# A DECOMPOSITION PROOF THAT THE DOUBLE SUSPENSION OF A HOMOTOPY 3-CELL IS A TOPOLOGICAL 5-CELL[1]

BY

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## 1. Introduction and definitions

In [5], the author proved that if $H^3$ is a $PL$ homotopy 3-sphere bounding a compact contractible $PL$ 4-manifold, then the double suspension of $H^3$ is topologically homeomorphic to the 5-sphere $S^5$. (We write this as $\Sigma^2 H^3 \approx S^5$, where $\Sigma^2$ denotes double suspension and $\approx$ means topologically homeomorphic.) In [10], Siebenmann gives an elegant proof that $\Sigma^2 H^3 \approx S^5$, for any homotopy 3-sphere $H^3$. However, this proof is somewhat unsatisfactory in that it has to make use of some deep results of the Kirby-Siebenmann triangulation theory, and a key theorem needed to obtain this result was given merely by a reference to a paper by Kirby and Siebermann, which apparently was not even in preprint form at the time.[3] In [6], the author made use of a completely geometrical, but quite involved, argument, outlined to him by Kirby, to show that if $F^3$ is a homotopy 3-cell, then $\Sigma^2 F^3 \approx I^5$. This requires a long and complicated argument, which depends quite heavily on the full work of [4]. In an addendum to [10], Siebenmann remarks that the same proof used to show that $\Sigma^2 H^3 \approx S^5$, also works to show that $\Sigma^2 F^3 \approx I^5$.

Here, we give an easy decomposition proof that $\Sigma^2 F^3 \approx I^5$, for any homotopy 3-cell $F^3$. The proof only requires a simple application of the engulfing lemma of [11], plus the fact that all homotopy 3-cells can be triangulated [1] and some basic fundamentals of geometric $PL$ theory. Moreover, by using the collaring theorem of [2] and the topological $h$-cobordism theory of [3] (which itself only requires [2] and the engulfing lemma), the proof given here actually can be used to show that $\Sigma^2 F^3 \approx I^5$, without even using the fact that 3-manifolds can be triangulated (also refer to the remarks at the beginning of §5).

In Corollary 4.3, we show that if $M^3$ is an arbitrary homotopy 3-sphere and $h : S^2 \to N^2 \subset M^3$ is a homeomorphism carrying $S^2$ onto the locally flat submanifold $N^2$ of $M^3$, then there exists a homeomorphism

$$H : (\Sigma^2(v_1 * S^2 * v_2), \Sigma^2 S^2) \to (\Sigma^2 M^3, \Sigma^2 N^2)$$

such that $H \mid \Sigma^2 S^2 = \Sigma^2 h$ (here $*$ denotes join and $\Sigma^2 h : \Sigma^2 S^2 \to \Sigma^2 N^2$ denotes

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[3] After this paper was written, Siebenmann informed the author that he and Kirby also know an "elementary" proof of this result using engulfing and an infinite meshing process of Černavskiĭ; however, this also is not written down anywhere.

the natural homeomorphism extending $h : S^2 \to N^2$ and carrying the suspension circle of $\Sigma^2 S^2$, by the identity map, to the suspension circle of $\Sigma^2 N^2$).

In Corollary 5.4, we show that if $N^3$ is a homotopy 3-sphere contained as a locally flat submanifold of the homotopy 4-sphere $M^4$, then there exists a homeomorphism

$$H : (\Sigma^2(v_1 * S^3 * v_2), \Sigma^2 S^3) \to (\Sigma^2 M^4, \Sigma^2 N^3)$$

such that $H \mid \Theta^1 =$ identity (here $\Theta^1$ denotes the suspension circle of each pair).

We now give a few additional definitions. We will use $\cong$ to denote *PL* (or combinatorially) homeomorphic, and as we have already noted, $\approx$ means topologically homeomorphic. By a homotopy 3-sphere $M^3$ (3-cell $F^3$), we will mean a closed (compact) topological 3-manifold (with nonempty boundary) such that $\pi_1(M^3) = 0$ ($F^3$ is contractible). By a homotopy 4-sphere $M^4$ (4-cell $F^4$), we will mean the above with 4 replacing 3, and $\pi_i(M^4) = 0$ $(i = 1, 2, 3)$ replacing $\pi_1(M^3) = 0$.

If $X$ is a metric space with metric $\rho$ and $Z$ is a subset of $X$, then, given $\varepsilon > 0$, we will use $V(Z, X, \varepsilon)$ to denote the set $\{x \in X \mid \rho(x, Z) < \varepsilon\}$. Also, if $Z$ is a compact subset of $X$, we will use $d(Z)$ to denote the diameter of $Z$, i.e.

$$d(Z) = \max \{\rho(z_1, z_2) \mid z_1, z_2 \in Z\}.$$

If $K$ is a compact subset of Euclidean $n$-space $E^n$, we define $\Sigma^2 K$ and $\Theta^1 \subset \Sigma^2 K$ explicitly in §2. Finally, if $N^k$ is a closed submanifold of the closed manifold $M^n$, we say $N^k$ is locally flat in $M^n$ if, for every $x \in N^k$, there exists an open set $U \subset M^n$ containing $x$ such that $(U, U \cap N^k) \approx (E^n, E^k)$.

## 2. Some preliminary notation and lemmas

Let $F^3$ be a homotopy 3-cell and let $N(\text{Bd}F^3, F^3)$ denote the regular neighborhood of $\text{Bd}F^3 \cong S^2$ in $F^3$ under the second barycentric subdivision of $F^3$. By [12], $N(\text{Bd}F^3, F^3) \cong \text{Bd}F^3 \times [0, 1]$. We identify $N(\text{Bd}F^3, F^3)$ with $\text{Bd}F^3 \times [0, 1]$, so that $x \in \text{Bd}F^3$ corresponds to $(x, 0) \in \text{Bd}F^3 \times [0, 1]$. Let $\Delta^2$ denote a 2-simplex in $\text{Bd}F^3$ and let $\{\Delta_i\}$ denote a sequence of concentric 2-simplexes in $\Delta^2$ so that $\Delta_1 \subset \text{int } \Delta^2$, $\Delta_{i+1} \subset \text{int } \Delta_i$ for each $i$, and $\bigcap_{i=1}^{\infty} \Delta_i = \{p\}$ is the barycenter of $\Delta^2$. For each $i = 1, 2, \cdots$, let

$$T_i = F^3 - (\text{Bd}F^3 \times [0, \tfrac{1}{2} - (\tfrac{1}{2})^{i+1})),$$

$$B_i = \Delta_i \times [\tfrac{1}{2} - (\tfrac{1}{2})^{i+1}, \tfrac{1}{2} + (\tfrac{1}{2})^{i+1}],$$

$$F_i = T_i - \{\text{int } B_i \cup [\text{int } \Delta_i \times (\tfrac{1}{2} - (\tfrac{1}{2})^{i+1})]\}$$

and

$$K = F^3 - (\text{Bd}F^3 \times [0, \tfrac{1}{2})).$$

We note, for each $i$, $T_i = B_i \cup F_i$,

$$B_i \cap F_i = \text{Bd}B_i \cap \text{Bd}F_i$$
$$= (\text{Bd}\Delta_i \times [\tfrac{1}{2} - (\tfrac{1}{2})^{i+1}, \tfrac{1}{2} + (\tfrac{1}{2})^{i+1}]) \cup (\Delta_i \times (\tfrac{1}{2} + (\tfrac{1}{2})^{i+1})$$

is a combinatorial 2-cell which we denote by $C_i$, $T_1 \subset \operatorname{int} F^3$, $T_{i+1} \subset \operatorname{int} T_i$, $B_{i+1} \subset \operatorname{int} B_i$, $K = \bigcap_{i=1}^{\infty} T_i$ and $z = (p, \frac{1}{2}) = \bigcap_{i=1}^{\infty} B_i$. Since $B_i \cap F_i = \mathrm{Bd}B_i \cap \mathrm{Bd}F_i = C_i$ is a 2-cell, $F_i \cong T_i \cong F^3$. Let $D_i$ denote the 2-cell

$$\mathrm{Bd}F_i - \operatorname{int} C_i = \mathrm{Bd}T_i - (\operatorname{int} \Delta_i \times (\tfrac{1}{2} - (\tfrac{1}{2})^{i+1})).$$

We now quote an elementary lemma proved in [5]. This only requires a simple application of the engulfing lemma of [11].

