FINITE GROUPS HAVING SUBGROUPS OF ODD ORDER WITH SMALL AUTOMISERS

BY
MICHAEL J. COLLINS

1. Introduction

The groups $SL(2, 2^a)$, $a \geq 2$, provide one of the easiest examples of finite simple groups in which all subgroups of odd order are abelian. Indeed, if K is such a subgroup, then $| N(K)/C(K) | \leq 2$. In this paper, we shall characterise these groups by this property. Actually, a weaker hypothesis will be taken since it will emerge as one more readily open to generalisations.

Let G be a finite group, and H a subgroup of G. Then the factor group $N_G(H)/C_G(H)$ is called the *automiser* of H in G. We define

$$a(H) = | N_G(H)/C_G(H) |.$$

The following is the main result of this paper.

THEOREM A. *Let G be a finite simple group such that whenever K is a subgroup of odd order which is either abelian or of prime power order, then $a(K) \leq 3$. Then G is isomorphic to one of the groups $SL(2, 2^a)$, $a \geq 2$, or $PSL(2, 7)$.*

Here of course, as throughout, "simple" means "non-abelian simple". The main steps in the proof are to construct maximal abelian Hall subgroups of odd order and exceptional characters for G corresponding to each, and to use a result of Brauer and Tuan [4] to fit together the fragments of character table so obtained. The arguments are then mainly number-theoretic, and culminate in showing that G is a CIT-group: that is, a group in which the centraliser of any involution is a 2-group. At this stage, Suzuki's classification could be invoked [12], [13]: however, so much additional information is available that the proof is very quickly completed. The precise calculations differ according to whether or not there is a subgroup K of odd order with $a(K) = 3$. If not, we show in Section 6 that G is isomorphic to $SL(2, 2^a)$: if there is, we show in the final section that only $PSL(2, 7)$ arises. It should be noted that these groups do satisfy the hypothesis of the theorem.

The methods should generalise, with restrictions, to larger automisers. For this reason, the character theory in Section 5 is considered in a more general setting, and the construction of the exceptional characters and the application of the Brauer-Tuan formula are discussed. However, three things distinguish the situation in this paper from that where automisers may have order 4. In the first place, we are able to show that when $a(K) \leq 3$, the action of the normaliser on K is of Frobenius-type: if $a(K) = 4$ were allowed,

Received May 28, 1970.

some such condition would have to be added to the hypothesis to allow the application of our character-theoretic methods. Secondly, in Section 5, we will define the idea of natural or unnatural situations in fitting together pieces of character table. In the proof of Theorem A, an unnatural situation is shown to be impossible: however, one does occur in the alternating groups A_6 and A_7. Finally, the corresponding general structure result would need to be of a different type, as the symmetric group S_5 shows: here we have the following as a consequence of Theorem A.

COROLLARY. *Let G be a finite group such that whenever K is a subgroup of odd order which is either abelian or of prime power order, then $a(K) \leq 3$. Then G has normal subgroups G_1 and G_2 such that*

(i) $G_2 = O_2(G)$;

(ii) $G_1 \supseteq G_2$, *and* $G_1/G_2 = A \times B$ *where A is abelian of odd order, and B is isomorphic to one of the groups* $SL(2, 2^a)$, $a \geq 2$, *or* $PSL(2, 7)$, *or one or both of A and B may be trivial; and*

(iii) $[G:G_1] \leq 3$: *if G is insoluble,* $G = G_1$.

The notation is, for the most part, standard, and standard results (as can be found in [5] and [10], for example) will be used without reference. All groups will be finite. Many definitions involving a prime p will be extended in a natural way to a set of primes π. π' will denote the complementary set of primes. An element whose order is divisible only by primes in π will be called a *π-element*. If G is a group and $g \in G$, g can be written uniquely as a product xy where x is a π-element, called the *π-part*, commuting with a π'-element y called the *π'-part*. An element with non-identity π-part is *π-singular*; a π'-element is *π-regular*. A *π-section* $S_\pi(g)$ is the set of all elements of G whose π-parts are conjugate to a given non-identity π-element g. π-sections thus consist of π-singular elements, and any element not lying in a π-section is p-regular for all $p \in \pi$.

Since we shall be concerned with principal blocks for various primes, we write $B_0(G; p)$ for the principal p-block of G, or just $B_0(p)$ if no confusion as to the group can arise. For H an abelian Hall subgroup of G, we shall in Section 5 define a set of characters $B(H)$ of G: this notation is deliberate since in the case that H is a p-group, this will coincide with the principal p-block. By character, we shall always mean an ordinary absolutely irreducible character.

If $K \subseteq H \subseteq G$, define $a_H(K) = |N_H(K)/C_H(K)|$. We write $a(K)$ for $a_G(K)$. As is normal, suffices in $N_G(K)$ and $C_G(K)$ will be suppressed if no confusion can occur.

2. Preliminary results

PROPOSITION 2.1. *Let P be a Sylow subgroup of a group G. Then any two elements of $C(P)$ conjugate in G are conjugate in $N(P)$.*

Proof. This is an immediate generalisation of the Burnside lemma [10, Theorem 7.1.1].

PROPOSITION 2.2. *Let G be a group and H an abelian Hall π-subgroup of G such that $N(H)/O_{\pi'}(N(H))$ is a Frobenius group. Then*

(i) *any two elements of H conjugate in G are already conjugate in $N(H)$; and*

(ii) *if $g \in H^{\#}$, then $C_G(g) = H \cdot O_{\pi'}(C(g))$.*

If, furthermore, every cyclic subgroup of H is normal in $N(H)$, then

(iii) *if L is a subgroup of G which contains a conjugate of every cyclic subgroup of H, then L contains a conjugate of H.*

Proof. By a result of Wielandt [14], if K is a subgroup of G containing H and a conjugate $H' = H^g$, then H and H' are conjugate in K. Parts (i) and (ii) are the generalisations of the corresponding results for a Sylow subgroup.

