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EXTREMAL PROPERTIES OF HILBERT FUNCTIONS

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1. Introduction

Recently there has been a lot of interest in the extremal properties of Hilbert functions. This subject is related to combinatorics, commutative algebra, and algebraic geometry. It was founded by Macaulay [12] who gave a characterization of the Hilbert functions of quotients of polynomial rings. His result can also be interpreted as a characterization of the h-vectors of multicomplexes [15, §2.2]. Kruskal [11] and Katona [10] characterized the f-vectors of simplicial complexes, or equivalently, the Hilbert functions of quotients of exterior algebras. Gotzmann proved a Persistence Theorem which states that an extremal (in the sense of Macaulay's theorem) vector space of homogeneous polynomials of degree d generates an extremal vector space in degree $d+1$ [6]. We will call such a vector space *Gotzmann.* Green [7] characterized the Hilbert functions of rings obtained by moding out quotients of polynomial rings with fixed Hilbert function by a general linear form. Recently, Aramova, Herzog, and Hibi [1] proved a Persistence Theorem for exterior algebras.

In §2 we introduce some notation. In §3 we study Gotzmann vector spaces and obtain:

- a Reverse Persistence Theorem similar to Gotzmann's;
- a Persistence Theorem for vector spaces which are extremal in the sense of Green's theorem;
- a structure theorem for Gotzmann vector spaces which generalizes structure results of Green [7] and Bigatti-Geramita-Migliore [4].

Macaulay's theorem can be stated in two equivalent ways: one is that for every homogeneous ideal there is a lexicographic ideal with the same Hilbert function; the other is numerical. The corresponding generalizations to modules over polynomial rings however are not equivalent. Hulett [8, 9] and Pardue [13, 14] showed that for every graded submodule of a free module over a polynomial ring there is a lexicographic submodule with the same Hilbert function.

In §4 we give a numerical generalization of Macaulay's theorem and generalizations of Green's and Gotzmann's theorems for finitely generated modules over polynomial rings. We also give generalizations of Kruskal-Katona's theorem and Aramova-Herzog-Hibi Persistence Theorem for finitely generated modules over exterior algebras.

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2. Preliminaries

Let d and a be positive integers. Then there exist unique positive integers $\delta = \delta(a, d)$ and $m_d, m_{d-1}, \ldots, m_\delta$ such that $m_d > m_{d-1} > \cdots > m_\delta \geq \delta$ and

$$a = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}. \tag{1}$$

We call (1) the *d-binomial representation* of a. Sometimes it will be inconvenient to specify what the value of δ is. For this reason we define the *non-reduced d-*binomial representation of a to be

$$a = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_1}{1}, \tag{2}$$

where $m_i = i - 1$ for $1 \leq i \leq \delta - 1$. If $\delta = 1$, then the d-binomial representation and the reduced d-binomial representation of a coincide. Note that the m_i's satisfy $m_d > m_{d-1} > \cdots > m_1 \geq 0$ and that this condition determines uniquely the non-reduced d-binomial representation of a. Note also that even though 0 does not have a d-binomial representation, it does have a non-reduced d-binomial representation, namely $0 = \binom{d-1}{d} + \binom{d-2}{d-1} + \cdots + \binom{0}{1}$. We let $\delta(0, d) = \infty$. For fixed d the bijection $a \leftrightarrow (m_d, m_{d-1}, \ldots, m_1)$ is order-preserving, where the order on the left-hand side is the usual order on the nonnegative integers and the order on the right-hand side is the lexicographic order.

There are three operations on nonnegative integers which will be important for us. If the non-reduced d-binomial representation of a is given by (2), then we set

$$a^{\langle d\rangle} = \binom{m_d+1}{d+1} + \binom{m_{d-1}+1}{d} + \cdots + \binom{m_1+1}{2},$$

$$a_{\langle d\rangle} = \binom{m_d-1}{d} + \binom{m_{d-1}-1}{d-1} + \cdots + \binom{m_1-1}{1},$$

$$a^{(d)} = \binom{m_d}{d+1} + \binom{m_{d-1}}{d} + \cdots + \binom{m_1}{2}.$$

It is easy to verify that $a \leq b$ is equivalent to $a^{\langle d\rangle} \leq b^{\langle d\rangle}$ and implies that $a_{\langle d\rangle} \leq b_{\langle d\rangle}$. In particular, $a = b$ is equivalent to $a^{\langle d\rangle} = b^{\langle d\rangle}$. Note that we can define $a^{\langle d\rangle}$, $a_{\langle d\rangle}$, and $a^{(d)}$ in exactly the same way as above by using the (reduced) d-binomial representation of a. Later we will need the following lemma which can be easily verified:

LEMMA 2.1. *If the d-binomial representation of a is* $a = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}$ *and* $\delta = 1$, *then* $(a+1)_{\langle d\rangle} = a_{\langle d\rangle} + 1$.

Throughout this paper k will be a field, $S = k[x_1, \dots, x_n]$ the polynomial ring over k in the variables $x_1, \dots, x_n$, and S_i the degree i homogeneous component of S. For a homogeneous ideal $I \subseteq S$ we denote by I_i the degree i component of I. If $V \subseteq S_d$ is a vector space, then we let $\delta(V) = \delta(\operatorname{codim}(V, S_d), d)$. When there is no danger of confusion we write $\operatorname{codim} V$ instead of $\operatorname{codim}(V, S_d)$. We denote by (V) the ideal generated by V.

Throughout x will be a general element of S_1. Fix d and let $V \subseteq S_d$ be a subspace. We denote by $\overline{V}$ the image of V in $\overline{S} = S/(x)$. Following [7] we set

$$c = \operatorname{codim}(V, S_d), \quad c_x = \operatorname{codim}(\overline{V}, \overline{S}_d),$$
$$c_1 = \operatorname{codim}(VS_1, S_{d+1}), \quad c_{1,x} = \operatorname{codim}(\overline{VS}_1, \overline{S}_{d+1}).$$

3. Gotzmann and Green vector spaces

By Macaulay's Theorem [12] $\operatorname{codim}(VS_1, S_{d+1}) \le \operatorname{codim}(V, S_d)^{\langle d\rangle}$. We call a vector space *Gotzmann* if equality holds. For such extremal space by Gotzmann Persistence Theorem [6] we have that the spaces VS_i are Gotzmann as well. Similarly, by Green's Theorem [7], $\operatorname{codim}(\overline{V}, \overline{S}_d) \le \operatorname{codim}(V, S_d)_{\langle d\rangle}$ and we call a vector space *Green* if equality holds.

THEOREM 3.1. *Let $V \subseteq S_d$ be a Gotzmann vector space. Then we have*:

1. *V is a Green vector space*;
2. *$\overline{V}$ is Gotzmann*;
3. $(VS_1 : x) = V$;
4. $(V : x) = (V : S_1)$.