LEMMA 2.1. *Suppose $M$ is a compact contractible combinatorial 5-manifold, $U$ is a contractible open subset of* Bd$M$, *$\delta$ is a positive number, and $Z$ is a closed subset of $M$ such that $Z \cap \mathrm{Bd}M \subset U$. If there exists a connected open subset $W$ of* Bd$M$ *missing $Z \cap \mathrm{Bd}M$ so that $\pi_1(W) = 0$ and $U \cup W = \mathrm{Bd}M$, then there is a piecewise linear isotopy $f_t$ $(0 \leq t \leq 1)$ taking $M$ onto itself such that*
(1) $f_0 =$ *identity,*
(2) $f_t =$ *identity on* Bd$M$ *for all $t$, and*
(3) $f_1(Z) \subset V(U, M, \delta)$.

Suppose $K$ is a finite complex or an arbitrary compact subset of $E^n$. Let $q_i$ and $-q_i$ $(i = 1 \text{ or } 2)$ be the points of $E^2$ given by $q_1 = (1, 0)$, $q_2 = (0, 1)$, $-q_1 = (-1, 0)$, and $-q_2 = (0, -1)$. Let $\theta_n$ denote the origin of $E^n$ and for $i = 1$ or 2, let $u_i$ and $v_i$ be the points of $E^{n+2} = E^n \times E^2$ defined by $u_i = (\theta_n, -q_i)$ and $v_i = (\theta_n, q_i)$. By the double suspension of $K (= K \times \theta_2)$ in $E^{n+2}$, we will mean the complex or compact set $\Sigma^2 K \subset E^{n+2}$ given by

$$\Sigma^2 K = u_2 * (u_1 * K * v_1) * v_2,$$

where $*$ denotes join (i.e. if $A$ and $B$ are two compact subsets of $E^m$, then

$$A * B = \{(1 - t)a + tb \mid a \in A, b \in B, \text{and } t \in [0, 1]\}).$$

Let $\Theta^1$ denote the polyhedral 1-sphere in $\theta_n \times E^2 \subset E^n \times E^2$ given by

$$\Theta^1 = u_2 * (u_1 \cup v_1) * v_2.$$

Then

$$\Sigma^2 K \cong K * \Theta^1.$$

Let $S^1$ denote the unit 1-sphere in $\theta_n \times E^2$. Let $\bar{p} : \Theta^1 \to S^1$ denote the projection of $\Theta^1$ onto $S^1$ from the origin $(\theta_n, \theta_2)$ of $\theta_n \times E^2$, and define

$$p_K : K * \Theta^1 \to K * S^1$$

to be the natural homeomorphism sending the interval $x * y \subset K * \Theta^1$ $(x \in K,\ y \in \Theta^1)$ to $x * \bar{p}(y) \subset K * S^1$ (i.e. $(1 - t)x + ty$ goes to $(1 - t)x + t\bar{p}(y)$ $(0 \leq t \leq 1)$).

Now every point of $(K * S^1) - S^1$ has a unique representation in the form $\langle x, ty \rangle$, where $x \in K$, $y \in S^1$ and $t \in [0, 1)$. That is,

$$\langle x, ty \rangle = (1 - t)x + ty \in x * y \quad \text{and} \quad \langle x, 0y \rangle = \langle x, 0 \rangle = x \in K.$$

Let $\Phi_K$ be the homeomorphism carrying $K \times E^2$ onto $(K * S^1) - S^1$ defined by sending

$$(x, w) \in K \times E^2 \quad \text{to} \quad \langle x, w/(1 + \| w \|)\rangle,$$

where $w = (w_1, w_2) \in E^2$ and $\| w \| = ((w_1)^2 + (w_2)^2)^{1/2}$.

LEMMA 2.2 *Suppose $K$ and $L$ are compact subsets of $E^n \times \theta_2 \subset E^n \times E^2$, and $\Sigma^2 K$ and $\Sigma^2 L$ are the double suspensions of $K$ and $L$, respectively, as defined above. If $f : K \times E^2 \to L \times E^2$ is a continuous map carrying $K \times E^2$ onto $L \times E^2$ such that $f$ is bounded on the $E^2$ factor (i.e. if $p_2 : L \times E^2 \to E^2$, then $\| w - p_2 \circ f(x, w) \| <$ constant), then $f$ induces a continuous map $g : \Sigma^2 K \to \Sigma^2 L$ such that $g \mid \Theta^1 =$ identity. Furthermore, if for some subset $B \subset K$, $f \mid B \times E^2$ is of the form $\bar{f} \times \mathrm{id}_{E^2}$, where $\bar{f} : B \to L$, then $g \mid \Sigma^2 B = \Sigma^2 \bar{f}$, i.e. if $x * y \in B * \Theta^1$, then*

$$g((1 - t)x + ty) = (1 - t)\bar{f}(x) + ty \in \bar{f}(x) * y.$$

*Proof.* Since $f : K \times E^2 \to L \times E^2$ is bounded on the $E^2$ factor, we claim that the map $\tilde{g} : (\Sigma^2 K) - \Theta^1 \to (\Sigma^2 L) - \Theta^1$, defined as the composition

$$\begin{aligned}(\Sigma^2 K) - \Theta^1 = (K * \Theta^1) - \Theta^1 &\xrightarrow{p_K} (K * S^1) - S^1 \xrightarrow{(\Phi_K)^{-1}} \\ K \times E^2 \xrightarrow{f} L \times E^2 &\xrightarrow{\Phi_L} (L * S^1) - S^1 \xrightarrow{(p_L)^{-1}} \\ &(L * \Theta^1) - \Theta^1 = (\Sigma^2 L) - \Theta^1,\end{aligned}$$

extends by the identity map on $\Theta^1$ to a continuous map $g : \Sigma^2 K \to \Sigma^2 L$.

