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EMBEDDING OF HARDY SPACES INTO WEIGHTED BERGMAN SPACES IN BOUNDED DOMAINS WITH C^2 BOUNDARY

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ABSTRACT. Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Let $H^p(D)$ be the Hardy space and $A^{p,\alpha}(D)$ be the space of holomorphic functions which are L^p-integrable with respect to the weighted measure $dV_\alpha(z) = \delta_D(z)^{\alpha-1}dV(z)$. We obtain some estimates on the mean growth of H^p functions in D. Using these estimates, we can embed the $H^p(D)$ space into $A^{q,\beta}(D)$ for $0 < p < q < \infty$, $\beta > 0$ satisfying $n/p = (n+\beta)/q$. We also show that the condition of C^2-smoothness of the boundary of D is an essential condition by giving a counter-example of a convex domain with $C^{1,\lambda}$ smooth boundary for $0 < \lambda < 1$ which does not satisfy the embedding result.

1. Introduction

Throughout this paper, D will be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. For $z \in D$ we let $\delta_D(z)$ denote the distance from z to ∂D. For $\alpha > 0$ we define a measure dV_α on D by $dV_\alpha(z) = \delta_D(z)^{\alpha-1}dV(z)$, where $dV(z)$ is the volume element. For $0 < p, \alpha < \infty$, we let $\|f\|_{p,\alpha}$ be the L^p-norm with respect to the measure dV_α and we define the weighted Bergman spaces $A^{p,\alpha}(D) = \{f \text{ holomorphic on } D : \|f\|_{p,\alpha} < \infty\}$. We will denote the usual Hardy space $H^p(D)$ by $A^{p,0}(D)$, and the associated norm by $\|f\|_{p,0}$. We can identify $A^{p,0}(D)$ in the usual way with a subspace of $L^p(\partial D : d\sigma)$ (see Section 4). In this paper we consider embedding results between $A^{p,\alpha}(D)$ spaces in a bounded domain in $\mathbb{C}^n$ with C^2 boundary.

The next embedding result is related to a classical estimate of Hardy-Littlewood on the growth of the means of holomorphic functions in the unit disc ([Du, p. 87]).

THEOREM 1.1. *Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Assume that $0 < p \leq q < \infty, \alpha, \beta \geq 0$, and $(n+\alpha)/p = (n+\beta)/q$. Then $A^{p,\alpha}(D) \subset A^{q,\beta}(D)$ and the inclusion is continuous.*

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The case $\alpha > 0$ was proved in [Be1] in general bounded domains with C^2 boundary. The case $\alpha = 0$ is the embedding of Hardy spaces $H^p(D)$ into the weighted Bergman spaces $A^{q,\beta}(D)$. As expected, the embedding of the Hardy space is the most difficult one. Even though Beatrous [Be1] proved the embedding $H^p(D) \subset A^{q,\beta}(D)$ for $0 < p < q < \infty$ with $n/p < (n+\beta)/q$, we can not prove the optimal embedding of the case $n/p = (n+\beta)/q$ by using his method. The optimal embedding of the case $\alpha = 0$ was proved only in some model domains such as the unit disc [Du], the unit ball [BB], and the strongly pseudoconvex domain [Be2]. Recently, the first author proved the case $\alpha = 0$ in the case of a convex domain of finite type [Ch]. The key point in the proof is the reproducing kernel with right estimate matching quasimetric on ∂D. Usually we study the behavior of holomorphic functions in terms of the basic invariant objects attached to the domain: the Bergman kernel and its metric, the Szegö kernel, and the Poisson-Szegö kernel, all of which naturally take into account simple geometric considerations. However, in general domains not enough is known about these domain functions and so we must use a different approach. Stein [St2] introduced the boundary behavior of H^p-functions in general bounded domains in $\mathbb{C}^n$ with C^2 boundary, without making use of any assumptions of pseudo-convexity. In our proofs we overcome the difficulty by using the Fatou theorem for H^p-functions proved by Stein [St2] and the growth space $A^{-\sigma}(D)$ introduced by Korenblum ([Ko1], [Ko2]).

