h_1 v + p x_1^p v = 0$ for any $v \in V_0$. Thus $x_1^p v \in V_0$, that is, x_1^p maps V_0 to V_0. Therefore there exist nonzero elements $v_0 \in V_0$ and $\mu \in K$ such that $x_1^p v_0 = \mu v_0$. When $i = 1, \ldots, p-1$, we have

$$\begin{aligned} h_1 v_0 \quad &= 0, \\ h_1 x_1^i v_0 &= i x_1^i v_0. \end{aligned}$$

Thus $x_1^j v_0 \in V_j$ for $j = 0, 1, \ldots, p-1$.

If $\mu = 0$, then $x_1^p v_0 = 0$. By the faithfulness of x_1 on V_s, there exists i, $1 \leq i \leq p-1$, such that $x_1^i v_0 \neq 0$, $x_1^{i+1} v_0 = 0$, and thus $x_1 \cdot (x_1^i v_0) = 0$. Since $x_1^i v_0 \in V_i$, this contradicts the fact that x_1 is faithful on V_i. Hence $\mu \neq 0$.

Let $U = \sum_{i\in\Lambda} K x_1^i v_0$. Then for $i = 0, 1, \ldots, p-2$,

$$x_1 \cdot x_1^i v_0 = x_1^{i+1} v_0 \in U,$$

and for $i = p-1$,

$$x_1 \cdot x_1^{p-1} v_0 = x_1^p v_0 = \mu v_0 \in U.$$

For $j = 2, \ldots, p-1$,

$$\begin{aligned} \partial_1 \cdot x_1^j v_0 &= \partial_1 x_1^{j-1} x_1 v_0 \\ &= x_1^{j-1} h_1 v_0 + (j-1) x_1^{j-1} v_0 \\ &= (j-1) x_1^{j-1} v_0 \in U, \end{aligned}$$

and for $j = 0, 1$,

$$\begin{aligned}\partial_1 \cdot x_1 v_0 &= h_1 v_0 = 0 \in U,\\ \partial_1 \cdot v_0 &= \partial_1 \left(\frac{1}{\mu} \cdot x_1^p v_0 \right)\\ &= \frac{1}{\mu} \cdot \left(x_1^{p-1} h_1 v_0 + (p-1) x_1^{p-1} v_0 \right)\\ &= \frac{p-1}{\mu} x_1^{p-1} v_0 \in U.\end{aligned}$$

Hence U is a nonzero submodule of V, and by the irreducibility of V it follows that $U = V$.

Since $x_1^i v_0 \in V_i$, $i = 0, 1, \ldots, p-1$, it follows from (3.3) that $\sum_{i \in \Lambda} K x_1^i v_0$ is a direct sum, and

$$V = \sum_{i \in \Lambda} K x_1^i v_0 = \bigoplus_{i \in \Lambda} K x_1^i v_0.$$

Let $v_i = x_1^i v_0, i = 1, \ldots, p-1$. Then it is easy to see that $V = \bigoplus_{i \in \Lambda} K v_i$, and

$$\begin{aligned}x_1 v_i &= v_{i+1}, \quad i = 0, 1, \ldots, p-2,\\ x_1 v_{p-1} &= \mu v_0,\\ \partial_1 v_j &= (j-1) v_{j-1}, \quad j = 1, \ldots, p-1,\\ \partial_1 v_0 &= \frac{p-1}{\mu} v_{p-1}.\end{aligned}$$

Hence $V \cong V(0, \mu)$. This completes the proof of the theorem. □

By the structure of $V(\lambda, \mu)$ and $\overline{V}$, we have $\dim_K V(\lambda, \mu) \le p$, and $\dim_K \overline{V}$ $\le p$. Thus Theorem 3.6 has the following corollary.

3.7 Corollary. *If V is any finite dimensional DA_1-module and $\dim_K V$ $> p$, then V is not a simple module.*

Theorems 3.5 and 3.6 give a complete classification of irreducible DA_1-modules in the case the field K is algebraically closed with characteristic $p > 0$. In the next section, we give an algorithm to determine, in finitely many steps, whether DA_1/L is a simple module. Corollary 3.7 above may reduce the complexity of the algorithm.

4. Algorithmic recognition of irreducible DA_1-modules

In this section we use the computing theory of DA_n to give an algorithm which can determine, in finitely many steps, whether DA_1/L is simple, where L is any left ideal of DA_1. In fact, the proof of Theorem 3.6 yields such an algorithm, and we have the following theorem.