We see this as follows: Suppose $\{\langle x_i, t_i y_i\rangle\}$ $(i = 1, 2, 3, \cdots)$ is a sequence of points of $(K * S^1) - S^1$ tending to $y_0 \in S^1$. We note that $\{t_i\} \to 1$, $\{y_i\}$ is a sequence of points in $S^1$ converging to $y_0$, and $\{x_i\}$ is a sequence of points of $K$. We consider a subsequence $\{\langle x_j, t_j y_j\rangle\}$ of $\{\langle x_i, t_i y_i\rangle\}$ so that $\{x_j\} \to x_0 \in K$ and $\{y_j\} \to y_0 \in S^1$. Then $\{(\Phi_K^{-1})\langle x_j, t_j y_j\rangle\}$ is an unbounded sequence of $K \times E^2$ of the form $\{(x_j, s_j y_j)\}$, where $\{s_j\} \to \infty$. Let

$$f(x_j, s_j y_j) = (z_j, \mathfrak{z}_j \tilde{y}_j)$$

where

$$p_1 \circ f(x_j, s_j y_j) = z_j, \quad \mathfrak{z}_j = \| p_2 \circ f(x_j, s_j y_j) \|$$

and

$$\tilde{y}_j = ((p_2 \circ f(x_j, s_j y_j))/\mathfrak{z}_j) \in S^1.$$

Since $\| s_j y_i - \mathfrak{z}_j \tilde{y}_j \| < M$, $\mathfrak{z}_j \to \infty$, and $\{\| y_j - \tilde{y}_j \|\} \to 0$. Thus

$$\{(\Phi_L) \circ f \circ (\Phi_K)^{-1}\langle x_j, t_j y_j\rangle\} = \{\Phi_L(z_j, \mathfrak{z}_j \tilde{y}_j)\} = \langle z_j, \mathfrak{t}_j \tilde{y}_j\rangle$$

is a sequence in $(L * S^1) - S^1$ converging to $y_0 \in S^1$, and our claim follows.

The final conclusion follows easily from the manner in which the various maps defining $\tilde{g}$ are defined.

## 3. A shrinking theorem and a pseudo-isotopy

THEOREM 3.1. *Suppose $F^3$ is a homotopy 3-cell and $\{T_i\}$ is the sequence of* *closed neighborhoods enclosing the contractible complex $K$ in* int $F^3$ *given in* §2. *Also, suppose $z = \bigcap_{i=1}^{\infty} B_i$ is the point of* Bd$K$ *as given in* §2. *Then for each $i$ and $\varepsilon > 0$ there is a piecewise linear isotopy $\mu_t$ of $F \times E^2$ onto itself such that $\mu_0$ = identity, $\mu_1$ is uniformly continuous, and*

(1) $\mu_t$ = *identity on* $\{(F^3 - \text{int } T_i) \times E^2\} \cup \{z \times E^2\}$ *for each $t$,*

(2) *$\mu_t$ changes $E^2$ coordinates $< \epsilon$, and*

(3) *for each $w \in E^2$, $d(\mu_1(T_{i+4} \times \{w\})) < \varepsilon$.*

*Proof. Step* 1. Let $F^3$ be subdivided so as to contain subdivisions of $T_{i+j}$, $B_{i+j}$, and $F_{i+j}$ as combinatorial submanifolds for $j = 1, \cdots, 4$. Let $\delta_1$ be a positive number less than $(\frac{1}{2})\ d(B_{i+1})$ (a further restriction will be placed on the size of $\delta_1$ later). Let $D$ be a combinatorial 3-cell contained in int $B_{i+1}$ such that $z \in \text{int } D$ and $d(D) < \delta_1$. Since each of $D$ and $B_{i+1}$ are combinatorial 3-cells contained in the interior of the combinatorial 3-cell $B_i$, given any closed neighborhood $N$ of $z$ in int $D$, it follows by [13] that there exists a piecewise linear isotopy $f_t$ carrying $B_i$ onto itself such that

$$f_0 = \text{identity},$$

$$f_t = \text{identity on } N \cup \text{Bd}B_i \text{ for all } t,$$

$$f_1(B_{i+1}) = D.$$

We extend $f_t$ to all of $F^3$ by the identity, and we denote the extended isotopy by $f_t$ also. Let $h_{1,t}$ be the isotopy of $F^3 \times E^2$ onto itself defined by $h_{1,t}(x, w) = (f_t(x), w)$, where $x, f_t(x) \in F^3$ and $w \in E^2$. We note that for all $t \in [0, 1]$ and $w \in E^2$, $h_{1,t}$ carries $B_i \times \{w\}$ onto itself and is the identity on $\{N \cup (F^3 - \text{int } B_i)\} \times \{w\}$. Also, for any $w \in E^2$ and $k \geq i + 1$, we have $d(h_{1,1}(B_k \times \{w\})) < \delta_1$.

*Step* 2. For each pair of integers $(m, n)$ and positive number $r$, let

$$D^2((m, n), r) = D^2(\alpha, r) \quad (\alpha = (m, n))$$

denote the 2-cell $[m - r, m + r] \times [n - r, n + r] \subset E^2$. Let $M_\alpha^5$ denote the combinatorial 5-manifold $h_{1,1}(F_{i+1} \times D^2(\alpha, \frac{1}{4}))$. Let $C_\alpha^4$ be the combinatorial 4-cell

$$h_{1,1}(C_{i+1}^2 \times D^2(\alpha, \tfrac{1}{4}))$$

(recall $C_{i+1}^2 = B_{i+1} \cap F_{i+1}$). Define $Z_\alpha$ to be

$$h_{1,1}(T_{i+2} \times D^2(\alpha, \tfrac{1}{8})) \cap M_\alpha^5 .$$

Since $Z_\alpha \cap \text{Bd}M_\alpha^5 \subset \text{int } C_\alpha^4$ and $\text{Bd}M_\alpha^5$ is simply connected, we can apply Lemma 2.1. (We can take $U$ and $W$ of Lemma 2.1 to be int $C_\alpha^4$ and

$$(\text{Bd}M_\alpha^5 - \text{int } C_\alpha^4) \cup \{\text{an open collar of Bd}C_\alpha^4 \text{ in } C_\alpha^4\},$$

respectively.) Let $\delta_2$ be a positive number less than $d(M_\alpha^5)$ (this number will also be restricted further later). We note that because of the way that $h_{1,1}$ was defined, $\delta_2$ is independent of the pair of integers $(m, n) = \alpha$. Thus, we obtain a piecewise linear isotopy $f_{\alpha,t}$ taking $M^5_\alpha$ onto itself such that

$$f_{\alpha,0} = \text{identity},$$
$$f_{\alpha,t} = \text{identity on } \mathrm{Bd}M^5_\alpha \text{ for all } t,$$
$$f_{\alpha,1}(Z_\alpha) \subset V(C^4_\alpha, M^5_\alpha, \delta_2).$$

Let $h_{2,t}$ be the isotopy of $F^3 \times E^2$ onto itself defined by

$$\begin{aligned} h_{2,t} &= f_{\alpha,t} \text{ on } M^5_\alpha \text{ for each pair of integers } \alpha = (m, n) \in E^2 \\ &= \text{identity outside } \cup\{M^5_\alpha \mid \alpha = (m, n)\}. \end{aligned}$$

We note that for each $w \in E^2$, $h_{2,t} = $ identity on $h_{1,1}(B_{i+1} \times \{w\})$. Also, for each $t \in [0, 1]$, $h_2,t$ moves no $E^2$ coordinates by more than $\frac{1}{2}$, as measured along either axis of $E^2$. Furthermore, for each pair of integers $(m, n) = \alpha$,

$$h_{2,1}(h_{1,1}(T_{i+2} \times D^2(\alpha, \tfrac{1}{8})) \subset V(h_{1,1}(B_{i+1} \times D^2(\alpha, \tfrac{1}{4})), F^3 \times E^2, \delta_2).$$

In particular,

$$d\{h_{2,1} \circ h_{1,1}(F_{i+2} \times \mathrm{Bd}\,(D^2(\alpha, \tfrac{1}{8})))\} < \delta_1 + 1 + 2\delta_2,$$

and $h_{2,1} \circ h_{1,1} = f_1$ (of Step 1) $\times$ identity on $B^2_{i+1} \times E^2$.