In Section 6 we observe that the assumption of C^2-smoothness of the boundary of D is an essential condition for the study of the behavior of holomorphic functions in a general bounded domain. We give a counter-example of a convex domain with $C^{1,\lambda}$ smooth boundary for $0 < \lambda < 1$ which does not satisfy our embedding results. Here a $C^{1,\lambda}$-function is a function whose first derivatives are Lipschitz continuous of order λ. The counter-example shows that even a small loss of derivatives of the boundary is not permitted for the sharp embedding in Theorem 1.1.

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2. Growth spaces $A^{-\sigma}(D)$

It is well-known that

$$|f(z)| \leq (1-|z|^2)^{-(n+\alpha)/p} \|f\|_{p,\alpha}, \quad f \in A^{p,\alpha}(\mathbb{B}^n),\ z \in \mathbb{B}^n,$$

where $\mathbb{B}^n$ is the unit ball (see [Ru], [Vu]). For a general bounded domain D with C^2 boundary, the same sharp estimates hold for $A^{p,\alpha}(D)$ functions. For the convenience of the reader we give here a proof. In [CK] we proved that the assumption of C^2-smoothness of the boundary of D is an essential condition for the growth estimates in Lemma 2.1.

LEMMA 2.1. *Let $\alpha \geq 0$ and $0 < p < \infty$. Then we have*

$$\sup\{\delta_D(z)^{(n+\alpha)/p}|f(z)| : z \in D\} \lesssim \|f\|_{p,\alpha} \quad \textit{for} \quad f \in A^{p,\alpha}(D).$$

Proof. For $p_0 \in D$ sufficiently near ∂D, we translate and rotate the coordinate system so that $z(p_0) = 0$ and the Im z_1 axis is perpendicular to ∂D. Let $\mathcal{B}_\epsilon(p_0)$ denote the non-isotropic ball

$$\mathcal{B}_\epsilon(p_0) = \left\{ \frac{|z_1|^2}{(\epsilon\delta_D(p_0))^2} + \sum_2^n \frac{|z_j|^2}{\epsilon\delta_D(p_0)} < 1 \right\}.$$

Since ∂D is C^2, it follows that there is an $\epsilon_0 > 0$ such that for p_0 sufficiently near ∂D and $z \in \mathcal{B}_{\epsilon_0}(p_0)$ we have $z \in D$ and

$$\frac{\delta_D(p_0)}{2} \leq \delta_D(z) \leq 2\delta_D(p_0) \tag{2.1}$$

(see [Be1]). Let $0 < p < \infty$ and $\alpha > 0$. Let $f \in A^{p,\alpha}(D)$. Since the plurisubharmonicity of $|f|^p$ is invariant under the affinity

$$(z_1, z_2, \ldots, z_n) \to \left(\frac{z_1}{\epsilon_0\delta_D(p_o)}, \frac{z_2}{\sqrt{\epsilon_0\delta_D(p_0)}}, \ldots, \frac{z_n}{\sqrt{\epsilon_0\delta_D(p_0)}} \right),$$

it follows that

$$\begin{aligned} |f(p_0)|^p &\lesssim \frac{1}{\mathrm{Vol}(\mathcal{B}_{\epsilon_0}(p_0))} \int_{\mathcal{B}_{\epsilon_0}(p_0)} |f(z)|^p dV(z) \\ &\lesssim \frac{1}{(\epsilon_0\delta_D(p_0))^{n+1}} \int_{\mathcal{B}_{\epsilon_0}(p_0)} |f(z)|^p dV(z). \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), it follows that

$$|f(p_0)| \lesssim \delta_D(p_0)^{-(n+\alpha)/p}\|f\|_{p,\alpha}.$$

Parameterizing ∂D locally by $x_1, z_2, \ldots, z_n$, we let $\tilde{\mathcal{B}}_{\epsilon_0}(p_0)$ denote the non-isotropic ball on ∂D:

$$\tilde{\mathcal{B}}_{\epsilon_0}(p_0) = \left\{ z \in \partial D : \frac{|x_1|^2}{(\epsilon_0\delta_D(p_0))^2} + \sum_{j=2}^n \frac{|z_j|^2}{\epsilon_0\delta_D(p_0)} < 1 \right\}.$$

For any $u \in L^p(\partial D)$ we denote by Λu the Hardy-Littlewood maximal function of u:

$$\Lambda u(z) = \sup_{\epsilon > 0} \frac{1}{\sigma(\tilde{\mathcal{B}}_\epsilon(z))} \int_{\tilde{\mathcal{B}}_\epsilon(z)} |u| d\sigma.$$