Suppose $g_1, g_2 \in H$, and $g_2 = g_1^x$ for some $x \in G$. Then $H, H^x \subseteq C(g_2)$ so that for some $y \in C(g_2)$, $H^{xy} = H$. Thus $xy \in N(H)$, and $g_1^{xy} = g_2^y = g_2$: hence (i) holds. Now if P is a Sylow p-subgroup of H, $p \in \pi$, $P \lhd N(H)$. On the other hand, if $u \in N(P)$, $H^u \subseteq C(P)$ so that for some $v \in C(P)$, $H^{uv} = H$: thus for some $w \in N(H)$, $u = wv^{-1}$ so that $u \in N(H) \cdot C(P)$. Hence $N(P) = N(H) \cdot C(P)$. With $N(H)/O_{\pi'}(N(H))$ a Frobenius group, $C(H) = H \times O_{\pi'}(N(H))$ and, for any $g \in H^{\#}$, $C_{N(H)}(g) = C(H)$. On the other hand, $N_{C(P)}(H) = C(H)$: thus no two distinct elements of H are conjugate in $C(P)$, and it follows that $C_G(g) \cap N(P) \subseteq C(P)$. By Burnside's transfer theorem, $C_G(g)$ has a normal p-complement for each $p \in \pi$ and so a normal π-complement: hence $C_G(g) = H \cdot O_{\pi'}(C(g))$.

Assume now that the hypothesis of (iii) also holds. Since, by (i), no two distinct cyclic subgroups of H are conjugate in G, a count of cyclic subgroups shows that L contains a Sylow p-subgroup of G for each $p \in \pi$. If $|\pi| = 1$ there is no more to prove, so suppose that $p, q \in \pi$, $p \neq q$. Let P be a Sylow p-subgroup of L, and Q a Sylow q-subgroup of H. If $g_1 \in P^{\#}$, $g_1^x \in H$ for some $x \in G$, and, for each $g_2 \in Q$, $\langle g_1^x g_2 \rangle^y \subseteq L$ for some $y \in G$, by hypothesis. Now, for some $z \in L$, $\langle g_1 \rangle^{xyz} = \langle g_1 \rangle$ and $\langle g_2 \rangle^{yz} \subseteq C_L(g_1)$. Thus $C_L(g_1)$ contains a conjugate of each cyclic subgroup of Q, and hence a conjugate of Q. Hence $|H|$ divides $|C_L(g_1)|$. On the other hand, $C_G(g_1)$, and hence also $C_L(g_1)$, has a normal π-complement, by (ii). Thus $C_L(g_1)$, and hence also L, contains a conjugate of H.

PROPOSITION 2.3. *Let G be a simple group, and let H be a Hall subgroup of odd order. If G has a non-linear character of degree dividing $|H|$, then no involution can centralise H.*

Proof. Suppose the contrary, and that y is an involution in $C(H)$, and χ a character with $\chi(1) = h \neq 1$ and h dividing $|H|$. Then

$$(h, |G|/|C(y)|) = 1.$$

$\chi(y)$ is a non-zero rational integer, so that $|G|/|C(y)|\cdot\chi(y)/\chi(1)$ is a rational integer since it must be an algebraic integer. Thus $\chi(y)/\chi(1)$ is a rational integer so that $\chi(y) = \pm\chi(1)$, which is impossible since χ is non-linear and G simple.

I am grateful to Professor J. G. Thompson for the particular case of the following result which is a trivial generalisation of that case.

PROPOSITION 2.4. *Let G be a group which is a semi-direct product of a 2-group T by a subgroup K, where $K = H\langle x\rangle$ is a Frobenius group in which x is an involution inverting every element of H. Then any involution in G outside T is conjugate to an element tx where $t \in C_T(H)$. In particular, if $C_T(H) = 1$, all involutions of G outside T are conjugate.*

Proof. If $C_T(H) = T$, the result is trivial, so we may suppose otherwise. Since x inverts H,

$$(*) \qquad h^{-1}(x^{-1}yx)h = x^{-1}hyh^{-1}x = x^{-1}yx$$

for all $h \in H$, $y \in C_T(H)$: thus $C_T(H)$ admits K. If U is a subgroup of T admitting K, then $N_T(U)$ admits K: thus there is a series of subgroups

$$C_T(H) = U_0 \subset U_1 \subseteq \cdots \subseteq U_n = T$$

such that $U_i \lhd U_{i+1}$ and U_i admits K, for each i. Furthermore, by refinement if necessary, we may suppose that K acts irreducibly on each factor U_i/U_{i-1}, and that all inclusions are proper. Let s be an involution in G outside T: we may suppose that $s \in \langle T, x\rangle$. If $s = tx$ for some element $t \in C_T(H)$ there is nothing to prove, so suppose that $s = t_0 x$ where $t_0 \in U_i$, but $t_0 \notin U_{i-1}$, and $i \geq 1$. Then it will be sufficient to prove that s is conjugate to an element tx with $t \in U_{i-1}$, for then sufficient repetition gives the result.