Proof. From the exact sequence

$$0 \to (VS_1 : x) \xrightarrow{x} VS_1 \to \overline{VS_1} \to 0$$

and the fact that $\dim S_d = \dim S_{d-1} + \dim \overline{S}_d$ we can conclude that $c_1 = c_{1,x} + \operatorname{codim}(VS_1 : x)$. Since $V \subseteq (VS_1 : x)$, it follows that

$$c_1 = c_{1,x} + \operatorname{codim}(VS_1 : x) \le c_{1,x} + \operatorname{codim} V = c_{1,x} + c, \tag{3}$$

so $c_1 - c \le c_{1,x}$. Then from the assumption that V is Gotzmann and Macaulay's and Green's theorems it follows that

$$(c_{\langle d\rangle})^{\langle d\rangle} = c^{\langle d\rangle} - c = c_1 - c \le c_{1,x} \le (c_x)^{\langle d\rangle} \le (c_{\langle d\rangle})^{\langle d\rangle}, \tag{4}$$

so all inequalities in (4) must be equalities, and in particular $(c_x)^{\langle d\rangle} = (c_{\langle d\rangle})^{\langle d\rangle}$. This implies that $c_x = c_{\langle d\rangle}$, so V is a Green vector space. It also follows from (4) that

$c_{1,x} = (c_x)^{\langle d \rangle}$, i.e., $\overline{V}$ is Gotzmann. We also have that $c_1 = c_{1,x} + c$, so the inequality in (3) is an equality, hence $(VS_1 : x) = V$. Since $((V : x)S_1)x = ((V : x)x)S_1 \subseteq VS_1$, we have that $(V : x)S_1 \subseteq (VS_1 : x) = V$. Therefore $(V : x) \subseteq (V : S_1)$, but we always have that $(V : S_1) \subseteq (V : x)$, so $(V : x) = (V : S_1)$. □

Remark 3.2. It can be shown that if I is a homogeneous saturated ideal in S generated in degrees $\leq d$ and I_d is Gotzmann, then a linear form is general in the sense of Theorem 3.1 exactly when it is a nonzerodivisor on the ring S/I. This shows that a result due to Bigatti, Geramita, and Migliore [4, Lemma 1.1] is equivalent to Theorem 3.1 (2). Moreover, they also noticed [4, Remark 1.2] that $c_{1,x} = c_{1\langle d+1 \rangle}$, which is a corollary to Theorem 3.1 (1).

Remark 3.3. It should be noted that not every Green vector space is Gotzmann. Take for example $V = \operatorname{span}\{x^2, y^2\} \subseteq k[x, y]_2$. Then $c = 1$ and $c_x = 0 = c_{\langle 2 \rangle}$, so V is a Green vector space, but $c_1 = 0 \lneqq c^{\langle 2 \rangle} = 1$, so V is not Gotzmann. It is also interesting to note that in this example V does not satisfy the conclusions (3) and (4) of Theorem 3.1.

THEOREM 3.4 (Reverse Persistence Theorem). *Let $V \subseteq S_d$ be a Gotzmann vector space and let the d-binomial representation of c be $c = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}$ with $\delta > 1$. Then $V = (V : S_1)S_1$ and $(V : S_1)$ is a Gotzmann vector space with* $\operatorname{codim}(V : S_1) = \binom{m_d - 1}{d-1} + \binom{m_{d-1}-1}{d-2} + \cdots + \binom{m_\delta - 1}{\delta - 1}$.

Proof. From the exact sequence

$$0 \to (V : x) \xrightarrow{x} V \to \overline{V} \to 0$$

and Theorem 3.1 it follows that

$$\begin{aligned}
\operatorname{codim}(V : x) &= c - c_x = c - c_{\langle d \rangle} \\
&= \left[\binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}\right] \\
&\quad - \left[\binom{m_d - 1}{d} + \binom{m_{d-1} - 1}{d-1} + \cdots + \binom{m_\delta - 1}{\delta}\right] \\
&= \binom{m_d - 1}{d-1} + \binom{m_{d-1} - 1}{d-2} + \cdots + \binom{m_\delta - 1}{\delta - 1}. \qquad (5)
\end{aligned}$$

The last expression is the $(d-1)$-binomial representation of $\operatorname{codim}(V : x)$, because $\delta > 1$. From Macaulay's theorem and (5) it follows that

$$\begin{aligned}
\operatorname{codim}(V : x)S_1 &\leq (\operatorname{codim}(V : x))^{\langle d-1 \rangle} = (c - c_x)^{\langle d-1 \rangle} \\
&= \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta} = c = \operatorname{codim} V.
\end{aligned}$$

Applying Theorem 3.1 we see that $(V : x) = (V : S_1)$, hence $(V : x)S_1 = (V : S_1)S_1 \subseteq V$. Thus $\operatorname{codim}(V : x)S_1 \geq \operatorname{codim} V$, hence $\operatorname{codim}(V : x)S_1 = \operatorname{codim} V$. Therefore $(V : S_1)S_1 = (V : x)S_1 = V$. Then $\operatorname{codim}(V : S_1)S_1 = \operatorname{codim} V = (\operatorname{codim}(V : S_1))^{\langle d-1\rangle}$, so $(V : S_1)$ is a Gotzmann vector space. □

The following theorem is an analog of Gotzmann Persistence Theorem for restrictions to general hyperplanes.

THEOREM 3.5. *Let $V \subseteq S_d$ be a Green vector space and the d-binomial representation of c be $c = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}$. If $\delta \geq 2$ or $\delta = 1$ and $m_1 \neq 1$, then $\overline{V}$ is also a Green vector space. Moreover, if $\overline{y}$ is a general element of $\overline{S}_1$ and y is any preimage of $\overline{y}$ in S_1, then $(\overline{V} : \overline{y}) = \overline{(V : y)}$.*

Proof. Let $\overline{\overline{V}}$ be the image of V in $\overline{\overline{S}}_d = (S/(x, y))_d$ and $c_{x,y} = \operatorname{codim}(\overline{\overline{V}}, \overline{\overline{S}}_d)$. Consider the exact sequences

$$0 \to (V : x) \xrightarrow{x} V \to \overline{V} \to 0 \quad \text{and} \quad 0 \to (\overline{V} : \overline{y}) \xrightarrow{\overline{y}} \overline{V} \to \overline{\overline{V}} \to 0.$$

We have that

$$c_{\langle d\rangle} = c_x = \operatorname{codim} \overline{V} = \operatorname{codim} \overline{\overline{V}} + \operatorname{codim}(\overline{V} : \overline{y}) = c_{x,y} + \operatorname{codim}(\overline{V} : \overline{y}) \tag{6}$$

$$c_{x,y} \leq (c_x)_{\langle d\rangle} = (c_{\langle d\rangle})_{\langle d\rangle}. \tag{7}$$

Also $\overline{(V : y)} \subseteq (\overline{V} : \overline{y})$ and $\operatorname{codim}(V : y) = \operatorname{codim}(V : x)$ (because x and y are general), so

$$\begin{aligned} \operatorname{codim}(\overline{V} : \overline{y}) &\leq \operatorname{codim} \overline{(V : y)} \leq (\operatorname{codim}(V : y))_{\langle d-1\rangle} \\ &= (\operatorname{codim}(V : x))_{\langle d-1\rangle} = (c - c_x)_{\langle d-1\rangle} = (c - c_{\langle d\rangle})_{\langle d-1\rangle}. \end{aligned} \tag{8}$$

It follows from (6), (7), and (8) that

$$c_{\langle d\rangle} = c_x = c_{x,y} + \operatorname{codim}(\overline{V} : \overline{y}) \leq (c_{\langle d\rangle})_{\langle d\rangle} + (c - c_{\langle d\rangle})_{\langle d-1\rangle} = c_{\langle d\rangle}.$$

Therefore all inequalities in (7) and (8) must be equalities, which completes the proof. □

The next example shows that it is necessary to assume that $m_1 \neq 1$ in Theorem 3.5.