4.1 Theorem. *Let K be an algebraically closed field with characteristic $p > 0$. Then, given any left ideal $L = \langle f_1, \ldots, f_s \rangle$ of $DA_1(K)$, there exists an algorithm that determines in finitely many steps whether DA_1/L is an irreducible DA_1-module. If DA_1/L is simple, this algorithm also gives the structure of DA_1/L as a Harish-Chandra module; i.e., the algorithm generates the parameters λ and μ, and the vector v_0, represented in terms of a basis of DA_1/L, in the representation of DA_1/L as a Harish-Chandra module $\overline{V}$ or $V(\lambda, \mu)$.*

Proof. By Theorem 3.1 and Corollary 3.7, we only need to determine the modules for the case when $\dim_K DA_1/L \le p$. By the last part of Section 2, $L' = \langle f_1', \ldots, f_s', y_1^p \rangle$ is the left ideal of the Weyl algebra A_1 corresponding to L, and we have $DA_1/L \cong A_1/L'$. Thus $\dim_K DA_1/L = \dim_K A_1/L'$ if $\dim_K DA_1/L < +\infty$.

Since the first Weyl algebra A_1 is a solvable polynomial algebra (see [7]), L' has a Groebner basis, say, $G = \{g_1, \ldots, g_t\}$. Let

$$B = \{x_1^i y_1^j \in SM(A_1) : x_1^i y_1^j \text{ is not divisible by } \mathrm{LT}(g_k), k = 1, \ldots, t\},$$

where $\mathrm{LT}(g)$ denotes the leading term of g in the graded lexicographic order. Let $[x_1^i y_1^j]$ be the coset of $x_1^i y_1^j$ modulo L'. Then

$$B' = \{[x_1^i y_1^j] : x_1^i y_1^j \in B\}$$

is a K-basis of A_1/L', and A_1/L' is finite dimensional if and only if there exist $g_{k_1}, g_{k_2} \in G$ such that $\mathrm{LT}(g_{k_1})$ is a power of x_1 and $\mathrm{LT}(g_{k_2})$ is a power of y_1 (see [7]). Since $DA_1/L \cong A_1/L'$ and B' is a K basis of A_1/L', the set

$$B'' = \left\{[x_1^i \partial_1^j] : i, j \text{ satisfies } [x_1^i y_1^j] \in B'\right\}$$

is a K-basis of DA_1/L. If the sets B' and B'' are finite, denote by $|B'|$ and $|B''|$ their respective cardinalities. Then $|B'| = |B''|$.

From the above analysis and the proof of Theorem 3.6, we have the following algorithm:

Define a boolean variable T so that `T=true` if DA_1/L is a simple module, and `T=false` if DA_1/L is not simple.

(1) Compute the Groebner basis $G = \{g_1, \ldots, g_t\}$ of $L' = \{f_1', \ldots, f_s', y_1^p\}$, and construct B, B', and B'' as above. If no element in the set $\mathrm{LT}(G) = \{\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t)\}$ is a power of x_1 or a power of y_1, or if $|B'| > p$, let `T=false`, and stop the algorithm. Otherwise go to Step (2).

(2) Let $|B''| = m$. Then $\dim_K DA_1/L = |B''| = m$. Using the multiplication in DA_1, compute the action of the operator h_1 on the elements of B'', and represent this operator by an $m \times m$ matrix H using the basis B''. Clearly, $H \in M_m(K)$, where $M_m(K)$ is the matrix ring over K. Determine

an eigenvalue λ of matrix H, and compute a basis of the eigenvector space V_λ corresponding to λ. Here the basis of V_λ is represented by vectors in K^m.

(3) If $\lambda \in \Lambda = \{0, 1, \dots, p-1\}$, then compute a basis of $\operatorname{Ker} H = \{a \in K^m : Ha = 0\}$ using the matrix H. It is obvious that $\operatorname{Ker} H$ is just $V_0 = \{v \in DA_1/L : h_1 v = 0\}$. Using the multiplication in DA_1, compute the action of the operators x_1 and ∂_1 on the basis B'', and represent x_1 and ∂_1 by two $m \times m$ matrices X and P. If $x_1 u = 0$ has nonzero solutions in V_0, i.e., if the linear system $Xa = 0$ has nonzero solutions in $\operatorname{Ker} H$, then take any such nonzero solution v_0. If the set of vectors $\{v_0, Pv_0, P^2 v_0, \dots, P^{p-1} v_0\}$ has rank m, then set `T=true`, output B'', v_0, $V \cong \overline{V}$, and stop the algorithm. Otherwise (i.e., if the above rank is not m), set `T=false`, and stop the algorithm.

(4) If $\lambda \notin \Lambda = \{0, 1, \dots, p-1\}$, or if $\lambda \in \Lambda$ and the equation $Xa = 0$ has no nonzero solution in $\operatorname{Ker} H$, then compute the action of the operator x_1^p on the elements of the basis B'' using the multiplication of DA_1, and represent x_1^p by an $m \times m$ matrix W. Compute a nonzero eigenvalue μ of W and a corresponding eigenvector $v_0 \neq 0$, such that μ and v_0 satisfy the following conditions:

- If $\lambda \notin \Lambda$, then $v_0 \in V_\lambda$.
- If $\lambda \in \Lambda$, then $v_0 \in V_0 = \operatorname{Ker} H$.