Since $t_0 = sx$, $t_0^x = t_0^{-1}$, so that x normalises $U_{i-1}\langle t_0\rangle$, and hence fixes the coset $U_{i-1}t_0$ of U_{i-1} in U_i. Let $M = U_i/U_{i-1}$. Then M may be regarded as an irreducible K-module. Choose $h \in H^{\#}$. Then $C_M(h)$ admits K (cf. $(*)$), and [10, Theorem 5.2.3]

$$M = C_M(h) \oplus M(1 - h).$$

Since M is certainly indecomposable, either $C_M(h) = 0$ or $C_M(h) = M$. Now $C_T(H) = U_0$ so that $C_M(H) = 0$: thus if $H_1 = C_H(M)$, $H_1 \neq H$. If $H/H_1 = A$, M may be regarded as an $A\langle x\rangle$-module with each element of A acting fixed-point-freely so that A is cyclic. Regarding M as a vector space, $C_M(x)$ and $M(1 - x)$ are the kernel and image respectively of the endomorphism $(1 - x)$: hence

$$\dim C_M(x) + \dim M(1 - x) = \dim M.$$

On the other hand, $M(1 - x) \subseteq C_M(x)$ and $\dim C_M(x) \leq \frac{1}{2}\dim M$: hence $C_M(x) = M(1 - x)$. Now $U_{i-1}t_0 \in C_M(x)$, and so $U_{i-1}t_0 \in M(1 - x)$. Returning to group notation, this means that there exists $t_1 \in U_i$ such that

$U_{i-1} t_0 = [U_{i-1} t_1, x]$, and so $t_0 = u[t_1, x]$ for some $u \in U_{i-1}$. Thus $t_0 = ut_1^{-1}xt_1 x$, and so $s = t_0 x = ut_1^{-1}xt_1$. Hence $t_1 st_1^{-1} = t_1 ut_1^{-1}x = tx$ for some $t \in U_{i-1}$ since t_1 normalises U_{i-1}. So s is conjugate to this element tx, and the result is proved, the special case being a trivial consequence.

3. Some general properties and the derivation of the corollary

In this section we shall consider properties of groups, not necessarily simple, which satisfy the following hypothesis:

(A_3) For any subgroup K of odd order which is either abelian or of prime power order, $a(K) \leq 3$.

The corresponding hypothesis with $a(K) \leq 2$ will be denoted by (A_2). Clearly any group satisfying (A_2) will satisfy (A_3): later we shall distinguish cases as G satisfies (A_2), or (A_3) but not (A_2).

As an obvious consequence of the definition of $a(K)$, we have the following result which will be frequently used without reference.

LEMMA 3.1. *Let G satisfy* (A_3). *Then any subgroup of G also satisfies* (A_3).

LEMMA 3.2. *Let G satisfy* (A_3). *Then all Sylow subgroups for odd primes are abelian, and if K is any subgroup of odd order, either K is abelian, or K has a normal abelian subgroup of index* 3.

Proof. Let P be a Sylow p-subgroup of G for p odd. Then if P were non-abelian, $|P/Z(P)|$ would be at least p^2 so that $a(P) > 3$: thus P is abelian. Now let K be any subgroup of G of odd order. If $3 \nmid |K|$, for each Sylow p-subgroup P_1 of K, $a_K(P_1) = 1$, so that K has a normal p-complement: hence K is abelian. If 3 does divide $|K|$, let T be a Sylow 3-subgroup of K. Then $a_K(T) = 1$ so that K has an abelian normal 3-complement L. If $a_K(L) = 1$, K is abelian; if $a_K(L) = 3$, $C_K(L)$ is a normal abelian subgroup of K of index 3.

LEMMA 3.3. *Let G be a group satisfying* (A_3). *If H is a soluble normal subgroup of G, then G/H satisfies* (A_3).

Proof. It is sufficient to prove the result for H elementary abelian. Since any p-subgroup of G is abelian if p is odd, it is sufficient to consider abelian subgroups of G/H of odd order. Let M/H be one. Let bars denote images in the factor group G/H. If $K = N_G(M)$, $\bar{K} = N_{\bar{G}}(\bar{M})$. First suppose that H and $\bar{M}$ have coprime order. Then by the Schur-Zassenhaus theorem, H has a complement M^* in M, and all such complements are conjugate in M. An analogue of the Frattini argument yields $K = M \cdot N_K(M^*)$. Since $C_{\bar{G}}(\bar{M}) \subseteq C_G(M^*) \cdot H/H$, $a_{\bar{G}}(\bar{M}) \leq 3$.

Suppose that $(|H|, |\bar{M}|) \neq 1$. Since H is elementary, M has odd order, and so is either abelian or non-abelian with a normal abelian subgroup L of index 3. If M is abelian, $C_{\bar{G}}(\bar{M}) \supseteq \overline{C_G(M)}$ and so $a_{\bar{G}}(\bar{M}) \leq 3$; if M is non-abelian, L is its Fitting subgroup so that $H \subseteq L \lhd K$. Since $a_M(L) = 3$,

$a_K(L) = 3$, and $C = C_K(L)$ is a normal subgroup of K of index 3 with $C \cap M = L$ and $K = CM$. Let T be a Sylow 3-subgroup of M. Then $M = LT$ so that $K = CT$ and $\bar{K} = \bar{C}\bar{T}$. Thus $\bar{L} \subseteq Z(\bar{K})$ so that $a_{\bar{G}}(\bar{M}) = a_{\bar{K}}(\bar{M}) = a_{\bar{K}}(\bar{T})$. If H has order coprime to 3, $a_{\bar{K}}(\bar{T}) \leq a_K(T) \leq 2$ by the previous paragraph: if H is a 3-group, $H \subseteq T$ and T is abelian, so that $a_{\bar{K}}(\bar{T}) \leq a_K(T)$ as above. Thus $a_{\bar{G}}(\bar{M}) \leq 3$.

LEMMA 3.4. *Let G be a soluble group satisfying* (A_3). *Then G satisfies the corollary.*

Proof. By the preceding lemma, $\bar{G} = G/O_2(G)$ satisfies (A_3). If $O_{2'}(\bar{G})$ is abelian, $C_{\bar{G}}(O_{2'}(\bar{G})) = O_{2'}(\bar{G})$ by the Hall-Higman centraliser lemma [10, Theorem 6.3.2], and the corollary is satisfied with $G_1 = O_{2,2'}(G)$ and $[G:G_1] \leq 2$. If $O_{2'}(\bar{G})$ is non-abelian, let H be the normal abelian subgroup of index 3. Then $H \lhd \bar{G}$ so that $K = C_{\bar{G}}(H) \lhd \bar{G}$. Since $O_2(\bar{G}) = 1$, $O_2(K) = 1$ and $O_{2'}(K) \subseteq O_{2'}(\bar{G})$: thus $H = O_{2'}(K)$ and, since $C_K(O_{2'}(K)) \subseteq O_{2'}(K)$, $H = K$. Hence the corollary is satisfied with G_1 the counterimage of H, and $[G:G_1] = 3$.