*Step* 3. This step will be quite similar to Step 2. For each pair of integers $(m, n) = \beta$, let $D^2_{\beta y}$ be the 2-cell

$$[m - \tfrac{1}{8}, m + \tfrac{1}{8}] \times [n + \tfrac{1}{8}, n + 1 - \tfrac{1}{8}]$$

and let $D^2_{\beta x}$ be the 2-cell

$$[m + \tfrac{1}{8}, m + 1 - \tfrac{1}{8}] \times [n - \tfrac{1}{8}, n + \tfrac{1}{8}].$$

We now want to consider the 5-manifolds

$$M^5_{\beta y} = h_{2,1} \circ h_{1,1}(F_{i+2} \times D^2_{\beta y}) \quad \text{and} \quad M^5_{\beta x} = h_{2,1} \circ h_{1,1}(F_{i+2} \times D^2_{\beta x}).$$

Let $C^4_{\beta y}$ and $C^4_{\beta x}$ be the contractible 4-manifolds in Bd $M^5_{\beta y}$ and in Bd $M^5_{\beta x}$, respectively, given by

$$\begin{aligned} C^4_{\beta y} = h_{2,1} \circ h_{1,1}(&\{F_{i+2} \times [m - \tfrac{1}{8}\, m + \tfrac{1}{8}] \times \{n + \tfrac{1}{8}\}\} \cup \{C^2_{i+2} \times D^2_{\beta y}\} \\ &\cup \{F_{i+2} \times [m - \tfrac{1}{8}, m + \tfrac{1}{8}] \times \{n + 1 - \tfrac{1}{8}\}\}), \\ C^4_{\beta x} = h_{2,1} \circ h_{1,1}(&\{F_{i+2} \times \{m + \tfrac{1}{8}\} \times [n - \tfrac{1}{8}, n + \tfrac{1}{8}]\} \cup \{C^2_{i+2} \times D^2_{\beta x}\} \\ &\cup \{F_{i+2} \times \{m + 1 - \tfrac{1}{8}\} \times [n - \tfrac{1}{8}, n + \tfrac{1}{8}]\}). \end{aligned}$$

It follows from last comment of Step 2, that each of $C^4_{\beta y}$ and $C^4_{\beta x}$ have diameter less than $(\delta_1 + 1 + 2\delta_2) + (\delta_1 + 1) + (\delta_1 + 1 + 2\delta_2) = 3\delta_1 + 3 + 4\delta_2$.

Let $Z_{\beta y}$ and $Z_{\beta x}$ be defined by

$$Z_{\beta y} = h_{2,1} \circ h_{1,1}(T_{i+3} \times [m - \tfrac{1}{16}, m + \tfrac{1}{16}] \times [n + \tfrac{1}{8}, n + 1 - \tfrac{1}{8}]) \cap M^5_{\beta y},$$
$$Z_{\beta x} = h_{2,1} \circ h_{1,1}(T_{i+3} \times [m + \tfrac{1}{8}, m + 1 - \tfrac{1}{8}] \times [n - \tfrac{1}{16}, n + \tfrac{1}{16}]) \cap M^5_{\beta x}.$$

Then $Z_{\beta y} \cap \operatorname{Bd} M^5_{\beta y} \subset \operatorname{int} C^4_{\beta y}$ and $Z_{\beta x} \cap \operatorname{Bd} M^5_{\beta x} \subset \operatorname{int} C^4_{\beta x}$. We again apply Lemma 2.1, where $U$ and $W$ of Lemma 2.1 correspond to int $C^4_{\beta\alpha}$ and

$$(\operatorname{Bd} M^5_{\beta\alpha} - \operatorname{int} C^4_{\beta\alpha}) \cup \{\text{an open collar of } \operatorname{Bd} C^4_{\beta\alpha} \text{ in } C^4_{\beta\alpha}\},$$

$\alpha = x$ or $y$. Let $\delta_3$ be a positive number less than both $d(M^5_{\beta x})$ and $d(M^5_{\beta y})$. We will add a further restriction in Step 5.

Thus by Lemma 2.1, for $\alpha = x$ or $y$, we obtain a piecewise linear isotopy $f_{\beta\alpha,t}$ taking $M^5_{\beta\alpha}$ onto itself such that

$$f_{\beta\alpha,0} = \text{identity},$$

$$f_{\beta\alpha,t} = \text{identity on } \operatorname{Bd} M^5_{\beta\alpha} \text{ for all } t,$$

$$f_{\beta\alpha,1}(Z_{\beta\alpha}) \subset V(C^4_{\beta\alpha}, M^5_{\beta\alpha}, \delta_3).$$

Let $h_{3,t}$ be the isotopy of $F^3 \times E^2$ onto itself defined by

$$h_{3,t} = f_{\beta\alpha,t} \text{ on } M^5_{\beta\alpha} \text{ for each pair of integers } \beta = (m, n) \text{ and } \alpha = x \text{ or } y$$

$$= \text{identity outside } \cup \{M^5_{\beta\alpha} \mid \beta = (m, n) \text{ and } \alpha = x \text{ or } y\}.$$

We note for each $w \in E^2$, $h_{3,t} = $ identity on $h_{2,1} \circ h_{1,1}(B_{i+2} \times \{w\})$. Hence

$$h_{3,1} \circ h_{2,1} \circ h_{1,1} = f_1 \text{ (of Step 1) } \times \text{ identity on } B_{1+2} \times E^2.$$

Also, for each $t \in [0, 1]$, $h_{3,1}$ changes no $E^2$ coordinate by more than $\frac{3}{2}$, as measured along either axis for $E^2$. Moreover, for each pair of integers $(m, n) = \beta$ and $\alpha = x$ or $y$, if $\hat{D}^2_{\beta\alpha}$ is the 2-cell used in defining $Z_{\beta\alpha}$, then

$$h_{3,1} \circ h_{2,1} \circ h_{1,1}(T_{i+3} \times \hat{D}^2_{\beta\alpha}) \subset \{h_{2,1} \circ h_{1,1}(B_{i+2} \times \hat{D}^2_{\beta\alpha})\} \cup \{V(C^4_{\beta\alpha}, M^5_{\beta\alpha}, \delta_3)\}.$$

*Step* 4. We note, if $w \in D^2(\alpha, \frac{1}{8})$ (defined in Step 2), since

$$h_{3,t} = \text{identity outside } \cup \{M^5_{\beta}\alpha \mid \beta = (m, n) \text{ and } \alpha = x \text{ or } y\},$$

$$h_{3,1} \circ h_{2,1} \circ h_{1,1}(T_{i+3} \times \{w\}) = h_{2,1} \circ h_{1,1}(T_{i+3} \times \{w\}) \text{ and hence}$$

$$d(h_{3,1} \circ h_{2,1} \circ h_{1,1}(T_{i+3} \times \{w\})) < \delta_1 + 1 + 2\delta_2 .$$

If $w \in D^2_{\beta\alpha}$, then it follows from the last comment of Step 3, that

$$\begin{aligned} d(h_{3,1} \circ h_{2,1} \circ h_{1,1}(T_{i+3} \times \{w\})) &< (\delta_1 + 1) + [(3\delta_1 + 3 + 4\delta_2) + 2\delta_3] \\ &= 4\delta_1 + 4 + 4\delta_2 + 2\delta_3 . \end{aligned}$$

For convenience, we will denote $h_{3,1} \circ h_{2,1} \circ h_{1,1}$ by $H_3$. For each pair of integers $(m, n) = \gamma$, let

$$D^2_\gamma = [m + \tfrac{1}{16}, m + 1 - \tfrac{1}{16}] \times [n + \tfrac{1}{16}, n + 1 - \tfrac{1}{16}].$$