By the proof of Theorem 3.6, such nonzero values of μ and v_0 do exist. If the set of vectors $\{v_0, Xv_0, X^2 v_0, \dots, X^{p-1} v_0\}$ has rank m, then $V \cong V(\lambda, \mu)$. In this case set `T=true`, output B'', $\lambda, \mu, v_0, V \cong V(\lambda, \mu)$, and stop the algorithm. Otherwise set `T=false`, and stop the algorithm.

From the proof of Theorem 3.6 and the above analysis it is easy to see that this algorithm satisfies the requirement of the theorem. This completes the proof. □

We conclude this paper with two examples which illustrate the above algorithm. In these examples, K is an algebraically closed field.

4.2 Example. Let $p = \operatorname{ch}(K) = 5$, $f_1 = x_1 \partial_1^2$, and let $L = \langle f_1 \rangle$ be a left ideal of $DA_1(K)$. We now determine whether DA_1/L is a simple module.

It is obvious that

$$L' = \langle f_1',\ y_1^5 \rangle = \langle x_1 y_1^2,\ y_1^5 \rangle,$$

and the Groebner basis of L' is $G = \{y_1^2\}$ (see [7]). Now there is no term in $\operatorname{LT}(G)$ which is a power of x_1. Hence, by Step (1) of the above algorithm, DA_1/L is not simple.

4.3 Example. Let $p = \operatorname{ch}(K) = 3$, $f_1 = x_1$, and let $L = \langle f_1 \rangle$ be a left ideal of DA_1. We now determine whether DA_1/L is simple.

It is obvious that $L' = \langle x_1, y_1^3 \rangle$ is the left ideal of A_1 corresponding to L, and its Groebner basis is $G = \{x_1, y_1^3\}$. By the above algorithm, A_1/L' is finite dimensional, and its basis is

$$B' = \left\{[1], [y_1], [y_1^2]\right\}.$$

Thus DA_1/L is also finite dimensional, and its basis is

$$B'' = \{u_1, u_2, u_3\},$$

where $u_1 = [1]$, $u_2 = [\partial_1]$, $u_3 = [\partial_1^2]$. Since $\dim_K DA_1/L = |B''| = 3 = p$, we move on to Step (2) of the algorithm.

It is obvious that the action of $h_1 = \partial_1 x_1$ on the basis $\{u_1, u_2, u_3\}$ is given by

$$\begin{aligned} h_1 u_1 &= [\partial_1 x_1] = 0, \\ h_1 u_2 &= [\partial_1 x_1 \partial_1] = -u_2, \\ h_1 u_3 &= \left[\partial_1 x_1 \partial_1^2\right] = \left[\partial_1^3 x_1\right] - \left[2\partial_1^2\right] = u_3. \end{aligned}$$

Hence h_1 can be represented by the following matrix with respect to the basis $\{u_1, u_2, u_3\}$:

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix H has eigenvalues $0, 1, -1$. Take the eigenvalue $\lambda = 0$. The eigenvector space V_0 corresponding to $\lambda = 0$ is

$$V_0 = \operatorname{Ker} H = \left\{ k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ : \ k \in K \right\}.$$

The action of the operator x_1 on $\{u_1, u_2, u_3\}$ is given by

$$\begin{aligned} x_1 u_1 &= [x_1] = 0, \\ x_1 u_2 &= [x_1 \partial_1] = [\partial_1 x_1 - 1] = [-1] = -u_1, \\ x_1 u_3 &= \left[x_1 \partial_1^2\right] = \left[\partial_1^2 x_1 - 2\partial_1\right] = [-2\partial_1] = -2u_2 = u_2. \end{aligned}$$

Thus x_1 can be represented by the following matrix with respect to the basis $\{u_1, u_2, u_3\}$:

$$X = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $v_0 = (1, 0, 0)^T$. Then $v_0 \in V_0$, and $X v_0 = 0$, so x_1 is not faithful on V_0.

The action of ∂_1 on $\{u_1, u_2, u_3\}$ is given by

$$\begin{aligned} \partial_1 u_1 &= [\partial_1] = u_2, \\ \partial_1 u_2 &= [\partial_1^2] = u_3, \\ \partial_1 u_3 &= [\partial_1^3] = 0. \end{aligned}$$

Hence ∂_1 can be represented by the following matrix with respect to the basis $\{u_1,u_2,u_3\}$:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since

$$Pv_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad P^2 v_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

the set of vectors $\{v_0, Pv_0, P^2v_0\}$ has rank 3. Since $\dim DA_1/L = 3$, it follows that DA_1/L is a simple module, and $DA_1/L \cong \overline{V}$, where

$$\overline{V} = \bigoplus_{i\in\{0,1,2\}} Kv_i, \quad v_1 = \partial_1 v_0, \quad v_2 = \partial_1^2 v_0.$$

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