We shall now show by induction that the corollary in general is a consequence of Theorem A. Let G be a group satisfying (A_3) which is a counterexample of minimal order to the corollary. Then G will be insoluble, and clearly $O_2(G) = 1$. Our aim will be to show that G is simple, and so no counterexample.

Suppose first that G has a non-trivial soluble normal subgroup. Let H be the maximal such subgroup. Then by inductive hypothesis, G/H is isomorphic to one of the groups $SL(2, 2^a)$, $a \geq 2$, or $PSL(2, 7)$. If H has odd order, either H is abelian or H has a normal abelian subgroup of index 3. If H is abelian, $H \subseteq Z(G)$: since the groups listed have Schur multipliers which are trivial or of order 2 [11], the extension splits, and there is no counterexample. If H is non-abelian, let K be its normal abelian subgroup of index 3. Then $K \lhd G$ and $[G:C(K)] = 3$ so that $C(K)$ is insoluble with K its maximal soluble normal subgroup. Then by the inductive hypothesis, $C(K) = K \times S$ with S simple and normal in G. Now H must centralise some Sylow 3-subgroup T of S, and $a_G(K \times T) = 6$, contrary to hypothesis.

Now suppose that H has even order. Since $O_2(H) = 1$, H has an abelian subgroup K of odd order and index 2 in H, and $C_H(K) = K$. Now $[G:C_G(K)] = 2$ and, as above, $C_G(K)$ splits over K with $C_G(K) = K \times S$ where S is simple and normal in G. With T a Sylow 3-subgroup of S, $a_G(K \times T) = 4$, contrary to hypothesis.

Thus, it may be assumed that G has no soluble normal subgroup. Let N be a minimal normal subgroup. Then N is simple. By Theorem A, $G \neq N$. Thus N is isomorphic to one of the listed simple groups. If $H = C_G(N)$, $H \lhd G$ and $H \cap N = 1$. Hence $H = 1$. Thus $N \subset G \subseteq \operatorname{Aut} N$, and it is readily verified that no such group G can be found to satisfy (A_3).

4. The Hall subgroups of a simple group

We now turn our attention to simple groups which satisfy the hypothesis (A_3) and, except in Section 5 where a more general situation will be considered, will let G be such a group for the remainder of this paper. The purpose in this section is essentially to find the maximal odd order subgroups.

LEMMA 4.1. *Let p be an odd prime divisor of $|G|$ and let P be a Sylow p-subgroup of G. Then*

(i) $C(P) \neq N(P)$;

(ii) *for any element $g \in P^{\#}$,*

$$C_{N(P)}(g) = C(P) \quad \textit{and} \quad C_G(g) = P \cdot O_{p'}(C(g));$$

and

(iii) *if $3 \mid |G|$, any element of G of order 3 is conjugate to its inverse.*

Proof. By Lemma 3.2, P is abelian. By a transfer argument, an element g of $P^{\#}$ must be conjugate in G, and hence in $N(P)$, to some other element of P so that $N(P) \supset C(P)$. Since $a(P) \leq 3$, the action of an element of $N(P)$ outside $C(P)$ must then be fixed-point-free, so that $C_{N(P)}(g) = C(P)$. By the Burnside transfer theorem, $C_G(g)$ must have a normal p-complement. Finally, if $p = 3$, $a(P) = 2$ so that any element of a Sylow 3-subgroup is conjugate to its inverse.

The main result of this section is the following, together with its corollary.

LEMMA 4.2. *Let p be an odd prime divisor of $|G|$ and let P be a Sylow p-subgroup of G. Then there is a subgroup H of odd order containing P such that*

(i) *H is an abelian Hall subgroup of G;*

(ii) $a(H) = 2$ *or* 3;

(iii) *if $g \in H^{\#}$, $C_{N(H)}(g) = C(H)$ and $C_G(g) = H \cdot O_2(C(g))$; and*

(iv) *if $a(H) = 2$, every cyclic subgroup of H is normal in $N(H)$.*

Proof. Put $C(P) = P \times K$ where K is a p'-group. Certainly $a(P) = 2$ or 3. Then if either $K = 1$ or K is a 2-group, (i), (ii) and the first part of (iii) are satisfied with $H = P$ by Lemma 4.1. Otherwise let Q be a Sylow q-subgroup of K, q odd. By the Frattini argument,

$$N(P) = C(P) \cdot N_{N(P)}(Q);$$

thus there exists an element g in $N(P) \backslash C(P)$ such that $g \in N(Q)$. Then $N_G(P \times Q) \supseteq \langle P, Q, g, N_K(Q) \rangle$. Since $a(P \times Q) \leq 3$, this implies that $N_K(Q) = C_K(Q)$: thus K has a normal q-complement for each odd prime divisor q of its order so that $K = LS$ where $S = O_2(K)$ and L is abelian of odd order. Since all complements of S in K are conjugate, $N_{N(P)}(L) \not\subseteq C(P)$. Let $\bar{Q}$ be a Sylow q-subgroup of G containing Q. Then $C_G(Q) = \bar{Q} \cdot O_{q'}(C(Q))$. Since $P \subseteq C(Q)$, we may suppose that $\bar{Q} \subseteq N(P)$. If $q \geq 5$ or $a(P) = 2$, then $\bar{Q} \subseteq C(P)$ so that $Q = \bar{Q}$. If $q = 3$ and $a(P) = 3$, then $N(P)$ has a normal 3-complement so that no two elements of Q can be conjugate in $N(P)$, and hence in G by Proposition 2.1: this is contrary to Lemma 4.1 (iii) so that the situation cannot arise. Thus L is a Hall subgroup of G of odd order, and we put $H = P \times L$. Since $N_{N(P)}(L) \nsubseteq C(P)$, $N(H) \neq C(H)$ so that H satisfies parts (i) and (ii) of the lemma, and any element of $N(H) \backslash C(H)$ acts fixed-point-freely on P. By construction, H is a Hall $2'$-subgroup of $C(P)$, and any two such subgroups are conjugate. Thus we may suppose that H is the subgroup constructed in this manner as a Hall $2'$-subgroup of $C(Q)$ for each Sylow subgroup Q of H. In particular, for each such Q, an element of $N(H) \backslash C(H)$ acts fixed-point-freely on Q, and hence on H. Thus for any element $g \in H^{\#}$, $C_{N(H)}(g) = C(H)$.