Example 3.6. Let $n \geq 4$ and consider the vector space

$$V = \operatorname{span}(x_1^d, x_2^d, \ldots, x_{n-1}^d) \subset k[x_1, x_2, \ldots, x_n]_d.$$

After a change of variables we can assume that

$$\overline{V} \cong \operatorname{span}(x_1^d, x_2^d, \ldots, x_{n-1}^d) \subset k[x_1, x_2, \ldots, x_{n-1}]_d.$$

We can also assume that $y = x_{n-1} - \sum_{i=1}^{n-2} a_i x_i$, so

$$\overline{\overline{V}} \cong \operatorname{span}\left(x_1^d, x_2^d, \ldots, x_{n-2}^d, \left(\sum_{i=1}^{n-2} a_i x_i\right)^d\right) \subset k[x_1, x_2, \ldots, x_{n-2}]_d.$$

Since the a_i's are general, we see that $\dim \overline{\overline{V}} = n-1$. We also have $\dim V = \dim \overline{V} = n-1$, so

$$c = \operatorname{codim} V = \binom{n+d-2}{d} + \binom{n+d-3}{d-1} + \cdots + \binom{n}{2} + \binom{1}{1},$$

$$c_x = \operatorname{codim} \overline{V} = \binom{n+d-3}{d} + \binom{n+d-4}{d-1} + \cdots + \binom{n-1}{2},$$

and

$$c_{x,y} = \operatorname{codim} \overline{\overline{V}} = \binom{n+d-4}{d} + \binom{n+d-5}{d-1} + \cdots + \binom{n-2}{2} - 1.$$

Therefore $m_1 = 1$, $c_x = c_{\langle d \rangle}$, and $c_{x,y} = (c_x)_{\langle d \rangle} - 1 \neq (c_x)_{\langle d \rangle}$, so this is a counterexample to the first part of Theorem 3.5 without the hypothesis $m_1 \neq 1$. To get a counterexample to the second part, consider $V = \operatorname{span}(x_1^d) \subset k[x_1, x_2]_d$. Then $c = \binom{d+1}{d} - 1 = \binom{d}{d} + \binom{d-1}{d-1} + \cdots + \binom{1}{1}$, so $m_1 = 1$ and $(V : y) = 0$, so $\overline{(V : y)} = 0$. We have that $\overline{V} \cong \operatorname{span}(z^d) = k[z]_d$, where z is an indeterminate, so $(\overline{V} : \overline{y}) = k[z]_{d-1} \neq \overline{(V : y)}$.

LEMMA 3.7. *Let $V \subseteq S_d$ be a Gotzmann vector space and the d-binomial representation of c be $c = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_\delta}{\delta}$ with $\delta \geq 2$ or $\delta = 1$ and $m_1 \neq 1$. Then $(\overline{V} : \overline{S}_1) = \overline{(V : S_1)}$ and $(V : S_1)$ is a Green vector space.*

Proof. Let y and $c_{x,y}$ be as in Theorem 3.5. By Theorem 3.1, V is a Green vector space, so by Theorem 3.5 we have that $\overline{V}$ is a Green vector space, i.e., $c_{x,y} = (c_x)_{\langle d \rangle}$.

If $\delta \geq 2$, apply Theorem 3.4. Now let $\delta = 1$. By Theorem 3.1, $(V : S_1) = (V : x)$ and $(\overline{V} : \overline{S}_1) = (\overline{V} : \overline{y})$, so

$$\operatorname{codim}(V : S_1) = c - c_x = \binom{m_d - 1}{d-1} + \binom{m_{d-1} - 1}{d-2} + \cdots + \binom{m_2 - 1}{1} + 1 \quad \text{and}$$

$$\begin{aligned}\operatorname{codim}(\overline{V} : \overline{S}_1) &= c_x - c_{x,y} = c_{\langle d \rangle} - (c_{\langle d \rangle})_{\langle d \rangle} \\ &= \binom{m_d - 2}{d-1} + \binom{m_{d-1} - 2}{d-2} + \cdots + \binom{m_2 - 2}{1} + 1.\end{aligned}$$

Using the fact that $(\overline{V} : \overline{S}_1) \supseteq \overline{(V : S_1)}$ and Lemma 2.1 we conclude that

$$\begin{aligned} \operatorname{codim}(\overline{V} : \overline{S}_1) &\leq \operatorname{codim} \overline{(V : S_1)} \leq (\operatorname{codim}(V : S_1))_{\langle d-1 \rangle} \\ &= \binom{m_d - 2}{d-1} + \binom{m_{d-1} - 2}{d-2} + \cdots + \binom{m_2 - 2}{1} + 1 \\ &= \operatorname{codim}(\overline{V} : \overline{S}_1). \end{aligned}$$

Therefore $\operatorname{codim}(\overline{V} : \overline{S}_1) = \operatorname{codim} \overline{(V : S_1)} = (\operatorname{codim}(V : S_1))_{\langle d-1 \rangle}$, so $(\overline{V} : \overline{S}_1) = \overline{(V : S_1)}$ and $(V : S_1)$ is a Green vector space. □

It is natural to ask whether we can say something about the structure of Gotzmann vector spaces. It was proved by Gotzmann in [6] that any homogeneous ideal $I \subseteq S$ has Hilbert polynomial of the form

$$P_{S/I}(t) = \binom{a_1 + t}{a_1} + \binom{a_2 + t - 1}{a_2} + \cdots + \binom{a_s + t - (s-1)}{a_s}, \tag{9}$$

where $a_1 \geq a_2 \geq \cdots \geq 0$. This implies that I_e is Gotzmann for $e \gg 0$. So we cannot hope to say much about the structure of arbitrary Gotzmann vector spaces V. However, in some cases the d-binomial representation of $\operatorname{codim} V$ determines the structure of V. One such case is treated in Theorem 3.8 below which was first proved by Green [7, Theorem 3] and was later given a different proof by Bigatti, Geramita, and Migliore [4, Lemma 3.1].

THEOREM 3.8. *Let $V \subseteq S_d$ be a Green vector space and I the saturation of* (V). *If $c = \binom{m+d}{d}$ for some $m \geq -1$, then I is generated by $n - m - 1$ linear forms, so in particular V is Gotzmann.*

It is not hard to see that if $V \subseteq S_d$ is a vector space and $h \neq 0$ a homogeneous form, then V is Gotzmann if and only if hV is. A vector space $V \subseteq S_d$ is called *reduced* if there is no vector space $\tilde{V} \neq 0$ and a homogeneous form $h \neq 0$ of degree ≥ 1 such that $V = h\tilde{V}$. So to study the structure of Gotzmann vector spaces it is enough to consider reduced vector spaces. The following theorem follows from [4, Proposition 2.7].

THEOREM 3.9. *Let $V \subseteq S_d$ be a Gotzmann vector space of* $\dim V \geq 2$. *Then V is reduced if and only if* $\dim V > \dim S_{d-1}$.

Now let I be a homogeneous ideal whose Hilbert polynomial is given by (9) and $r = r(I)$ be the least integer such that I_e is Gotzmann for all $e \geq r$. If I is saturated, then by Gotzmann Persistence Theorem [6] and Theorem 3.4 it follows that $r = s$ and I is the saturation of (I_r). In particular, the r-binomial representation of $\operatorname{codim} I_r$ is

$$\operatorname{codim} I_r = \binom{a_1 + r}{r} + \binom{a_2 + r - 1}{r-1} + \cdots + \binom{a_r + 1}{1},$$

so $\delta(I_r) = 1$. Thus there is a one-to-one correspondence between saturated homogeneous ideals I and Gotzmann vector spaces V with $\delta(V) = 1$. Namely, I corresponds to $I_{r(I)}$ and V corresponds to the saturation of (V).

Next we give a structure result about saturated homogeneous ideals, which by the discussion above can be interpreted as a structure result about Gotzmann vector spaces.