Let $M^5_\gamma = H_3(F_{i+3} \times D^2_\gamma)$ and let $C^4_\gamma \subset \operatorname{Bd} M^5_\gamma$ be defined by the equation

$$C^4_\gamma = H_3((C^2_{i+3} \times D^2_\gamma) \cup (F_{i+3} \times \operatorname{Bd} D^2_\gamma)).$$

We note,

$$\operatorname{Bd} M^5_\gamma - \operatorname{int} C^4_\gamma = H_3(D^2_{i+3} \times D^2_\gamma)$$

(we recall that $D^2_{i+3} = \text{Bd}\, F_{i+3} - \text{int}\, C^2_{i+3}$). Thus

$$\text{Bd}\, C^4_\gamma = H_3(\text{Bd}\,(D^2_{i+3} \times D^2_\gamma))$$

is a 3-sphere and $C^4_\gamma$ is contractible. Also,

$$\begin{aligned} d(C^4_\gamma) &< (\delta_1 + \sqrt{2}) + (\sqrt{(\tfrac{3}{2})^2 + (\tfrac{3}{2})^2} + 2[4\delta_1 + 4 + 4\delta_2 + 2\delta_3]) \\ &< 13 + 9\delta_1 + 8\delta_2 + 4\delta_3 . \end{aligned}$$

Let $Z_\gamma = (H_3(T_{i+4} \times D^2_\gamma)) \cap M^5_\gamma$. Then $Z_\gamma \cap \text{Bd}\, M^5_\gamma \subset C^4_\gamma$ and we can apply Lemma 2.1 a final time. Let $\delta_4$ be a positive number less than $\text{d}(M^5_\gamma)$. Hence, for $\gamma = (m, n)$, we obtain a piecewise linear isotopy $f_{\gamma,t}$ taking $M^5_\gamma$ onto itself such that

$$f_{\gamma,0} = \text{identity},$$

$$f_{\gamma,t} = \text{identity on Bd}\, M^5_\gamma \text{ for all } t,$$

$$f_{\gamma,1}(Z_\gamma) \subset V(C^4_\gamma, M^5_\gamma, \delta_4).$$

Let $h_{4,t}$ be the isotopy of $F^3 \times E^2$ onto itself defined by

$$\begin{aligned} h_{4,t} &= f_{\gamma,t} \text{ on } M^5_\gamma \text{ for each pair of integers } \gamma = (m, n) \\ &= \text{identity outside } \bigcup \{M^5_\gamma \mid \gamma = (m, n)\}. \end{aligned}$$

For each $w$, contained in $E^2$,

$$h_{4,t} = \text{identity on } H_3(B_{i+3} \times \{w\})$$

and

$$h_{4,1} \circ H_3 = f_1 \text{ (of Step 1)} \times \text{identity on } B_{i+3} \times E^2.$$

For $w \in \bigcup \{D^2_\gamma \mid \gamma = (m, n)\}$,

$$h_{4,1} \circ H_3(T_{i+4} \times \{w\}) \subset H_3(B_{i+3} \times \{w\}) \cup V(C^4_\gamma, M^5_\gamma, \delta_4),$$

for some $\gamma = (m, n)$. Thus, for $w \in \cup \{D^2_\gamma \mid \gamma = (m, n)\}$,

$$\begin{aligned} d(h_{4,1} \circ H_3(T_{i+4} \times \{w\})) &< \delta_1 + ((13 + 9\delta_1 + 8\delta_2 + 4\delta_3) + 2\delta_4) \\ &= 13 + 10\delta_1 + 8\delta_2 + 4\delta_3 + 2\delta_4 . \end{aligned}$$

By the first paragraph of Step 4, since $h_{4,t} = $ identity outside $\bigcup \{M^5_\gamma \mid \gamma = (m, n)\}$, if $w \in E^2 - (\bigcup \{D^2_\gamma \mid \gamma = (m, n)\})$, then

$$d(T_{i+4} \times \{w\}) < 4\delta_1 + 4 + 4\delta_2 + 2\delta_3 .$$

Also $h_{4,t}$ changes no $E^2$ by more than $\frac{3}{2}$, as measured along either axis of $E^2$.

*Step* 5. We now can obtain the desired isotopy $\mu_t$ of Theorem 3.1. Let $\varepsilon > 0$ be given. We modify our scale on each axis of $E^2$ so that $1 < (\frac{1}{18})(\varepsilon/5)$, and then apply Steps 1–4, where we further restrict the various $\delta$'s used in these steps as follows:

$$\delta_1 < (\tfrac{1}{10})(\varepsilon/5), \quad \delta_2 < (\tfrac{1}{8})(\varepsilon/5), \quad \delta_3 < (\tfrac{1}{4})(\varepsilon/5) \quad \text{and} \quad \delta_4 < (\tfrac{1}{2})(\varepsilon/5).$$

We define the isotopy $\mu_t$ of $F^3 \times E^2$ onto itself by

$$\begin{aligned} \mu_t &= h_{1,4t} && \text{if } 0 \leq t \leq \tfrac{1}{4}, \\ &= h_{2,4t-1} \circ h_{1,1} && \text{if } \tfrac{1}{4} \leq t \leq \tfrac{1}{2}, \\ &= h_{3,4t-2} \circ h_{2,1} \circ h_{1,1} && \text{if } \tfrac{1}{2} \leq t \leq \tfrac{3}{4}, \\ &= h_{4,4t-3} \circ h_{3,1} \circ h_{2,1} \circ h_{1,1} && \text{if } \tfrac{3}{4} \leq t \leq 1. \end{aligned}$$

Clearly, $\mu_t$ is well defined and $\mu_0$ = identity. Also, if during each step, we just don't arbitrarily define the various $f_{(m,n),t}$'s, applying Lemma 2.1 separately for each $(m, n)$, but obtain one "model" function via Lemma 2.1 (we need two such functions in Step 3) and then translate this "model" function around to obtain the various $f_{(m,n),t}$'s of the given step, it will follow that, for each $i = 1, \cdots, 4$ and $t \in [0, 1]$, $h_{i,t}$ is uniformly continuous (also, the diameters of the various $M^5_{(m,n)}$'s of a given step would be independent of $(m, n)$). Hence, $\mu_1$ is uniformly continuous.

It is also clear from the way the $h_{i,t}$'s have been defined that

(1) $\mu_t$ = identity on $\{(F^3 - \text{int } T_i) \times E^2\} \cup \{z \times E^2\}$ for each $t$.

Since $\mu_t$ changes $E^2$ — coordinates $<3$, as measured each axis of $E^2$, it follows that

(2) $\mu_t$ changes $E^2$ coordinates $< \sqrt{(3)^2 + (3)^2} < 5 < \varepsilon/13$. Finally, by the last paragraph of Step 4, we see, for all $w \in E^2$, that

$$d(T_{i+4} \times \{w\}) < 13 + 10\delta_1 + 8\delta_2 + 4\delta_3 + 2\delta_4 .$$

Hence, by the further restrictions on the $\delta_i$'s above, we get that

(3) for each $w \in E^2$, $d(\mu_1(T_{i+4} \times \{w\})) < \varepsilon$,

and this completes the proof of Theorem 3.1.

Let $F^3$ be an arbitrary homotopy 3-cell and let $z \in \text{Bd } K$ be the point $z = \bigcap_{i=1}^{\infty} B_i$ as defined in §2. Let $G'$ denote the decomposition of $F^3$ given by

$$G' = \{g' \mid g' \text{ is a point of } F^3 - K \text{ or } g' = K\}$$

and let $G$ denote the decomposition of $F^3 \times E^2$ given by

$$G = \{g = g' \times w \mid g' \in G' \quad \text{and} \quad w \in E^2\}.$$

The following result is modeled after Theorem 3 of [14] and is included for completeness.