Let $f \in A^{p,0}(D)$. For $1 < p < \infty$ we let f^* be a boundary value function in $L^p(\partial D)$. For $z \in \mathcal{B}_{\epsilon_0}(p_0)$ we let $\pi(z)$ denote the projection of z onto ∂D. Then it follows that

$$|f(z)| \le C\Lambda f^*(\pi(z)) \quad \text{for} \quad z \in \mathcal{B}_{\epsilon_0}(p_0).$$

From (2.2) we obtain that

$$\begin{aligned} |f(p_0)|^p &\lesssim \frac{1}{\delta_D(p_0)^{n+1}} \int_{\mathcal{B}_{\epsilon_0(p_0)}} |f(z)|^p dV(z) \\ &\lesssim \frac{1}{\delta_D(p_0)^{n+1}} \int_{\mathcal{B}_{\epsilon_0(p_0)}} \Lambda f^*(\pi(z))^p dV(z) \\ &\lesssim \frac{1}{\delta_D(p_0)^{n+1}} \int_{\tilde{\mathcal{B}}_{\epsilon_0(p_0)}} \int_{\delta_D(p_0)/2}^{2\delta_D(p_0)} \Lambda f^*(\zeta)^p dt d\sigma(\zeta) \\ &\lesssim \frac{1}{\delta_D(p_0)^n} \|\Lambda f^*\|^p_{L^p(\partial D)} \\ &\lesssim \frac{1}{\delta_D(p_0)^n} \|f\|^p_{p,0}. \end{aligned} \tag{2.3}$$

If $0 < p \le 1$, we apply the estimate (2.3) above to the function $|f|^{1/s}$, where s is a large positive number, and with p replaced by sp, and obtain the required inequality in this case as well. □

For any $\sigma > 0$, the space $A^{-\sigma}(D)$ consists of holomorphic functions f in D such that

$$\|f\|_{-\sigma} = \sup\{\delta_D(z)^\sigma |f(z)| : z \in D\} < \infty.$$

It is easy to verify that $A^{-\sigma}(D)$ is a Banach space with the norm defined above. Each space $A^{-\sigma}(D)$ clearly contains all the bounded holomorphic functions. The growth spaces $A^{-\sigma}$ were introduced by Korenblum (see [Ko1] and [Ko2]) in the unit disc case.

For $0 < p < \infty$ and $\alpha \ge 0$ we define

$$A^{p,\alpha}_{-\sigma}(D) = A^{p,\alpha}(D) \cap A^{-\sigma}(D).$$

Then $A^{p,\alpha}_{-\sigma}(D)$ is a Banach space with the norm defined by

$$\|f\|_{p,\alpha,-\sigma} = \max\{\|f\|_{p,\alpha}, \|f\|_{-\sigma}\}.$$

Applying Lemma 2.1, we get the following result.

Proposition 2.2. *Let $0 < p < \infty$ and $\alpha \ge 0$. Then we have*

$$A^{p,\alpha}_{-(n+\alpha)/p}(D) = A^{p,\alpha}(D).$$

3. Inclusion relations between weighted Bergman spaces

Let $\alpha > 0$ and $0 < p \leq q < \infty$. Then

$$\begin{aligned} \int_D |f|^q dV_{\alpha+\sigma(q-p)} &= \int_D |f|^p |f|^{q-p} \delta^{\alpha-1} \delta^{\sigma(q-p)} dV \\ &\leq \left(\int_D |f|^p dV_\alpha \right) (\sup\ \delta^\sigma |f|)^{q-p} . \end{aligned} \tag{3.1}$$

By (3.1), we have that

$$\begin{aligned} \|f\|_{q,\alpha+\sigma(q-p)} &\leq \|f\|_{p,\alpha}^{p/q} \|f\|_{-\sigma}^{1-p/q} \\ &\leq \|f\|_{p,\alpha} + \|f\|_{-\sigma} \\ &\lesssim \|f\|_{p,\alpha,-\sigma}. \end{aligned}$$

Hence it follows that

$$A^{p,\alpha}_{-\sigma}(D) \subset A^{q,\alpha+\sigma(q-p)}(D).$$

If we choose $\sigma = (n+\alpha)/p$, by Proposition 2.2, we obtain the following result.