In any case, there is a subgroup H of G satisfying parts (i) and (ii), and the first half of (iii) of the lemma. Let π be the set of prime divisors of $| H |$. $N(H)/O_{\pi'}(N(H))$ is a Frobenius group and so, by Proposition 2.2, $C_G(g) = H \cdot O_{\pi'}(C(g))$ for any $g \in H^{\#}$. By construction, for each $p \in \pi$, $C(P) = HS_p$ where P is the Sylow p-subgroup of H and $S_p = O_2(C(P))$. Thus for $g \in H^{\#}$, if r is an odd prime divisor of $| O_{\pi'}(C(g)) |$, each Sylow subgroup of H must normalise a Sylow r-subgroup of $O_{\pi'}(C(g))$ but cannot centralise it. Thus (iii) holds unless H is a 3-group and there is a Sylow r-subgroup R of $O_{\pi'}(C(g))$ normalised, but not centralised, by H, with r odd. Now let H' be a Hall $2'$-subgroup of $C(\bar{R})$ where $\bar{R}$ is a Sylow subgroup of G containing R. Then $3 \nmid | H' |$ so that $3 \nmid | C(R) |$ by the above argument, contrary to the choice of R in $C(g)$. Thus (iii) always holds.

Finally, if $a(H) = 2$, any element of $N(H) \backslash C(H)$ must invert every element of H so that (iv) holds.

From (iii) it follows that H is a maximal abelian subgroup of odd order, and also that every maximal abelian subgroup of odd order in G can be realised in this manner. Thus as an immediate consequence we have

COROLLARY 4.2.1. *The odd prime divisors of $| G |$ form disjoint sets $\pi_1, \cdots, \pi_n$, with $n \geq 2$, such that for each i,*

(i) *G has an abelian Hall π_i-subgroup H_i; and*

(ii) *if $p_i \in \pi_i$ and $p_j \in \pi_j$, $i \neq j$, then G has no element of order $p_i p_j$.*

Proof. For the odd prime divisors of $| G |$, define the relation $p \sim q$ if and only if there are elements of orders p and q that commute. By (iii) this is an equivalence relation, so let $\pi_1, \cdots, \pi_n$ be the equivalence classes and $H_1, \cdots, H_n$ the corresponding Hall subgroups that appear (one for each class). If $n = 1$, each element of $H^{\#}$ lies in a conjugacy class with 2^{α} elements, denying simplicity of G by a result of Burnside [10, Lemma 4.3.2]: thus $n \geq 2$.

5. The exceptional character theory

In this section a situation more general than that occurring under the hypothesis (A_3) is considered. We shall use the Dade isometry [6] in order to generalise a result of Brauer [3, Theorems 9A and 9B], constructing a complete set of exceptional characters corresponding to a Hall subgroup rather than a Sylow subgroup. The methods are those of Feit: the non-linear characters to be lifted are coherent. (See [9], especially pages 174–175.) However, we shall need the precise values of the exceptional characters and also certain non-exceptional characters: these together will contain the principal block for each prime divisor of the Hall subgroup. We shall then use a result of Brauer and Tuan to show how these sets of characters may overlap for different Hall subgroups.

The notation, except where defined here or earlier, follows [9].

LEMMA 5.1. *Let G be a group in which, for a prime p, the centraliser of every non-identity p-element has a normal p-complement. Then an irreducible character of G lies in $B_0(p)$ if and only if it is constant on p-sections, and non-zero on at least one.*

Proof. Let x be a non-identity p-element of G, and let y be a p-regular element of $C(x)$. Then if χ is a character of G,

$$\chi(xy) = \textstyle\sum_{\varphi} d_{\varphi}\varphi(y),$$

where the $\{\varphi\}$ are Brauer characters of $C(x)$, $\{d_{\varphi}\}$ are generalised decomposition numbers, and the sum is restricted to those φ lying in blocks dominated by the block of G containing χ. In particular, $B_0(G;p)$ dominates only $B_0(C(x);p)$, and $B_0(C(x);p)$ has only one Brauer character, the principal character φ_0, since $C(x)$ has a normal p-complement.

If $\chi \in B_0(G;p)$, χ cannot be zero on all p-singular elements. Suppose g is a p-singular element with p-part x such that $\chi(g) \neq 0$. Then for any p-regular element $y \in C(x)$,

$$\chi(xy) = d_{\varphi_0}\varphi_0(y) = d_{\varphi_0},$$

and since $\chi(g) \neq 0$, $d_{\varphi_0} \neq 0$. In any case, χ is necessarily constant on the p-section $S_p(x)$, and so constant on each p-section of G.

Conversely, if χ is constant on p-sections, but not always zero, let x be a non-identity p-element such that $\chi(x) \neq 0$. Since $\chi(xy)$ is constant as y ranges over all p-regular elements of $C(x)$,

$$\chi(xy) = \chi(x)\cdot\varphi_0(y).$$

Since the generalised decomposition numbers are unique, $d_{\varphi_0} = \chi(x)$, so that $\chi \in B_0(G;p)$.