THEOREM 3.10. *Let I be a homogeneous saturated ideal in S. Then the Hilbert polynomial of S/I has the form* $P_{S/I}(t) = \binom{a+t}{a} + \binom{a+t-1}{a} + \cdots + \binom{a+t-(d-2)}{a} + \binom{b+t-(d-1)}{b}$ *with $a \geq b \geq 1$ if and only if* $\dim I_1 = n - a - 2$ *and one of the following is satisfied*:

1. *$a > b$ and there exist a vector space $W \subseteq S_1$ with $I_1 \cap W = 0$ and an element $h \in S_{d-1} \setminus (I_1)_{d-1}$ such that* $\dim W = a - b + 1$ *and* $I = (I_1) + (hW)$.
2. *$a = b$ and there exists an element $f \in S_d \setminus (I_1)_d$ such that* $I = (I_1) + (f)$.

Proof. The "if" part is easy to prove. To prove the "only if" part, note that $r(I) = d$, so I_d is a Gotzmann vector space with

$$\begin{aligned}\operatorname{codim} I_d &= \binom{a+d}{a} + \binom{a+d-1}{a} + \cdots + \binom{a+2}{a} + \binom{b+1}{b} \\ &= \binom{a+d}{d} + \binom{a+d-1}{d-1} + \cdots + \binom{a+2}{2} + \binom{b+1}{1}.\end{aligned}$$

Since I is saturated, this implies that $I_{d-1} = (I_d : S_1)$. By Theorem 3.1, $(I_d : S_1) = (I_d : x)$, so from the exact sequence $0 \to (I_d : x) \xrightarrow{x} I_d \to \overline{I}_d \to 0$ we get $\operatorname{codim} I_{d-1} = \operatorname{codim} I_d - \operatorname{codim} \overline{I}_d$. By Theorem 3.1 $\operatorname{codim} \overline{I}_d = (\operatorname{codim} I_d)_{\langle d\rangle}$, so $\operatorname{codim} I_{d-1} = \operatorname{codim} I_d - (\operatorname{codim} I_d)_{\langle d\rangle} = \binom{a+d-1}{d-1} + \binom{a+d-2}{d-2} + \cdots + \binom{a+1}{1} + 1 = \binom{a+d}{d-1}$. By Lemma 3.7, I_{d-1} is a Green vector space, so, by Theorem 3.8, I_{d-1} is Gotzmann and I_1 is spanned by $n - a - 2$ linear forms. Then

$$\operatorname{codim} I_{d-1} S_1 = (\operatorname{codim} I_{d-1})^{\langle d-1\rangle} = \binom{a+d+1}{d}$$

and

$$\dim I_d - \dim I_{d-1} S_1 = a - b + 1.$$

We can assume without loss of generality that I_1 is spanned by $x_{a+3}, x_{a+4}, \ldots, x_n$. Then we can write $I_d = I_{d-1} S_1 \oplus K$, where K is a vector subspace of $k[x_1, x_2, \ldots, x_{a+2}]_d$ of dimension $a - b + 1$. If $a = b$, then K is spanned by a single element $f \in S_d \setminus (I_1)_d$ and we are done. If $a > b$, then let L be any subspace of $k[x_1, x_2, \ldots, x_{a+2}]_d$ of dimension $a - b + 1$. Then

$$\begin{aligned}(I_{d-1} S_1) S_1 \cap L S_1 &= (x_{a+3}, x_{a+4}, \ldots, x_n)_{d+1} \cap L S_1 = L(x_{a+3}, x_{a+4}, \ldots, x_n)_1 \\ &= L x_{a+3} \oplus L x_{a+4} \oplus \cdots \oplus L x_n.\end{aligned}$$

Hence $\dim[(I_{d-1}S_1)S_1 \cap LS_1] = (n-a-2)\dim L = (n-a-2)(a-b+1)$, so

$$\begin{aligned}\dim[(I_{d-1}S_1 \oplus L)S_1] &= \dim[(I_{d-1}S_1)S_1 + LS_1] \\ &= \dim(I_{d-1}S_1)S_1 + \dim LS_1 - \dim[(I_{d-1}S_1)S_1 \cap LS_1] \\ &= \dim LS_1 + \binom{n+d}{d+1} - \binom{a+d+2}{d+1} \\ &\quad - (n-a-2)(a-b+1)\end{aligned}$$

and we can conclude that $\dim[(I_{d-1}S_1 \oplus L)S_1] - \dim LS_1$ does not depend on the choice of L. If L is generated by a lex-segment in $k[x_1, x_2, \ldots, x_{a+2}]_d$, then $I_{d-1}S_1 \oplus L$ is generated by a lex-segment in $k[x_1, x_2, \ldots, x_n]_d$ (we order $x_{a+3} < x_{a+4} < \cdots < x_n < x_1 < x_2 < \cdots < x_{a+2}$), thus $I_{d-1}S_1 \oplus L$ is Gotzmann. Since $I_{d-1}S_1 \oplus K = I_d$ is Gotzmann, it follows that

$$\dim[(I_{d-1}S_1 \oplus L)S_1] = \dim[(I_{d-1}S_1 \oplus K)S_1],$$

so $\dim LS_1 = \dim KS_1$. But L is Gotzmann, so K is Gotzmann.

Since $\dim K = a-b+1$ and $2 \le a-b+1 < n = \dim S_1$, it follows by Theorem 3.9 that there exists a subspace W in $k[x_1, x_2, \ldots, x_{a+2}]_1$ with $\dim W = a-b+1$ and an element $h \in k[x_1, x_2, \ldots, x_{a+2}]_{d-1}$ such that $K = hW$. Then $I = (x_{a+3}, x_{a+4}, \ldots, x_n) + (hW) = (I_1) + (hW)$. □

Green proved the special case $a = b = 1$ of Theorem 3.10 in [7, Theorem 4] and Bigatti, Geramita, and Migliore proved the more general special case $a = b$ in [4, Corollary 3.2]. Theorem 3.10 shows that in suitable coordinates I is "almost" lexicographic. It is also clear that the generic initial ideal of I, $\mathrm{gin}(I)$, is lexicographic:

COROLLARY 3.11. *If I is as in Theorem* 3.10, *then* $\mathrm{gin}(I)$ *is lexicographic.*

Remark 3.12. If $V \subseteq S_2$ is a Gotzmann vector space, then the saturation of the ideal generated by V satisfies the hypothesis of Theorem 3.10, so the structure of Gotzmann vector subspaces of S_2 is completely determined by Theorem 3.10. Also, by Theorem 3.9, a Gotzmann vector space V with $\dim V \le \dim S_2 = \binom{n+1}{2}$ has the form $V = hW$, where h is a homogeneous form and W is a subspace of S_e for some $e \in \{0, 1, 2\}$. Thus, Theorem 3.10 also determines the structure of any Gotzmann vector space of dimension $\le \binom{n+1}{2}$.

4. Hilbert functions of modules

Here we generalize Macaulay's Theorem [12], Green's theorem [7] and Gotzmann Persistence Theorem [6] for S-modules. We also give generalizations of Kruskal-Katona's Theorem [10, 11] and the Persistence Theorem of Aramova-Herzog-Hibi [1] for modules over an exterior algebra.

Remark 4.1. Hulett [8], [9] and Pardue [13], [14] generalized Macaulay's Theorem as follows: if F is a finitely generated free S-module and $V, L \subseteq F_d$ vector spaces of the same dimension such that L is generated by an initial lex-segment, then $\operatorname{codim}(VS_1, F_{d+1}) \leq \operatorname{codim}(LS_1, F_{d+1})$. However, unlike the ideal case, we no longer have $\operatorname{codim}(LS_1, F_{d+1}) = \operatorname{codim}(L, F_d)^{\langle d\rangle}$ when $L \subseteq F_d$ is generated by an initial lex-segment. Take for example $S = k[x]$, $F = S \oplus S$, $d = 1$, and $L = 0 \subseteq F_1$. Then $\operatorname{codim}(LS_1, F_2) = 2 \lneqq \operatorname{codim}(L, F_1)^{\langle 1\rangle} = 3$. Nevertheless, there exists a numerical generalization of Macaulay's theorem for S-modules which we give in part 2 of the next theorem.