THEOREM 3.1. *Assume that* $0 < p \leq q < \infty$, $\alpha, \beta > 0$, *and* $(n+\alpha)/p = (n+\beta)/q$. *Then* $A^{p,\alpha}(D) \subset A^{q,\beta}(D)$ *and the inclusion is continuous.*

Theorem 3.1 above was proved by Beatrous in [Be1] by a different method.

4. Embedding of Hardy spaces

Let $\mathcal{N}$ be a real vector field in a neighborhood of ∂D which agrees with the outward unit normal vector field on ∂D. For $z \in \partial D$ and $t > 0$ sufficiently small, say $0 < t < \delta_0$, the integral curve for $\mathcal{N}$ through z has a unique intersection point with the hypersurface $\{z \in D : \delta_D(z) = t\}$. We denote this intersection point by z_t.

For any function f on D we define f_t on ∂D by $f_t(z) = f(z_t)$ for $z \in \partial D$. For $f \in A^{p,0}(D)$ we have that

$$\|f\|_{p,0} \simeq \sup_{0<t<\delta_0} \left(\int_{\partial D} |f_t|^p d\sigma \right)^{1/p} .$$

Let $\theta > 0, z \in \partial D$. Let ν_z be the unit outward complex normal vector at z. Define

$$\mathcal{A}_\theta(z) = \{\zeta \in D : |(\zeta - z) \cdot \bar{\nu}_z| < (1+\theta)\delta_z(\zeta), |z-\zeta|^2 < \theta \delta_z(\zeta)\},$$

where $\delta_z(\zeta)$ is the minimum of the distance from ζ to ∂D and from ζ to the tangent plane at z.

Definition 4.1. We say that f has an admissible limit at $z, z \in \partial D$, if

$$\lim_{\mathcal{A}_\theta(z) \ni \zeta \to z} f(\zeta) \quad \text{exists, for all} \quad \theta > 0.$$

Theorem 4.2 ([St2]). *Let $f \in H^p(D), p > 0$. Then f has admissible limits at almost every boundary point and*

$$\int_{z \in \partial D} \sup_{\zeta \in \mathcal{A}_\theta(z)} |f(\zeta)|^p d\sigma(z) \leq C_{\theta,p} \|f\|_{p,0}^p.$$

For $0 \leq \sigma < \infty$ we define the function $\mathcal{M}_\theta^\sigma f(z)$ on ∂D by

$$\mathcal{M}_\theta^\sigma f(z) = \sup\{\delta_D(\zeta)^\sigma |f(\zeta)| : \zeta \in \mathcal{A}_\theta(z) \cap (D \setminus D_{\delta_0})\},$$

where $D_{\delta_0} = \{z \in D : \delta_D(z) > \delta_0\}$.

Note that

$$\mathcal{M}_\theta^\sigma f(z) \leq \delta_0^\sigma \mathcal{M}_\theta^0 f(z) \tag{4.1}$$

and

$$\int_{\partial D} \mathcal{M}_\theta^0 f(z)^p d\sigma \lesssim \|f\|_{p,0}^p. \tag{4.2}$$

The following estimates on the mean growth of $H^p(D)$ functions were proved in [KK] for the case of the unit disc.

Theorem 4.3. *Let $0 < p, q, s < \infty, 0 < \alpha < q, \sigma > 0$, and $(q-p)s \leq p$. Let $\gamma = p\alpha s/(p-(q-\alpha)s)$. Let u be a non-negative function on D such that $\mathcal{M}_\theta^0 u \in L^p(\partial D)$ and $\mathcal{M}_\theta^\sigma u \in L^\gamma(\partial D)$. Then we have*

$$\int_{\partial D} \left(\int_0^{\delta_0} u_t(z)^q t^{\alpha\sigma - 1} dt \right)^s d\sigma \lesssim \|\mathcal{M}_\theta^0 u\|_{L^p(\partial D)}^{(q-\alpha)s} \|\mathcal{M}_\theta^\sigma u\|_{L^\gamma(\partial D)}^{\alpha s}.$$