THEOREM 5.2. *Let G be a group and H an abelian Hall π-subgroup of odd order such that $N(H)/O_{\pi'}(N(H))$ is a Frobenius group of order $q \mid H \mid$, with $q \geq 2$. Suppose either that $q \leq 4$ or that H has at least one cyclic Sylow subgroup. Then G has irreducible characters $\zeta_0 = 1, \zeta_1, \cdots, \zeta_{q-1}$ and $\chi_1, \cdots, \chi_s$ where $s = (\mid H \mid - 1)/q$ for which, if g is an element of G with non-identity*

π-part $x \in H$,

$$\zeta_l(g) = \delta_l, \qquad l = 1, \cdots, (q-1),$$

and

$$\chi_j(g) = \varepsilon\varphi_j(x), \qquad j = 1, \cdots, s,$$

where $\delta_1, \cdots, \delta_{q-1}, \varepsilon = \pm 1$, and $\varphi_1, \cdots, \varphi_s$ are the non-linear characters of $N(H)$ having $O_{\pi'}(N(H))$ in their kernel. If y is any π'-element of G, all $\chi_j(y)$ are equal, and

$$1 + \textstyle\sum_{l=1}^{q-1} \delta_l \zeta_l(y) = \varepsilon\chi_j(y).$$

In particular, ε must be such that $\varepsilon(1 + \sum_{l=1}^{q-1} \delta_l \zeta_l(1))$ is positive.

Furthermore, for each $p \in \pi$, $B_0(G;p)$ consists of the characters $\zeta_0, \cdots, \zeta_{q-1}$ together with those characters χ_j for which $\chi_j(h) = q\varepsilon$ for all p-regular elements $h \in H^{\#}$.

Remark. The hypothesis on $N(H)$ together with the restriction ensures that $N(H)/C(H)$ is cyclic. The set of all characters $\{\zeta_l\} \cup \{\chi_j\}$ will be denoted by $B(H)$. If $q < |H| - 1$, the characters $\{\chi_j\}$ are called *exceptional*, and we denote this set by $\mathcal{E}(H)$, and the characters $\{\zeta_0, \cdots, \zeta_{q-1}\}$ are called *non-exceptional*. If $q = |H| - 1$, the character χ_1 is essentially indistinguishable from $\zeta_1, \cdots, \zeta_{q-1}$: in this case, all characters in $B(H)$ will be called non-exceptional.

Proof of Theorem 5.2. If $q = |H| - 1$, then H is necessarily cyclic of prime order p, and the result is due to Brauer [1]: the characters in question are precisely those of $B_0(p)$. Indeed, if H is a p-group, the result follows immediately from Theorem (9A) of [3] once it is shown that there is a complete set of q non-exceptional characters. If $q \leq 4$, this follows by direct computation: if $q \geq 5$ and H is cyclic, it is a result of Dade [7, Theorem 1, part 1].

Thus we may suppose that H is not of prime power order. Let $s = (|H| - 1)/q$, and let $\psi_0, \cdots, \psi_{q-1}$ be the linear characters of $N(H)$ and $\varphi_1, \cdots, \varphi_s$ those of degree q, all having $O_{\pi'}(N(H))$ in their kernel. Since H must have at least two Sylow subgroups of orders at least $(q + 1)$ and $(2q + 1)$ respectively,

$$s \geq 2q + 3. \tag{5.1}$$

Now $(\varphi_1 - \sum_{i=0}^{q-1} \psi_i)$ together with the set $\{(\varphi_1 - \varphi_j)\}_{j=2,\cdots,s}$ form a basis for the module of integral combinations of $\psi_0, \cdots, \psi_{q-1}, \varphi_1, \cdots, \varphi_s$ which vanish on all π-regular elements of $N(H)$. By Proposition 2.2 the conditions for the application of the Dade isometry are met: denoting that map by $*$, we obtain the equations

$$\|(\varphi_1 - \varphi_j)^*\|_G = 2, \qquad j = 2, \cdots, s,$$

and

$$((\varphi_1 - \varphi_j)^*, (\varphi_1 - \varphi_k)^*)_G = 1, \qquad j, k = 2, \cdots, s, \quad j \neq k.$$

Since $(\varphi_1 - \varphi_j)^*(1) = 0$ for each j, there must be irreducible characters $\chi_1, \cdots, \chi_s$ of G and a sign $\varepsilon = \pm 1$ such that for all $j, k = 1, \cdots, s$,

$$(\varphi_j - \varphi_k)^* = \varepsilon(\chi_j - \chi_k) \qquad \text{(cf. (23.3) of [9])}.$$

For each $j = 2, \cdots, s$, the Dade isometry also yields that

$$((\varphi_1 - \textstyle\sum_{i=0}^{q-1} \psi_i)^*, (\varphi_1 - \varphi_j)^*)_G = ((\varphi_1 - \sum_{i=0}^{q-1} \psi_i), (\varphi_1 - \varphi_j))_{N(H)} = 1.$$

Hence either

$$(\varphi_1 - \textstyle\sum_{i=0}^{q-1} \psi_i)^* = \varepsilon\chi_1 - \sum_{l=0}^{r} n_l \zeta_l$$

for non-zero integers n_l and irreducible characters $\{\zeta_l\}$ different from the $\{\chi_j\}$, from which linearity of the isometry yields that

$$(5.2) \qquad (\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i)^* = \varepsilon\chi_j - \sum_{l=0}^{r} n_l \zeta_l, \qquad j = 1, \cdots, s:$$

or

$$(5.3) \qquad (\varphi_1 - \textstyle\sum_{i=0}^{q-1} \psi_i)^* = \varepsilon\chi_1 - \varepsilon \sum_{k=1}^{s} \chi_k - \sum_m n_m' \zeta_m',$$

for integers n_m' and irreducible characters $\{\zeta_m'\}$ different from the $\{\chi_j\}$. However, we also have that

$$(5.4) \qquad \| (\varphi_1 - \textstyle\sum_{i=0}^{q-1} \psi_i)^* \|_G = \| (\varphi_1 - \sum_{i=0}^{q-1} \psi_i) \|_{N(H)} = q + 1.$$

Since (5.3) implies that

$$(5.5) \qquad \| (\varphi_1 - \textstyle\sum_{i=0}^{q-1} \psi_i)^* \|_G \geq s - 1,$$

(5.1), (5.4) and (5.5) together give the inequality

$$q + 1 \geq 2q + 2,$$

which is clearly impossible. Thus (5.2) must hold.