THEOREM 4.2. *Let $S = k[x_1, \ldots, x_n]$ and $F = S\xi_1 + \cdots + S\xi_\nu$ be a finitely generated free S-module. Let $N \subseteq F$ be a graded submodule, $l = \max\{\deg \xi_i \mid i = 1, \ldots, \nu\}$, and $M = F/N$. Let x be a general element in S_1, $\overline{S} = S/(x)$, and $\overline{M} = F/(N + xF)$. Then for any pair (p, d) such that $p \geq 0$ and $d \geq p + l + 1$ we have*:

1. $\dim \overline{M}_d \leq (\dim M_d)_{\langle d-l-p\rangle}$;
2. $\dim M_{d+1} \leq (\dim M_d)^{\langle d-l-p\rangle}$;
3. *If N is generated in degrees $\leq d$ and* $\dim M_{d+1} = (\dim M_d)^{\langle d-l-p\rangle}$, *then* $\dim M_{d+2} = (\dim M_{d+1})^{\langle d+1-l-p\rangle}$.

Note that Theorem 4.2 (2) implies that for any $p \geq 0$ there exists a number $D = D(p)$ such that $\dim M_{d+1} = (\dim M_d)^{\langle d-l-p\rangle}$ whenever $d \geq D$. To see why this is true, set $h_d = \dim M_{d+l+p}$, so $h_{d+1} \leq h_d^{\langle d\rangle}$. There exists a polynomial ring P and a lexicographic ideal $L \subseteq P$ such that $\dim(P/L)_d = h_d$. If D is the largest degree of a minimal generator of L, then $\dim(P/L)_{d+1} = (\dim(P/L)_d)^{\langle d\rangle}$ for any $d \geq D$.

THEOREM 4.3. *Let $F = E\xi_1 + \cdots + E\xi_\nu$ be a finitely generated free module over an exterior algebra E and let $N \subseteq F$ be a graded submodule. Let $l = \max\{\deg \xi_i \mid i = 1, \ldots, \nu\}$ and $M = F/N$. Then for any pair (p, d) such that $p \geq 0$ and $d \geq p+l+1$ we have*:

1. $\dim M_{d+1} \leq (\dim M_d)^{(d-l-p)}$;
2. *If N is generated in degrees $\leq d$ and* $\dim M_{d+1} = (\dim M_d)^{(d-l-p)}$, *then* $\dim M_{d+2} = (\dim M_{d+1})^{(d+1-l-p)}$.

We will omit the proof of Theorem 4.3 because it is similar to that of Theorem 4.2. To prove the latter theorem we need some preliminary results.

LEMMA 4.4. *Let $a, b \geq 0$ and $d \geq 1$ be integers. Then*:

1. $a_{\langle d\rangle} + b_{\langle d\rangle} \leq (a + b)_{\langle d\rangle}$;

2. $a^{\langle d\rangle} + b^{\langle d\rangle} \le (a+b)^{\langle d\rangle}$;
3. $a^{(d)} + b^{(d)} \le (a+b)^{(d)}$;
4. *If* $a^{\langle d\rangle} + b^{\langle d\rangle} = (a+b)^{\langle d\rangle}$, *then* $(a^{\langle d\rangle})^{\langle d+1\rangle} + (b^{\langle d\rangle})^{\langle d+1\rangle} = (a^{\langle d\rangle} + b^{\langle d\rangle})^{\langle d+1\rangle}$;
5. *If* $a^{(d)} + b^{(d)} = (a+b)^{(d)}$, *then* $(a^{(d)})^{(d+1)} + (b^{(d)})^{(d+1)} = (a^{(d)} + b^{(d)})^{(d+1)}$.

Proof. Let $S = k[x_1, x_2, \ldots]$, $T = k[y_1, y_2, \ldots]$ be polynomial rings and $I \subseteq S$, $J \subseteq T$ homogeneous lex-segment ideals generated in degree d such that $H_{S/I}(d) = a$ and $H_{T/J}(d) = b$, where $H_{S/I}$ and $H_{T/J}$ are the Hilbert functions of S/I and T/J respectively. Then $H_{S/I}(d+1) = a^{\langle d\rangle}$, $H_{T/J}(d+1) = b^{\langle d\rangle}$, $H_{\overline{S}/\overline{I}}(d) = a_{\langle d\rangle}$, and $H_{\overline{T}/\overline{J}}(d) = b_{\langle d\rangle}$, where $\overline{S} = S/(x)$, $\overline{I} = I+(x)/(x)$ for some general element $x \in S_1$ and similarly for $\overline{T}$ and $\overline{J}$. Let $U = k[x_1, x_2, \ldots, y_1, y_2, \ldots]$ and K be the ideal of U generated by the elements of I, J, and all monomials of the form $x_i y_j$. Then

$$(U/K)_n \cong (S/I)_n \oplus (T/J)_n \text{ for } n \ge 1,$$

so

$$H_{U/K}(n) = H_{S/I}(n) + H_{T/J}(n) \text{ for } n \ge 1.$$

Let $z = \sum \alpha_i x_i + \sum \beta_j y_j$ be a general element in U_1 and let $x = \sum \alpha_i x_i$ and $y = \sum \beta_j y_j$. (Then x is a general element in S_1 and y is a general element in T_1.) For $d \ge 1$ we have the following sequence of maps of k-vector spaces:

$$(U/(K,z))_d \xrightarrow{\phi} (U/(K,x,y))_d \xrightarrow{\psi} (S/(I,x))_d \oplus (T/(J,y))_d, \tag{10}$$

where ϕ is a surjection and ψ is an isomorphism. Also ϕ is an isomorphism for $d \ge 2$. Since $I \subseteq S$ and $J \subseteq T$ are lex-segment ideals generated in degree d, we have

$$\begin{aligned} a_{\langle d\rangle} &= \dim(S/(I,x))_d, \ b_{\langle d\rangle} = \dim(T/(J,y))_d, \\ a^{\langle d\rangle} &= \dim(S/I)_{d+1}, \text{ and } b^{\langle d\rangle} = \dim(T/J)_{d+1}. \end{aligned}$$

So from (10) and Green's theorem [7] we get

$$\begin{aligned} a_{\langle d\rangle} + b_{\langle d\rangle} &= \dim(S/(I,x))_d + \dim(T/(J,y))_d = \dim(U/(K,x,y))_d \\ &\le \dim(U/(K,z))_d \le (\dim(U/K)_d)_{\langle d\rangle} = (H_{U/K}(d))_{\langle d\rangle} \\ &= (H_{S/I}(d) + H_{T/J}(d))_{\langle d\rangle} = (a+b)_{\langle d\rangle}. \end{aligned} \tag{11}$$

This proves part (1) of Lemma 4.4. To prove part (2), note that

$$\begin{aligned} a^{\langle d\rangle} + b^{\langle d\rangle} &= H_{S/I}(d+1) + H_{T/J}(d+1) = H_{U/K}(d+1) \\ &\le (H_{U/K}(d))^{\langle d\rangle} = (a+b)^{\langle d\rangle}. \end{aligned} \tag{12}$$