Proof. Let $z \in \partial D$. First we suppose that $0 < \mathcal{M}_\theta^0 u(z) < \infty$. From (4.1) it follows that

$$0 < \frac{\mathcal{M}_\theta^\sigma u(z)}{\mathcal{M}_\theta^0 u(z)} \leq \delta_0^\sigma.$$

Take

$$t_0(z) = \left(\frac{\mathcal{M}_\theta^\sigma u(z)}{\mathcal{M}_\theta^0 u(z)} \right)^{1/\sigma}.$$

Then we have

$$\begin{aligned} \int_0^{\delta_0} u_t(z)^q t^{\alpha\sigma - 1} dt \leq \mathcal{M}_\theta^0 u(z)^q & \int_0^{t_0(z)} t^{\alpha\sigma - 1} dt \\ & + \mathcal{M}_\theta^\sigma u(z)^q \int_{t_0(z)}^{\delta_0} t^{\sigma(\alpha - q) - 1} dt. \end{aligned} \tag{4.3}$$

We note that

$$\int_0^{t_0(z)} t^{\alpha\sigma-1}dt = \frac{1}{\alpha\sigma}\left(\frac{\mathcal{M}_\theta^\sigma u(z)}{\mathcal{M}_\theta^0 u(z)}\right)^\alpha \tag{4.4}$$

and

$$\begin{aligned}\int_{t_0(z)}^{\delta_0} t^{\sigma(\alpha-q)-1}dt &= \frac{1}{\sigma(\alpha-q)}\left\{\delta_0^{\sigma(\alpha-q)} - t_0(z)^{\sigma(\alpha-q)}\right\}\\ &\le \frac{1}{\sigma(q-\alpha)}\left(\frac{\mathcal{M}_\theta^\sigma u(z)}{\mathcal{M}_\theta^0 u(z)}\right)^{\alpha-q}.\end{aligned} \tag{4.5}$$

By (4.3), (4.4), and (4.5), it follows that

$$\int_0^{\delta_0} u_t(z)^q t^{\alpha\sigma-1}dt \lesssim \mathcal{M}_\theta^0 u(z)^{q-\alpha}\mathcal{M}_\theta^\sigma u(z)^\alpha. \tag{4.6}$$

Indeed, the inequality (4.6) is trivial if $\mathcal{M}_\theta^0 u(z) = \infty$. Now suppose that $\mathcal{M}_\theta^0 u(z) = 0$. Since $z_t \in \mathcal{A}_\theta(z) \cap (D \setminus D_{\delta_0})$, we have in this case $u_t(z) = 0$ for $0 < t < \delta_0$. Thus (4.6) holds for every $z \in \partial D$.

By Hölder's inequality, we have

$$\begin{aligned}\int_{\partial D} \mathcal{M}_\theta^0 u(z)^{s(q-\alpha)}\mathcal{M}_\theta^\sigma u(z)^{s\alpha}d\sigma &\lesssim \left(\int_{\partial D} \mathcal{M}_\theta^0 u(z)^p d\sigma\right)^{(q-\alpha)s/p}\\ &\quad\times\left(\int_{\partial D} \mathcal{M}_\theta^\sigma u(z)^\gamma d\sigma\right)^{\alpha s/\gamma}.\end{aligned} \tag{4.7}$$

By (4.2), (4.6), and (4.7), it follows that

$$\int_{\partial D}\left(\int_0^{\delta_0} u_t(z)^q t^{\alpha q-1}dt\right)^s d\sigma \lesssim \|\mathcal{M}_\theta^0 u\|_{L^p(\partial D)}^{(q-\alpha)s}\|\mathcal{M}_\theta^\sigma u\|_{L^\gamma(\partial D)}^{\alpha s}. \qquad \square$$

COROLLARY 4.4. *Let $\sigma > 0$ and $0 < p \le q < \infty$. Then we have*

$$A_{-\sigma}^{p,0}(D) \subset A^{q,\sigma(q-p)}(D).$$

Proof. We note that

$$\begin{aligned}\|\mathcal{M}_\theta^\sigma f\|_{L^\infty(\partial D)} &= \sup_{z\in\partial D}\sup\{\delta_D(\zeta)^\sigma|f(\zeta)| : \zeta \in \mathcal{A}_\theta(z)\cap(D\setminus D_{\delta_0})\}\\ &\le \sup\{\delta_D(\zeta)^\sigma|f(\zeta)| : \zeta \in D\setminus D_{\delta_0}\}\\ &\le \|f\|_{-\sigma}.\end{aligned}$$