THEOREM 3.2. *Suppose $F^3$ is an arbitrary homotopy 3-cell and $G$ is the decomposition of $F^3 \times E^2$ defined above. Then, given $\varepsilon > 0$, there is a pseudo-isotopy $f(x, t)$ $(x \in F^3 \times E^2,\ 0 \leq t \leq 1)$ of $F^3 \times E^2$ onto itself such that*

(a) *$f(x, 0)$ is the identity* (i.e. $f(x, 0) = x$),

(b) *for each fixed* t $< 1, f(x, t)$ *is a homeomorphism of* $F^3 \times E^2$ *onto itself,*

(c) *for each* t$(0 \leq t \leq 1), f(x, t) =$ *identity on*

$$\{(F^3 = V(K, F^3, \varepsilon)) \times E^2\} \cup \{z \times E^2\}$$

*and changes* $E^2$ *coordinates* $< \varepsilon$, *and*

(d) $f(x, 1)$ *takes* $F^3 \times E^2$ *onto itself and each element of* $G$ *onto a distinct point.*

*Proof.* We will obtain the isotopy promised above by a sequence of applications of Theorem 3.1. Let $\{T_i\}$ be the sequence of compact neighborhoods in int $F^3$ enclosing m as given in Theorem 3.1. We suppose $2\varepsilon <$ distance $(K, \text{Bd } F^3)$. Let $\varepsilon_1, \varepsilon_2, \cdots$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon/2$. We will define a monotone increasing sequence $n_1, n_2, \cdots$ of integers and a sequence of isotopies

$$f(x,t)\ (x \in F^3 \times E^2, 0 \leq t \leq \tfrac{1}{2}), \quad f(x,t)\ (x \in F^3 \times E^2, \tfrac{1}{2} \leq t \leq \tfrac{2}{3}), \cdots$$

such that

$T_{n_1} \subset V(K, F^3, \varepsilon)$,
$f(x, 0) = x$,
two adjacent $f(x, t)$'s agree on their common end,
each $f(x, i/(i+1))$ is uniformly continuous,

(1) $f(x, (i-1)/i) = f(x, t)((i-1)/i \leq t \leq i/(i+1))$ except possibly on $(T_{n_i} \times E^2) - (z \times E^2)$.

(2) $f(x, t)$ changes $E^2$ coordinates $< \varepsilon_i\ ((i-1)/i \leq t \leq i/(i+1))$,

(3) $d(f(T_{n_{i+1}} \times w, i/(i+1))) < \varepsilon_i$ for all $w \in E^2$,

(4) no point moves more than $2\varepsilon_{i-1}$ during

$$f(x, t)\ ((i-1)/i \leq t \leq i/(i+1)),$$

and

(5) $f(F^3 \times V(w, E^2, \varepsilon_i), (i-1)/i) \supset f(F^3 \times w, i/(i+1))$.

Before defining the sequence of $f(x, t)$'s, we show that the existence of such a sequence is enough to guarantee the truth of Theorem 3.2. Since $f(x, 0) = x$, it follows by (1), that $f(x, t) =$ identity on

$$\{(F^3 - T_{n_1}) \times E^2\} \cup \{z \times E^2\} \quad (0 \leq t < 1).$$

Condition (4) and the above fact, along with the fact that each $f(x, i/(i+1))$ is uniformly continuous, implies that $f(x, 1) = \lim\ (t \to 1) f(x, t)$ is a continuous map of $F^3 \times E^2$ onto itself. Conditions (1) and (2) insure that for each $t$ $(0 \leq t \leq 1)$,

$$f(x, t) = \text{identity on } \{(F^3 - V(K, F^3, \varepsilon)) \times E^2\} \cup \{z \times E^2\}$$

and changes $E^2$ coordinates $< \varepsilon$.

Condition (3) insures that $f(g, 1)$ is a point for each element $g$ of $G$. Condition (1) implies that, if $f(g_1, 1) = f(g_2, 1)(g_1, g_2 \in G)$, then each $g_i$ must be of the form $K \times w_i$ ($w_i \in E^2$, $i = 1, 2$). The reason is as follows. If one of $g_1$ or $g_2$ is a point, say $g_1$, then there is an integer $i$ so large that $f(x, (i-1)/i) = f(x, 1)$ in a neighborhood of $g_1$. Finally, Condition (5) implies that no two points with different $w$ coordinates to into the same point under $f(x, 1)$. That is, if $w_1 \neq w_2$, there is an $i$ such that

$$\varepsilon_i + \varepsilon_{i+1} + \cdots < \delta = \| w_2 - w_1 \|/2$$

and Condition (5) implies that

$$f(F^3 \times V(w, E^2, \epsilon_i + \epsilon_{i+1} + \cdots), (i-1)/i) \supset f(F^3 \times V(w, E^2, \epsilon_{i+1} + \cdots), i/(i+1)) \supset \cdots \supset f(F^3 \times w, 1).$$

Thus, $f(F^3 \times w_1, 1)$ and $f(F^3 \times w_2, 1)$ lie respectively in the mutually exclusive curved "tubes"

$$f(F^3 \times V(w_1, E^2, \delta), (i-1)/i) \quad \text{and} \quad f(F^3 \times V(w_2, E^2, \delta), (i-1)/i).$$

The existence of the desired $f(x, t)$ ($x \in F^3 \times E^2$, $0 \leq t \leq \frac{1}{2}$) and $n_2$ follow directly from Theorem 3.1. (Clearly, $n_1$ exists so that $T_{n_1} \subset V(K, F^3, \varepsilon)$. The $\varepsilon$ and $i$ used in Theorem 3.1 is $\varepsilon_1$ and $n_1$, respectively, and $n_2 = n_1 + 4$. We ignore Condition (4), since $\varepsilon_0$ is not defined.) We now proceed, inductively, to define $f(x, t)$ ($(i-1)/i \leq t \leq i/(i+1)$) and $n_{i+1}$.

Let $\gamma$ be a positive number so small that

$$d(T_{n_i} \times V(w, E^2, \gamma)) < 2\, \varepsilon_{i-1},$$

for all $w \in E^2$. The existence of such a $\gamma$ follows from Condition (3) and the uniform continuity of $f(x, (i-1)/i)$. Let $\delta$ be a positive number so small that for each set $S$ of diameter $<\delta$, $d(f(S, (i-1)/i)) < \varepsilon_i$. It follows from Theorem 3.1 that there is an isotopy

$$\mu_t(x) \quad (x \in F^3 \times E^2, (i-1)/i \leq t \leq i/(i+1))$$

and an integer $n_{i+1} = n_i + 4$ such that

$\mu(i-1)/i(x) = x$,
$\mu_t(x) = x$ unless $x \in (T_{n_i} \times E^2) - (z \times E^2)$,
$\mu_t$ moves no point with respect to the $E^2$ factor by more than the minimum of $(\gamma, \delta)$,
$d(\mu_{i/(i+1)}(T_{n_{i+1}} \times w)) < \delta$, and
$\mu_{i/(i+1)}$ is uniformly continuous.