Now, for each i, $\psi_i \in B_0(N(H);p)$ for all $p \in \pi$. Since for each such prime p some $\varphi_j \in B_0(N(H);p)$ also, a result of Wong [15, Theorems 6 and 7] implies that each constituent of $(\varphi_j - \sum_{i=0}^{q-1} \psi_i)^*$ lies in $B_0(G;p)$. Thus each character ζ_l lies in $B_0(G;p)$ and so, by Lemma 5.1, must be constant on each p-section of G, for all $p \in \pi$. By assumption, $|\pi| \geq 2$. Thus "lying in the same p-section for some $p \in \pi$" generates an equivalence relation under which all non-identity elements of H are equivalent. Hence

$$(5.6) \qquad \zeta_l(g) = d_l, \qquad l = 0, \cdots, r,$$

for all π-singular elements $g \in G$. In particular, each d_l is real. By (33.3) of [9], for any j,

$$(5.7) \qquad ((\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i)^*, \eta)_G = ((\varphi_j - \sum_{i=0}^{q-1} \psi_i), \eta\,|_{N(H)})_{N(H)}$$

for any irreducible character η of G which is constant on π-sections. In particular, we may put $\eta = 1$ to obtain a multiplicity of -1: so put $\zeta_0 = 1$

and $n_0 = 1$. For $\eta = \zeta_l$,

$$\begin{aligned} n_l &= -((\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i)^*, \zeta_l)_G \\ &= -((\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i), \zeta_l|_{N(H)})_{N(H)} \\ &= -d_l((\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i), 1_{N(H)})_{N(H)} \end{aligned}$$

since $\zeta_l(x) = d_l$ for all elements x of $N(H)$ for which

$$(\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i)(x) \neq 0.$$

Hence for each $l = 1, \cdots, r$,

$$n_l = d_l : \tag{5.8}$$

as a consequence, each d_l is a rational integer.

Since $\varphi_j \in B_0(N(H);p)$ implies that the corresponding $\chi_j \in B_0(G;p)$, and, by (5.2), all the characters χ_j have the same degree, no character χ_j can vanish on all p-singular elements. Also from (5.2), χ_j is constant on p-sections of G for $p \in \pi$ if and only if φ_j is constant on p-sections of $N(H)$. Thus, by Lemma 5.1, $\chi_j \in B_0(G;p)$ if and only if $\varphi_j \in B_0(N(H);p)$: that is, if and only if $\varphi_j(h) = q$ for all p-regular elements of H.

For any $p \in \pi$, if P is the Sylow p-subgroup of H and $\{\tilde{\psi}_i\}$, $\{\tilde{\varphi}_j\}$ are the characters of $B_0(N(P);p)$ which take the same values as $\{\psi_i\}$ and those $\{\varphi_j\}$ in $B_0(N(H);p)$ on H, then

$$(\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i)^* = (\tilde{\varphi}_j - \textstyle\sum_{i=0}^{q-1} \tilde{\psi}_i)^*$$

where $*$ also denotes the Dade isometry for $N(P)$: that it may be applied follows from Proposition 2.2. If (5.7) is applied with P in place of H, and η a character of $B_0(G;p)$ which takes the same value (necessarily non-zero) on all p-singular elements of G, then

$$((\tilde{\varphi}_j - \textstyle\sum_{i=0}^{q-1} \tilde{\psi}_i)^*, \eta)_G \neq 0 \qquad \text{(cf. derivation of (5.8))}.$$

By (5.2), if $|P| > q + 1$, no distinct χ_j and χ_k in $B_0(G;p)$ can be equal on all p-singular elements: thus such a character η is some ζ_l (or χ_j if $|P| = q + 1$). Hence, by Dade's theorem on cyclic defect groups [7], if $q \geq 5$ and P is chosen as a cyclic Sylow subgroup of H, we see that $r = (q - 1)$ and

$$n_l = d_l = \delta_l = \pm 1, \qquad l = 0, \cdots, (q-1). \tag{5.9}$$

If $q \leq 4$, since

$$\| (\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i)^* \|_G = q + 1 \quad \text{and} \quad (\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i)^*(1) = 0,$$

the same conclusion holds since $\zeta_0 = 1$.

It remains, therefore, only to show that the set of characters $\{\chi_j\}$ take the values prescribed in the theorem, and that the complete set of characters $B(H)$ contains $B_0(G;p)$ for each $p \in \pi$. From (5.2) and (5.9),

$$\varepsilon\chi_j = (\varphi_j - \textstyle\sum_{i=0}^{q-1} \psi_i)^* + \textstyle\sum_{l=0}^{q-1} \delta_l \zeta_l :$$

hence for a π-singular element $g \in S_\pi(x)$, $x \in H^\#$,

$$\varepsilon\chi_j(g) = \varphi_j(x) - \sum_{i=0}^{q-1} \psi_i(x) + q = \varphi_j(x),$$

and for a π-regular element $y \in G$,

$$\varepsilon\chi_j(y) = \sum_{l=0}^{q-1} \delta_l \zeta_l(y) = 1 + \sum_{l=1}^{q-1} \delta_l \zeta_l(y),$$

as required. Finally, since for any $p \in \pi$ and any non-identity p-element $x \in H^\#$,

$$\sum_{l=0}^{q-1} \zeta_l(x)^2 + \sum_{j \in J_p} | \chi_j(x) |^2 = q + \sum_{j \in J_p} | \varphi_j(x) |^2 = | P |,$$

where P is a Sylow p-subgroup of H, and the set J_p indexes those $\chi_j \in B_0(G;p)$ and $\varphi_j \in B_0(N(H);p)$, and since the Cartan matrix of $B_0(C(x);p)$ is just $(| P |)$, $B_0(G;p)$ can contain no further characters.