We choose $\alpha = q - p$ and $s = 1$ in Theorem 4.3. Then it follows that

$$\int_{\partial D}\left(\int_0^{\delta_0} |f_t(z)|^q t^{(q-p)\sigma-1}dt\right)d\sigma \lesssim \|f\|_{p,0,-\sigma}^q.$$

Hence we have

$$\int_{D\setminus D_{\delta_0}} |f|^q dV_{\sigma(q-p)} \lesssim \|f\|_{p,0,-\sigma}^q.$$

Since $|f|^q$ is subharmonic, it follows that

$$\int_{D_{\delta_0}} |f|^q dV_{\sigma(q-p)} \lesssim \int_{D\setminus D_{\delta_0}} |f|^q dV_{\sigma(q-p)}.$$

Thus we have the result. □

If we choose $\sigma = n/p$ in Corollary 4.4 and apply Proposition 2.2, we obtain the following result.

Corollary 4.5. *Let $0 < p \le q < \infty, \beta \ge 0$, and $n/p = (n+\beta)/q$. Then $A^{p,0}(D) \subset A^{q,\beta}(D)$ and the inclusion is continuous.*

Theorem 1.1 is a consequence of Theorem 3.1 and Corollary 4.5.

5. A counter-example

In this section we observe that the assumption of C^2-smoothness of the boundary of D is an essential condition for the sharp embedding of Theorem 1.1 in a general bounded domain. Let $C^{m,\lambda}$ be the space of C^m-functions whose m-th derivatives are Lipschitz continuous of order λ.

Lemma 5.1. *Let $u(z) = |z|^{m+\lambda}$ be a function in one complex variable $z = x + iy \in \mathbb{C}$, where m is a non-negative integer and $0 < \lambda < 1$. Let $R > 0$. Then $u \in C^{m,\lambda}(\overline{D_R(0)})$, but $u \notin C^{m,\nu}(\overline{D_R(0)})$, where $\lambda < \nu \le 1$.*

Proof. For $0 \le |\alpha| \le m$ we have

$$D^\alpha u(z) = P_\alpha(x,y)|z|^{m+\lambda-2|\alpha|},$$

where $P_\alpha(x,y)$ is a homogeneous polynomial of degree $|\alpha|$ in x and y. Hence it follows that

$$|D^\alpha u(z)| \le K_\alpha |z|^\lambda, \quad z \in \overline{D_R(0)},$$

and so $u \in C^{m,\lambda}(\overline{D_R(0)})$.

For $|\alpha| = m$ we have

$$\begin{aligned}\frac{|D^\alpha u(x) - D^\alpha u(0)|}{|x-0|^\nu} &= \frac{|P_\alpha(x,0)||x|^{\lambda-m}}{|x|^\nu}\\ &= K'_\alpha \frac{1}{|x|^{\nu-\lambda}}.\end{aligned}$$

Thus $u \notin C^{m,\nu}(\overline{D_R(0)})$ when $\lambda < \nu \le 1$. □

EXAMPLE 5.2. We consider the domain defined by

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{1+\lambda} < 1\}, \quad \text{where} \quad 0 < \lambda < 1.$$

Applying Lemma 5.1, we see that D is a bounded convex domain with $C^{1,\lambda}$ boundary, but it has no C^2 boundary.

Let $0 < p \leq q < \infty, \alpha, \beta \geq 0$, and $(n+\alpha)/p = (n+\beta)/q$. Let $f(z_1, z_2)$ be a branch of $(1 - z_1)^{-d}$ on $\bar{D}$, where $(1 + 2/(1+\lambda) + \beta)/q < d < (1 + 2/(1 + \lambda) + \alpha)/p$. We will prove that

$$f \in A^{p,\alpha}(D), \quad \text{but} \quad f \notin A^{q,\beta}(D).$$

These two facts imply that $A^{p,\alpha}(D)$ cannot be embedded into $A^{q,\beta}(D)$.