To prove part (4), note that the equality $a^{\langle d\rangle} + b^{\langle d\rangle} = (a+b)^{\langle d\rangle}$ implies that the inequality in (12) is an equality, so

$$H_{U/K}(d+1) = (H_{U/K}(d))^{\langle d\rangle}.$$

Since K is generated in degrees $\leq d$ we can apply the Gotzmann Persistence Theorem and conclude that $H_{U/K}(d+2) = (H_{U/K}(d+1))^{\langle d+1\rangle}$. Hence

$$\begin{aligned}(a^{\langle d\rangle})^{\langle d+1\rangle} + (b^{\langle d\rangle})^{\langle d+1\rangle} &= H_{S/I}(d+2) + H_{T/J}(d+2) = H_{U/K}(d+2)\\ &= (H_{U/K}(d+1))^{\langle d+1\rangle} = (a^{\langle d\rangle} + b^{\langle d\rangle})^{\langle d+1\rangle}.\end{aligned}$$

To prove (3) and (5) we replace the polynomial rings $S = k[x_1, x_2, \ldots]$ and $T = k[y_1, y_2, \ldots]$ by the exterior algebras on the x's and on the y's respectively and argue exactly as in the proofs of (2) and (4). □

LEMMA 4.5. *For any $a \geq 0$ and $d \geq 1$ we have*:

1. $a_{\langle d+1\rangle} \leq a_{\langle d\rangle}$;
2. $a^{\langle d+1\rangle} \leq a^{\langle d\rangle}$;
3. *If* $a^{\langle d+1\rangle} = a^{\langle d\rangle}$, *then* $(a^{\langle d+1\rangle})^{\langle d+2\rangle} = (a^{\langle d\rangle})^{\langle d+1\rangle}$;
4. $a^{(d+1)} \leq a^{(d)}$;
5. *If* $a^{(d+1)} = a^{(d)}$, *then* $(a^{(d+1)})^{(d+2)} = (a^{(d)})^{(d+1)}$.

Proof. By induction on a and d.

For $a = 0$ the lemma is obvious. Now we will prove parts (1) and (2) for $a > 0$. First let $d = 1$ and let $a = \binom{k_2}{2} + \binom{k_1}{1}$ be the (possibly non-reduced) 2-binomial representation of a. Note that $k_2 \geq 2$ since $a > 0$. We have

$$a_{\langle 1\rangle} = \binom{a}{1}_{\langle 1\rangle} = a - 1 \qquad \text{and} \qquad a_{\langle 2\rangle} = \binom{k_2 - 1}{2} + \begin{cases} k_1 - 1, & \text{if } k_1 \geq 1\\ 0, & \text{if } k_1 = 0.\end{cases}$$

Hence

$$\begin{aligned}a_{\langle 1\rangle} - a_{\langle 2\rangle} &= a - 1 - a_{\langle 2\rangle} = \binom{k_2 - 1}{1} + \begin{cases} 0, & \text{if } k_1 \geq 1\\ -1, & \text{if } k_1 = 0\end{cases}\\ &\geq \binom{k_2 - 1}{1} - 1 \geq \binom{2-1}{1} - 1 = 0,\end{aligned}$$

which proves part (1) when $d = 1$. Now assume we have already proved that $b^{\langle 2\rangle} \leq b^{\langle 1\rangle}$ for $b < a$. It is easy to see that $a_{\langle 2\rangle} < a$, so the inductive hypothesis implies that $(a_{\langle 2\rangle})^{\langle 2\rangle} \leq (a_{\langle 2\rangle})^{\langle 1\rangle}$. We already proved that $a_{\langle 2\rangle} \leq a_{\langle 1\rangle}$, so $(a_{\langle 2\rangle})^{\langle 1\rangle} \leq (a_{\langle 1\rangle})^{\langle 1\rangle}$. Since $a^{\langle 2\rangle} = (a_{\langle 2\rangle})^{\langle 2\rangle} + a$ and $a^{\langle 1\rangle} = (a_{\langle 1\rangle})^{\langle 1\rangle} + a$ it follows that $a^{\langle 2\rangle} \leq a^{\langle 1\rangle}$ which proves part (2).

Now let $d \geq 2$ and $a \geq 1$. Assume we have already proved that $b_{\langle d\rangle} \leq b_{\langle d-1\rangle}$ and $b^{\langle d\rangle} \leq b^{\langle d-1\rangle}$ for any b, and $b_{\langle d+1\rangle} \leq b_{\langle d\rangle}$ and $b^{\langle d+1\rangle} \leq b^{\langle d\rangle}$ for $b < a$. Let

$$a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta} = \binom{l_{d+1}}{d+1} + \binom{l_d}{d} + \cdots + \binom{l_\gamma}{\gamma}$$

be the d and $(d+1)$-binomial representations of a. If $b = a - a_{\langle d\rangle}$ and $c = a - a_{\langle d+1\rangle}$, then

$$b = \binom{k_d - 1}{d-1} + \binom{k_{d-1} - 1}{d-2} + \cdots + \binom{k_\delta - 1}{\delta - 1}$$

$$c = \binom{l_{d+1} - 1}{d} + \binom{l_d - 1}{d-1} + \cdots + \binom{l_\gamma - 1}{\gamma - 1}.$$

(These expressions are not necessarily the $(d-1)$ and d-binomial representations of b and c respectively.) To prove that $a_{\langle d+1\rangle} \le a_{\langle d\rangle}$ it is equivalent to prove that $b \le c$. Assume that on the contrary, $b > c$. We consider 4 cases:

Case 1. $\delta \ge 2, \gamma \ge 2$. In this case $a = b^{\langle d-1\rangle} = c^{\langle d\rangle}$, but $b > c$, so the induction hypothesis implies that $a = b^{\langle d-1\rangle} > c^{\langle d-1\rangle} \ge c^{\langle d\rangle} = a$, which is a contradiction.

Case 2. $\delta = 1, \gamma \ge 2$. In this case we have that $b - 1 \ge c$, so

$$a > \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_2}{2} = (b-1)^{\langle d-1\rangle} \ge c^{\langle d-1\rangle} \ge c^{\langle d\rangle} = a,$$

which is a contradiction.

Case 3. $\delta \ge 2, \gamma = 1$. In this case

$$\begin{aligned} a &= b^{\langle d-1\rangle} > c^{\langle d-1\rangle} \ge c^{\langle d\rangle} = \left[\binom{l_{d+1} - 1}{d+}\binom{l_d - 1}{d-1} + \cdots + \binom{l_2 - 1}{1} + 1\right]^{\langle d\rangle} \\ &= \binom{l_{d+1}}{d+1} + \binom{l_d}{d} + \cdots + \binom{l_2}{2} + (l_2 - 1) + 1 > a, \end{aligned}$$

which is a contradiction.

Case 4. $\delta = 1, \gamma = 1$. In this case

$$\begin{aligned} a &> \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_2}{2} = (b-1)^{\langle d-1\rangle} \ge c^{\langle d-1\rangle} \ge c^{\langle d\rangle} \\ &= \binom{l_{d+1}}{d+1} + \binom{l_d}{d} + \cdots + \binom{l_2}{2} + (l_2 - 1) + 1 > a, \end{aligned}$$

which is a contradiction.