Then

$$f(\mu_t(x), (i-1)/i) = f(x, t)((i-1)/i \leq t \leq i/(i+1)).$$

The $f(x, t)$ ($x \in F^3 \times E^2$, $(i-1)/i \leq t \leq i/(i+1)$) we have defined satisfies Condition (1) because $\mu_{(i-1)/i}(x) = x$ except possibly on

$$(T_{n_i} \times E^2) - (z \times E^2).$$

It satisfies Condition (2), since $\mu_t$ changes $E^2$ coordinates $< \delta$, and satisfies Condition (3) because $d(\mu_{i/(i+1)}\ (T_{n_{i+1}} \times w)) < \delta$. It satisfies Condition (4) because

$$d(T_{n_i} \times V(w, E^2, \gamma)) < 2\varepsilon_{i-1},$$

and $\mu_t$ moves no point along the $E^2$ factor by more than $\gamma$. Finally, it satisfies Condition (5) because $\mu_t(F^3 \times w) \subset F^3 \times V(w, E^2, \delta)$ and

$$\begin{aligned} f(F^3 \times w, i/(i+1)) &= f(\mu_{i/(i+1)}(F^3 \times w), (i-1)/i) \\ &\subset f(F^3 \times V(w, E^2, \epsilon_i), (i-1)/i). \end{aligned}$$

## 4. The main results

THEOREM 4.1. *Suppose $F^3$ is an arbitrary homotopy 3-cell, and*

$$h : S^2 \to \mathrm{Bd}F^3$$

*is a homeomorphism carrying $S^2$ onto $\mathrm{Bd}F^3$. Then, given $\varepsilon > 0$, there exist a point $z \in \mathrm{int}\, F^3$ and a homeomorphism*

$$H : (v * S^2) \times E^2 \to F^3 \times E^2$$

*such that $H \mid S^2 \times E^2 = h \times \mathrm{id}_{E^2}$, $H(v, w) = (z, w)$ for all $w \in E^2$, and*

$$\| w - P_2 \circ H(x, w) \| < \varepsilon$$

*for all $w \in E^2$.*

*Proof.* Let $K$ and $z \in \mathrm{Bd}K$ denote the subcomplex of int $F^3$ and the point of int $F^3$ described in §2 and used in Theorem 3.2. If $G'$ and $G$ are the decompositions of $F^3$ and $F^3 \times E^2$, as given just before the proof of Theorem 3.2, then $F^3/G' = F^3/K \approx z * \mathrm{Bd}F^3$ and

$$(F^3 \times E^2)/G \approx (F^3/G') \times E^2 = (F^3/K) \times E^2 \approx (z * \mathrm{Bd}F^3) \times E^2.$$

Let $\tilde{h} : (v * S^2) \times E^2 \to (F^3/K) \times E^2$ denote the homeomorphism defined by

$$\tilde{h}(((1-t)x + tv), w) = ((1-t)h(x) + t\{K\}, w),$$

where $x \in S^2$, $w \in E^2$, and $\{K\} \in F^3/K$ corresponds to $z \in z * \mathrm{Bd}F^3$ under the natural homeomorphism $z * \mathrm{Bd}F^3 \approx F^3/K$. Then

$$\tilde{h} \mid S^2 \times E^2 = h \times \mathrm{id}_{E^2}, \quad \tilde{h}(v, w) = (\{K\}, w)$$

and

$$\tilde{h}((v * S^2) \times w) = (F^3/K) \times w.$$

Let $f : F^3 \times E^2 \to F^3 \times E^2$ denote the map of $F^3 \times E^2$ onto itself given by Theorem 3.2, where $f = f(\quad, 1)$ described there. We see that $f =$ identity on $(\mathrm{Bd}F^3 \times E^2) \cup (z \times E^2)$ and $\| w - p_2 \circ f(x, w) \| < \varepsilon$ for all $w \in E^2$. Also, $G = \{f^{-1}(x, w) \mid (x, w) \in F^3 \times E^2\}$ and hence $f$ factors through

$(F^3/K) \times E^2$. That is, if

$$\rho : F^3 \times E^2 \to (F^3/K) \times E^2$$

is the quotient map, then $g = f \circ (\rho^{-1})$ is a 1-1 continuous map taking $(F^3/K) \times E^2$ onto $F^3 \times E^2$. Since $(F^3/K) \times E^2$ is a manifold $(\approx (v * S^2) \times E^2)$ and $g$ is a compact map (preimage of compact sets compact), $g$ is a homeomorphism carrying $(F^3/K) \times E^2$ onto $F^3 \times E^2$. We note, $g$ = identity on $\mathrm{Bd}F^3 \times E^2$, $g(\{K\}, w) = (z, w)$ and

$$g((F^3/K) \times w) \subset F^3 \times V(w, E^2, \varepsilon).$$

It follows immediately that $H = g \circ \tilde{h}$ is the desired homeomorphism carrying $(v * S^2) \times E^2$ onto $F^3 \times E^2$.

COROLLARY 4.2. *If $F^3$ is an arbitrary homotopy 3-cell, and*

$$h : S^2 \to \mathrm{Bd}F^3$$

*is a homeomorphism carrying $S^2$ onto $\mathrm{Bd}F^3$, then $\Sigma^2 h : \Sigma^2 S^2 \to \Sigma^2(\mathrm{Bd}F^3)$ extends to a homeomorphism $\hat{H} : \Sigma^2(v * S^2) \to \Sigma^2 F^3$.*

The proof follows immediately from Theorem 4.1 and Lemma 2.2.

COROLLARY 4.3. *If $M^3$ is an arbitrary homotopy 3-sphere and*

$$h : S^2 \to N^2 \subset M^3$$

*is a homeomorphism carrying $S^2$ onto the locally flat submanifold $N^2$ of $M^3$, then there exists a homeomorphism*

$$H : (\Sigma^2(v_1 * S^2 * v_2), \Sigma^2 S^2) \to (\Sigma^2 M^3, \Sigma^2 N^2)$$

*such that $H \mid \Sigma^2 S^2 = \Sigma^2 h$.*

The proof follows immediately from Corollary 4.2, since $N^2$ decomposes $M^3$ into the union of two homotopy 3-cells $F_1^3 \cup F_2^3$, where $F_1^3 \cap F_2^3 = N^2$. That is, if $\hat{H}_i : \Sigma^2(v_i * S^2) \to \Sigma^2 F_i^3$ $(i = 1, 2)$ is the homeomorphism extending $\Sigma^2 h$, then $H$ is defined by

$$H \mid \Sigma^2(v_i * S^2) = \hat{H}_i \quad (i = 1, 2).$$

## 5. Some corresponding results involving homotopy 4-cells and 4-spheres

Clearly, the proof of Theorem 3.1 applies, as given, to PL homotopy 4-cells $F^4$, where $\mathrm{Bd}F^4$ is a homotopy 3-sphere. Moreover, it is not necessary to assume that $F^4$ is a *PL* 4-manifold. That is, Lemma 2.1 holds for all compact contractible topological $n$-manifolds $M^n$ $(n \geq 5)$, since we really only need (and, in fact, only use) the hypothesis that int $M^n$ is a *PL* manifold (and this fact follows from [3]). Also, by [3], int $F^4 \times E^2 \approx E^6$. Thus, int $F^4 \times E^2$

has a $PL$ structure and the interior of any compact 6-manifold in int $F^4 \times E^2$ has an induced $PL$ structure. Therefore, the following result, corresponding to Theorem 3.2, will also hold.