First we consider the case $\alpha = 0$. Set $r(z_1) = (1-|z_1|^2)^{1/(1+\lambda)}$. By Fubini's theorem, we have

$$\begin{aligned}
&\int_{\partial D} \frac{d\sigma}{|1 - z_1|^{dp}} \\
&\quad = \int_{|z_1|<1} \frac{dA}{|1 - z_1|^{dp}} \\
&\qquad \times \int_{|z_2|=r(z_1)} \left(1 + \left(\frac{2}{1+\lambda} \right)^2 (1 - |z_1|^2)^{2/(1+\lambda)-2} |z_1|^2 \right)^{1/2} ds \\
&\quad \lesssim \int_{|z_1|<1} \frac{dA}{|1 - z_1|^{dp}} \int_{|z_2|=r(z_1)} (1 - |z_1|^2)^{1/(1+\lambda)-1} |z_1| ds \\
&\quad \lesssim \int_{|z_1|<1} \frac{(1 - |z_1|^2)^{2/(1+\lambda)-1}}{|1 - z_1|^{dp}} dA \\
&\quad = \lim_{r \to 1^-} \int_{|z_1|<1} \frac{(1 - |z_1|^2)^{2/(1+\lambda)-1}}{|1 - r z_1|^{dp}} dA \\
&\quad \simeq \lim_{r \to 1^-} \frac{1}{(1 - r^2)^{dp-1-2/(1+\lambda)}} < \infty,
\end{aligned}$$

since $dp - 1 - 2/(1 + \lambda) < 0$. Hence $f \in A^{p,0}(D)$.

Since D is a Lipschitz domain, we have

$$1 - |z_1|^2 - |z_2|^{1+\lambda} \simeq \delta_D(z_1, z_2) \quad \text{for} \quad (z_1, z_2) \in D \tag{5.1}$$

(see Lemma 2 in [St1], Section 3.2.1 of Chapter VI).

By (5.1), it follows that

$$\begin{aligned}
\int_D \frac{1}{|1 - z_1|^{dq}} dV_\beta \simeq &\int_{|z_1|<1} \frac{dA}{|1 - z_1|^{dq}} \\
&\times \int_{|z_2|<r(z_1)} (1 - |z_1|^2 - |z_2|^{1+\lambda})^{\beta-1} dA.
\end{aligned}$$

We now estimate the integral

$$I(z_1) = \int_{|z_2|<r(z_1)} (1 - |z_1|^2 - |z_2|^{1+\lambda})^{\beta-1} dA.$$

Changing to polar coordinates, we have

$$\begin{aligned} I(z_1) &\simeq \int_0^{r(z_1)} (1 - |z_1|^2 - r^{1+\lambda})^{\beta-1} r dr \\ &\simeq \int_0^{1-|z_1|^2} (1 - |z_1|^2 - s)^{\beta-1} s^{2/(1+\lambda)-1} ds \\ &\simeq (1 - |z_1|^2)^{2/(1+\lambda)+\beta-1} \int_0^1 (1-\tau)^{\beta-1} \tau^{2/(1+\lambda)-1} d\tau. \end{aligned}$$

Note that

$$\int_0^1 (1-\tau)^{\beta-1} \tau^{2/(1+\lambda)-1} d\tau = B\left(\frac{2}{1+\lambda}, \beta\right),$$

where $B(\cdot,\cdot)$ is the beta function. Hence we have

$$\begin{aligned} \int_D \frac{1}{|1-z_1|^{dq}} dV_\beta &\simeq \int_{|z_1|<1} \frac{(1-|z_1|^2)^{2/(1+\lambda)+\beta-1}}{|1-z_1|^{dq}} dA \\ &= \lim_{r\to 1^-} \int_{|z_1|<1} \frac{(1-|z_1|^2)^{2/(1+\lambda)+\beta-1}}{|1-rz_1|^{dq}} dA \\ &\simeq \lim_{r\to 1^-} \frac{1}{(1-r^2)^{dq-2/(1+\lambda)-\beta-1}} = \infty, \end{aligned}$$

since $dq - 2/(1+\lambda) - \beta - 1 > 0$. Thus $f \notin A^{q,\beta}(D)$.

By similar calculations as above, we see that in the case $\alpha > 0$ we also have

$$f \in A^{p,\alpha}(D), \quad \text{but} \quad f \notin A^{q,\beta}(D)$$

for $(1 + 2/(1+\lambda) + \beta)/q < d < (1 + 2/(1+\lambda) + \alpha)/p$.

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