This proves part (1) of Lemma 4.5 for all a and d. Now we will prove part (2). Assume that we have already proved part (2) for all integers $< a$. It is not hard to see that $a_{\langle d\rangle} < a$ for all $a > 0$, so by the induction hypothesis we have $(a_{\langle d\rangle})^{\langle d+1\rangle} \le (a_{\langle d\rangle})^{\langle d\rangle}$. It follows from part (1) that $a_{\langle d+1\rangle} \le a_{\langle d\rangle}$, so

$$a^{\langle d+1\rangle} = (a_{\langle d+1\rangle})^{\langle d+1\rangle} + a \le (a_{\langle d\rangle})^{\langle d+1\rangle} + a \le (a_{\langle d\rangle})^{\langle d\rangle} + a = a^{\langle d\rangle}, \tag{13}$$

which proves Lemma 4.5 (2). Now we will prove part (3) for $a > 0$. This is clear for $a = 1$ and d arbitrary. Now let $d = 1$, so $a^{\langle 1\rangle} = a^{\langle 2\rangle}$. Then all inequalities in (13) are equalities and in particular $(a_{\langle 2\rangle})^{\langle 2\rangle} = (a_{\langle 1\rangle})^{\langle 2\rangle}$. This implies that $a_{\langle 1\rangle} = a_{\langle 2\rangle}$ and an easy calculation shows that this in turn implies that $a = 0$ or 1, so we are done in this case. Now let $d \geq 2$ and $a \geq 1$. We have

$$(a_{\langle d+1\rangle})^{\langle d+1\rangle} = a^{\langle d+1\rangle} - a = a^{\langle d\rangle} - a = (a_{\langle d\rangle})^{\langle d\rangle}$$

and by Lemma 4.5 (1) and (2) we also have that $(a_{\langle d+1\rangle})^{\langle d+1\rangle} \leq (a_{\langle d+1\rangle})^{\langle d\rangle} \leq (a_{\langle d\rangle})^{\langle d\rangle}$. Hence

$$(a_{\langle d+1\rangle})^{\langle d+1\rangle} = (a_{\langle d+1\rangle})^{\langle d\rangle} = (a_{\langle d\rangle})^{\langle d\rangle}.$$

The second of these equalities (as well as (13)) implies that $a_{\langle d+1\rangle} = a_{\langle d\rangle}$, while the first implies by induction on a that

$$((a_{\langle d+1\rangle})^{\langle d+1\rangle})^{\langle d+2\rangle} = ((a_{\langle d+1\rangle})^{\langle d\rangle})^{\langle d+1\rangle} = ((a_{\langle d\rangle})^{\langle d\rangle})^{\langle d+1\rangle}.$$

Since $(a_{\langle d+1\rangle})^{\langle d+1\rangle} = (a^{\langle d+1\rangle})_{\langle d+2\rangle}$ and $(a_{\langle d\rangle})^{\langle d\rangle} = (a^{\langle d\rangle})_{\langle d+1\rangle}$, we get

$$\begin{aligned}(a^{\langle d+1\rangle})^{\langle d+2\rangle} &= ((a^{\langle d+1\rangle})_{\langle d+2\rangle})^{\langle d+2\rangle} + a^{\langle d+1\rangle} = ((a_{\langle d+1\rangle})^{\langle d+1\rangle})^{\langle d+2\rangle} + a^{\langle d+1\rangle}\\ &= ((a_{\langle d\rangle})^{\langle d\rangle})^{\langle d+1\rangle} + a^{\langle d+1\rangle} = ((a^{\langle d\rangle})_{\langle d+1\rangle})^{\langle d+1\rangle} + a^{\langle d\rangle} = (a^{\langle d\rangle})^{\langle d+1\rangle}.\end{aligned}$$

Parts (4) and (5) follow from (2) and (3) and the facts that $a^{(d)} = a^{\langle d\rangle} - a$ and $(a^{(d)})^{(d+1)} = (a^{\langle d\rangle})^{\langle d+1\rangle} - 2a^{\langle d\rangle} + a$. □

In [1], Aramova, Herzog, and Hibi developed Gröbner basis theory for exterior algebras. They showed that with minor modifications Gröbner basis theory known from polynomial rings carries over. So in what follows we will let R be the polynomial ring S or the exterior algebra E on $x_1, \dots, x_n$. We will freely cite results proved only in the case of polynomial rings, since the proofs in the case of exterior algebras are identical. We extend the definition of a Gotzmann vector space given in §2 to subspaces of E: A vector space $V \subseteq E_d$ is called *Gotzmann*, if $\operatorname{codim}(VE_1, E_{d+1}) = \operatorname{codim}(V, E_d)^{(d)}$. We use the term syzygies to denote a minimal set of generators for the first syzygy module.

LEMMA 4.6. *Let $V \subseteq R_d$ be a Gotzmann vector space. Let $I = (V)$ and $J = \operatorname{in}(I)$. Suppose that $g_1, \dots, g_r$ is a basis for $V = I_d$ such that the syzygies on $\operatorname{in}(g_1), \dots, \operatorname{in}(g_r)$ are linear. Then the syzygies on $g_1, \dots, g_r$ are linear.*

Proof. As J_d is a Gotzmann vector space, J is generated by $\operatorname{in}(g_1), \dots, \operatorname{in}(g_r)$. Thus the syzygies of J are linear.

Let $\omega = (\omega_1, \dots, \omega_n)$ be a weight vector such that $\operatorname{in}(I) = \operatorname{in}_{<\omega}(I)$. We add a new variable t and homogenize I with respect to ω, as in [5, p.343]. We denote by $\tilde{I}$ the ideal obtained in this way. By [5, §15.17] we have that $R[t]/\tilde{I}$ is a flat family over $k[t]$ whose fiber over 0 is R/J. Therefore the syzygies of $\tilde{I}$ over $R[t]$ are linear in $x_1, \ldots, x_n$. They provide a (non-minimal) generating set of syzygies when we set $t = 1$. The fiber of $R[t]/\tilde{I}$ over 1 is R/I, hence the syzygies of I over R are linear. □

PROPOSITION 4.7. *Let $V \subseteq R_d$ be a Gotzmann vector space. If $g_1, \ldots, g_r$ is a basis for V, then the syzygies on $g_1, \ldots, g_r$ are linear.*

Proof. Let $I = (V)$ and $J = \text{gin}(I)$, where $\text{gin}(I)$ denotes the generic initial ideal of I. Assume that we are in general coordinates, so that $\text{gin}(I) = \text{in}(I)$. We have that J_d is Gotzmann and is generated by $\text{in}(g_1), \ldots, \text{in}(g_r)$. The ideal J is generated by J_d; this follows from Gotzmann and Aramova-Herzog-Hibi Persistence Theorems. If R is an exterior algebra or $\text{char}(k) = 0$, then J is a strongly stable ideal ([1], [5, Ch. 15]). So the syzygies on $\text{in}(g_1), \ldots, \text{in}(g_r)$ are linear. Applying Lemma 4.6, we get that the syzygies on $g_1, \ldots, g_r$ are linear.

It remains to consider the case when R is a polynomial ring and $\text{char}(k) \neq 0$. Following [1], let $\text{inm}(g_i)$ be the monomial such that $\text{in}(g_i) = \alpha_i \text{inm}(g_i)$ for some $\alpha_i \in k$. Then $\text{inm}(g_1), \ldots, \text{inm}(g_r)$ form a basis of J_d. We will show that the syzygies on the $\text{inm}(g_i)$'s are linear. Since the syzygies on the $\text{inm}(g_i)$'s do not depend on k [2, Corollary 5.3], [3, Theorem 1.3 (b)], we can replace k with any field of characteristic 0. By the first part of the proof we have that the syzygies on the $\text{inm}(g_i)$'s are linear. This implies that the syzygies on the $\text{in}(g_i)$'s are linear, so by Lemma 4.6 the syzygies on the g_i's are linear. □

We are ready to prove Theorem 4.2. One of the steps in the proof of Theorem 4.2 (3) is to show that we can assume that the module M has the form (14). In this step we use ideas from Bigatti's dissertation 1995 which were also used by Aramova, Herzog, and Hibi [1].