THEOREM 5.1. *Suppose $F^4$ is an arbitrary homotopy 4-cell such that* $\mathrm{Bd}F^4$ *is a homotopy 3-sphere. Also, suppose that $N$ is a collared neighborhood of* $\mathrm{Bd}F^4$ *in $F^4$. For convenience, we identify $N$ with* $\mathrm{Bd}F^4 \times [0, 1]$, *with $x \in \mathrm{Bd}F^4$ corresponding to $(x, 0)$ (such an $N$ exists by* [2]*). If $K$ is the subset of* int $F^4$ *defined by*

$$K = F^4 - (\mathrm{Bd}F^4 \times [0, \tfrac{1}{2})),$$

*$G'$ is the decomposition of $F^4$ given by*

$$G' = \{g' \mid g' \text{ is a point of } F^4 - K \text{ or } g' = K\},$$

*and $G$ is the decomposition of $F^4 \times E^2$ given by*

$$G = \{g = g' \times w \mid g' \in G' \quad \text{and} \quad w \in E^2\},$$

*then, given $\varepsilon > 0$, there is a pseudo-isotopy $f_t$ of $F^4 \times E^2$ onto itself such that*

(a) *$f_0 =$ identity,*

(b) *for each fixed $t < 1$, $f_t$ is a homeomorphism of $F^4 \times E^2$ onto itself,*

(c) *for each $t$ $(0 \leq t \leq 1)$ $f_t =$ identity on $(F^4 - V(K, F^4, \varepsilon)) \times E^2$ and changes $E^2$ coordinates $< \varepsilon$, and*

(d) *$f_1$ takes $F^4 \times E^2$ onto itself and each element of $G$ onto a distinct point.*

*Remark* 5.2. In [5], we show that an analogous result holds for $F^4 \times E^1$ (where $E^1$ replaces $E^2$ above, and $F^4$ is an arbitrary $PL$ homotopy 4-cell such that $\mathrm{Bd}F^3$ is a homotopy 3-sphere). Since int $F^4 \times E^1 \approx E^5$ [3], we also did not really need the fact that $F^4$ was a $PL$ 4-manifold, and this corresponding result, in [5], was used to show that $\Sigma(\mathrm{Bd}F^4) \approx S^5$.

THEOREM 5.2. *Suppose $F^4$ is an arbitrary homotopy 4-cell such that* $\mathrm{Bd}F^4$ *is a homotopy 3-sphere. Then, given $\varepsilon > 0$, there exists a homeomorphism $h$ carrying $(v * \mathrm{Bd}F^4) \times E^2$ onto $F^4 \times E^2$ such that*

$$\| w - p_2 \circ h(x, w) \| < \varepsilon \quad \text{for all } w \in E^2$$

*and $h \mid \mathrm{Bd}F^4 \times E^2 =$ "identity". Furthermore, $h$ induces a homeomorphism*

$$H : \Sigma^2(v * \mathrm{Bd}F^4) \to \Sigma^2 F^4$$

*such that $H \mid \Sigma^2\mathrm{Bd}F^4 =$ "identity".*

*Remark* 5.3. If we further assume that $\mathrm{Bd}F^4 \approx S^3$, then it follows from [5], when extended by [3], that $\Sigma^1 F^4 \approx \Sigma^1(v * S^3) \cong I^5$. This requires the use of a difficult result of [7]. (By also using [3] and [7], this same result was obtained in [8]. Also, refer to [9].)

*Proof.* Let $g$ be a homeomorphism of $\mathrm{Bd}F^4 \times [0, 1]$ onto a closed neighborhood $N$ of $\mathrm{Bd}F^4$ in $F^4$ such that $g(x, 0) = x \in \mathrm{Bd}F^4 \subset N \subset F^4$ [2]. Let

$$K = F^4 - g(\mathrm{Bd}F^4 \times [0, \tfrac{1}{2})).$$

By Theorem 5.1, there exists a map $f_1$ taking $F^4 \times E^2$ onto itself such that $f_1 =$ identity on $(F^4 - g(\mathrm{Bd}F^4 \times [0, \tfrac{1}{4}))) \times E^2$, $f_1$ changes $E^2$ coordinates $< \varepsilon$, and $f_1$ factors through $(F^4/K) \times E^2$ (i.e. $\{f_1^{-1}(x, w) \mid x \in F^4, w \in E^2\} = G$, as defined in Theorem 5.1).

If $\rho : F^4 \times E^2 \to (F^4/K) \times E^2$ is the quotient map, then

$$\tilde{h} = f_1 \circ \rho^{-1} : (F^4/K) \times E^2 \to F^4 \times E^2$$

is a 1-1 continuous compact map carrying $(F^4/K) \times E^2$ onto $F^4 \times E^2$ such that $\tilde{h} =$ identity on $\mathrm{Bd}F^4 \times E^2$ and

$$\tilde{h}((F^4/K) \times w) \subset F^4 \times V(w, E^2, \varepsilon).$$

Since $f =$ identity on {a neighborhood of $\mathrm{Bd}F^4$} $\times E^2$ and $f$ is a compact map, $\tilde{h}$ is a homeomorphism. Let $k$ denote the natural homeomorphism carrying $(v * \mathrm{Bd}F^4) \times E^2$ onto $(F^4/K) \times E^2$ (i.e.

$$k((v * \mathrm{Bd}F^4) \times w) = (F^4/K) \times w$$

with $k((v, w)) = (\{K\}, w)$ and $k \mid \mathrm{Bd}F^4 \times w =$ identity). The desired homeomorphism

$$h : (v * \mathrm{Bd}F^4) \times E^2 \to F^4 \times E^2$$

is given by $h = \tilde{h} \circ k$.

It follows by Lemma 2.2, that $h$ induces a homeomorphism

$$H : \Sigma^2(v * \mathrm{Bd}F^4) \to \Sigma^2 F^4$$

such that $H \mid \Sigma^2\mathrm{Bd}F^4 =$ identity.

Corollary 5.3. *If $\tilde{k} : S^3 \times E^2 \to \mathrm{Bd}F^4 \times E^2$ is any homeomorphism carrying $S^3 \times E^2$ onto $\mathrm{Bd}F^4 \times E^2$ that is bounded on the $E^2$ factor, then $\tilde{k}$ induces (by Lemma 2.2) a homeomorphism $k : \Sigma^2 S^3 \to \Sigma^2(\mathrm{Bd}F^4)$ such that $k \mid \Theta^1 =$ identity (recall $\Theta^1$ is the suspension circle of each set), and $k$ extends to a homeomorphism $K : \Sigma^2(v * S^3) \to \Sigma^2 F^4$.*

*Proof.* Since

$$\Sigma^2(v * S^3) = v * (\Sigma^2 S^3) \quad \text{and} \quad \Sigma^2(v * \mathrm{Bd}F^4) = v * (\Sigma^2\mathrm{Bd}F^4),$$

$k : \Sigma^2 S^3 \to \Sigma^2(\mathrm{Bd}F^4)$ extends to a homeomorphism

$$f = v * k : v * (\Sigma^2 S^3) \to v * (\Sigma^2\mathrm{Bd}F^4).$$

We define $K : \Sigma^2(v * S^3) \to \Sigma^2 F^4$ by $K = H \circ f$, where

$$H : \Sigma^2(v * \mathrm{Bd}F^4) \to \Sigma^2 F^4$$

is the homeomorphism of Theorem 5.2. Since $H \mid \Sigma^2\mathrm{Bd}F^4 =$ identity, $K \mid \Sigma^2 S^3 = f \mid \Sigma^2 S^3 = k$.

COROLLARY 5.4. *If $N^3$ is a homotopy 3-sphere contained as a locally flat submanifold of the homotopy 4-sphere $M^4$, then there exists a homeomorphism $H : (\Sigma^2(v_1 * S^3 * v_2), \Sigma^2 S^3) \to (\Sigma^2 M^4, \Sigma^2 N^3)$ such that $H \mid \Theta^1 =$ identity.*

This follows immediately from Corollary 5.3, just as Corollary 4.3 followed from Corollarv 4.2.

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