Proof of Theorem 4.2. First we will show that it is enough to assume that M has the form

$$M = (S/I_1)\xi_1 \oplus (S/I_2)\xi_2 \oplus \cdots \oplus (S/I_k)\xi_k, \tag{14}$$

where $I_1, I_2, \ldots, I_k$ are ideals in S. That we can make this assumption with respect to parts (1) and (2) of Theorem 4.2 follows from the Hulett-Pardue theorem [8], [9], [13], [14]. However, there is a very simple direct proof, so we present it here. Define a partial order $\succ$ on the elements of F of the form fe_i (where $0 \neq f \in S$) as follows:

$$fe_i \succ ge_j \text{ iff } i < j.$$

For a nonzero element $r = \sum_{i \geq 1} f_i e_i$ define the initial form of r with respect to $\succ$, $\text{in}_{\succ}(r)$, to be $f_j e_j$, where $j = \min\{i \mid f_i \neq 0\}$. For any $x \in S_1$ we have $\text{in}_{\succ}(N_d \cap xF_{d-1}) \subseteq \text{in}(N)_d \cap xF_{d-1}$, so $\dim(N_d \cap xF_{d-1}) = \dim(\text{in}_{\succ}(N_d \cap$

$xF_{d-1})) \leq \dim(\mathrm{in}_{\succ}(N)_d \cap xF_{d-1})$. This implies that $\dim \overline{N}_d \geq \dim(\overline{\mathrm{in}_{\succ}(N)_d})$. Let $M' = F/\mathrm{in}_{\succ}(N)$. For any d we have $\dim M'_d = \dim M_d$ and from the above discussion it also follows that $\dim \overline{M}'_d \geq \dim M_d$, so to prove Theorem 4.2 (1) and (2) we can replace M by M' and assume that M has the desired form (14).

Now assume that the hypothesis of Theorem 4.2 (3) is satisfied. Let $>_{\mathrm{hlex}}$ denote the homogeneous lexicographic order on monomials in S. Define the homogeneous lexicographic order $>$ on monomials in F to be the lexicographic product of $\succ$ and $>_{\mathrm{hlex}}$; i.e.,

$$me_i > ne_j \text{ if } i < j \text{ or } i = j \text{ and } m >_{\mathrm{hlex}} n.$$

Let $\tilde{N} = \mathrm{in}_>(N) = I^{(1)}\xi_1 \oplus I^{(2)}\xi_2 \oplus \cdots \oplus I^{(k)}\xi_k$, where the $I^{(j)}$'s are monomial ideals in S. The hypothesis of Theorem 4.2 (3) implies that for $1 \leq j \leq k$, $I^{(j)}_{d-\deg \xi_j+1} = I^{(j)}_{d-\deg \xi_j} S_1$ and $I^{(j)}_{d-\deg \xi_j} \subseteq S_{d-\deg \xi_j}$ is a Gotzmann vector space. Then Proposition 4.7 implies that if $g_1, \ldots, g_r$ form a basis of N_d, then the syzygies on $\mathrm{in}(g_1), \ldots, \mathrm{in}(g_r)$ are linear. For $1 \leq i, j \leq r$ let $m_{ij} = \mathrm{in}(g_i)/\mathrm{GCD}(\mathrm{in}(g_i), \mathrm{in}(g_j))$ and for $1 \leq i < j \leq r$ let h_{ij} be the remainder of $m_{ji}g_i - m_{ij}g_j$ with respect to $g_1, \ldots, g_r$ when we perform the division algorithm [5, 15.6 and 15.7]. Then $\deg h_{ij} = d + 1$ whenever $h_{ij} \neq 0$. But $\mathrm{in}(h_{ij})$ is a minimal generator of $\mathrm{in}_>(N)$ when $h_{ij} \neq 0$ and $\mathrm{in}_>(N)$ does not have minimal generators in degree $d + 1$, so all $h_{ij} = 0$. By the Buchberger criterion [5, Theorem 15.8] this implies that the g_i's form a Gröbner basis for N, hence the $\mathrm{in}(g_i)$'s generate $\mathrm{in}_>(N)$. This shows that to prove Theorem 4.2 (3) we can replace M by $F/\mathrm{in}_>(N)$ and thus assume that M has the form (14).

Let $H_j(n) = H_{S/I_j}(n)$ be the Hilbert function of S/I_j and $\overline{H}_j(n) = H_{\overline{S}/\overline{I}_j}(n)$ be the Hilbert function of $\overline{S}/\overline{I}_j$, where $\overline{I}$ is the image of I in $\overline{S}$. Let $a_j = \deg \xi_j$. We can assume without loss of generality that $a_1 = 0 \geq a_2 \geq a_3 \geq \cdots \geq a_k$. Then $l = 0$ and for any $n \geq 0$,

$$\dim M_n = \sum_{j=1}^{k} \dim(S/I_j)_{n-a_j} = \sum_{j=1}^{k} H_j(n - a_j) \quad \text{and}$$

$$\dim \overline{M}_n = \sum_{j=1}^{k} \dim(\overline{S}/\overline{I}_j)_{n-a_j} = \sum_{j=1}^{k} \overline{H}_j(n - a_j).$$

By Green's theorem [7], Lemma 4.5 (1), and Lemma 4.4 (1) it follows that

$$\begin{aligned}\dim \overline{M}_d &= \sum_{j=1}^{k} \overline{H}_j(d - a_j) \leq \sum_{j=1}^{k} H_j(d - a_j)_{\langle d-a_j \rangle} \leq \sum_{j=1}^{k} H_j(d - a_j)_{\langle d-p \rangle} \\ &\leq \left(\sum_{j=1}^{k} H_j(d - a_j)\right)_{\langle d-p \rangle} = (\dim M_d)_{\langle d-p \rangle},\end{aligned}$$

which proves part (1) of Theorem 4.2. By Macaulay's theorem [12], Lemma 4.5 (2), and Lemma 4.4 (2) it follows that

$$\begin{aligned}\dim M_{d+1} &= \sum_{j=1}^{k} H_j(d+1-a_j) \leq \sum_{j=1}^{k} H_j(d-a_j)^{\langle d-a_j\rangle} \leq \sum_{j=1}^{k} H_j(d-a_j)^{\langle d-p\rangle} \\ &\leq \left(\sum_{j=1}^{k} H_j(d-a_j)\right)^{\langle d-p\rangle} = (\dim M_d)^{\langle d-p\rangle}, \qquad (15)\end{aligned}$$

which proves part (2) of Theorem 4.2.

To prove part (3) note that $\dim M_{d+1} = (\dim M_d)^{\langle d-p\rangle}$ implies that all inequalities in (15) are equalities. Then for $1 \leq j \leq k$ we have $H_j(d+1-a_j) = H_j(d-a_j)^{\langle d-a_j\rangle} = H_j(d-a_j)^{\langle d-p\rangle}$, so by the Gotzmann Persistence Theorem [6] and Lemma 4.5 (3) it follows that $H_j(d+2-a_j) = (H_j(d-a_j)^{\langle d-a_j\rangle})^{\langle d+1-a_j\rangle} = (H_j(d-a_j)^{\langle d-p\rangle})^{\langle d+1-p\rangle}$. Applying Lemma 4.4 (4) we get

$$\begin{aligned}\dim M_{d+2} &= \sum_{j=1}^{k} H_j(d+2-a_j) = \sum_{j=1}^{k} (H_j(d-a_j)^{\langle d-p\rangle})^{\langle d+1-p\rangle} \\ &= \left(\sum_{j=1}^{k} H_j(d-a_j)^{\langle d-p\rangle}\right)^{\langle d+1-p\rangle} = (\dim M_{d+1})^{\langle d+1-p\rangle},\end{aligned}$$

which proves Theorem 4.2 (3). □

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