As a special case, we have:

COROLLARY 5.2.1. *If* $q = 2$, $\varepsilon = \delta_1$.

By restricting each character to H, and considering the multiplicity of the principal character, we obtain the following:

COROLLARY 5.2.2. *For each* $l = 1, \cdots, (q-1)$,

$$\zeta_l(1) \equiv \delta_l \pmod{| H |},$$

and for each $j = 1, \cdots, s$,

$$\chi_j(1) \equiv q\varepsilon \pmod{| H |}.$$

The inequality (5.1) gives:

COROLLARY 5.2.3. *If* $| \mathcal{E}(H) | \leq 6$, *then H is of prime power order.*

COROLLARY 5.2.4. *Let p be a prime not dividing $| H |$. Suppose that no element of $H^\#$ commutes with any element of order p. Then either the exceptional characters all lie in the same p-block of G, or they lie in p-blocks of defect zero.*

Proof. For any p-singular element $g \in G$, $p \nmid | C(g) |$, so that if p^a is the full power of p dividing $| G |$, $p^a | [G:C(g)]$. All exceptional characters have the same degree: if the degree is divisible by p^a, all lie in blocks of defect zero. Otherwise

$$| G |/| C(g) | \cdot \chi_j(g)/\chi_j(1) \equiv 0 \bmod \mathfrak{p} \quad (j = 1, \cdots, s)$$

where g is any π-singular element, and $\mathfrak{p}$ is a prime ideal containing p in a suitable algebraic number field. Since the exceptional characters take the same values on π-regular elements, they must all lie in the same p-block of G [5, 85.12].

COROLLARY 5.2.5. *Suppose that G is a simple group with a subgroup H satisfying the hypothesis of Theorem 5.2 with $q = 2$. Then if $p \in \pi$ and g is* *any non-identity p-element of G, every involution of G is conjugate to an involution in $C^*(g)$ lying outside $C(g)$.*

Proof. If y is an involution of G which does not satisfy the conclusion, g can never be inverted by a conjugate of y. Thus, by Corollary 1 to Proposition 4 of [2],

$$\textstyle\sum_{\chi\epsilon B_0(p)} \chi(g)\chi(y)^2/\chi(1) = 0.$$

Hence, with J_p the indexing set as in the proof of the theorem,

$$1 + \frac{\delta\zeta_1(y)^2}{\zeta_1(1)} + \sum_{j\epsilon J_p} \frac{\delta\varphi_j(g)\cdot\chi_j(y)^2}{\chi_j(1)} = 0 \quad (\delta = \delta_1),$$

so that

$$1 + \frac{\delta\zeta_1(y)^2}{\zeta_1(1)} + \frac{\delta(\zeta_1(y) + \delta)^2}{\zeta_1(1) + \delta} \sum_{j\epsilon J_p} \varphi_j(g) = 0.$$

Evaluating $\sum \varphi_j(g)$ in $N(H)$, we obtain the equation

$$1 + \frac{\delta\zeta_1(y)^2}{\zeta_1(1)} - \frac{\delta(\zeta_1(y) + \delta)^2}{\zeta_1(1) + \delta} = 0,$$

or

$$(\zeta_1(1) - \zeta_1(y))^2/\zeta_1(1)(\zeta_1(1) + \delta) = 0.$$

Thus $\zeta_1(y) = \zeta_1(1)$, which is impossible in a simple group since ζ_1 is non-principal: so the corollary holds.

Suppose now that a group G contains a Hall π_1-subgroup H_1 and a Hall π_2-subgroup H_2, each satisfying the hypothesis of Theorem 5.2 with $q = q_1$ and $q = q_2$ respectively, with $\pi_1 \cap \pi_2 = \emptyset$, and with no element of order $p_1 p_2$ in G for any $p_1 \epsilon \pi_1$ and $p_2 \epsilon \pi_2$. Then a result of Brauer and Tuan applies [4, (4.12)]: namely, for any p_1-singular element x and any p_2-singular element y,

(5.10) $$\textstyle\sum_{\chi\epsilon B_0(p_1)\cap B_0(p_2)} \chi(x)\chi(y) = 0, \qquad p_1 \epsilon \pi_1, \quad p_2 \epsilon \pi_2.$$

In particular, $B_0(p_1) \cap B_0(p_2)$ cannot consist of the principal character alone. To apply this formula, we first make the following observation.

PROPOSITION 5.3. *$\mathcal{E}(H_1) \cap \mathcal{E}(H_2)$ is empty.*

Proof. Suppose otherwise, and that η is a character of G in the intersection. If χ_1 is another exceptional character for H_1, and χ_2 for H_2 (χ_1 and χ_2 not necessarily distinct), then

$$\eta(g) = \chi_2(g)$$

for all π_1-singular elements $g \epsilon G$ since no element of G can be both π_1-singular and π_2-singular. Since $(\eta - \chi_1)$ vanishes on all π_1-regular elements of G,

$$(\eta - \chi_1, \chi_2)_G = (\eta - \chi_1, \eta)_G = 1:$$

this is clearly impossible since η, χ_1 and χ_2 are irreducible characters of G, and $\eta \neq \chi_2$.

Now if the number of exceptional characters for H_1 exceeds q_2 (or indeed $(q_2 - 1)$ if $|H_2| > (q_2 + 1)$), no such character can lie in $B(H_2)$, and similarly for the exceptional characters for H_2. Thus if q_1 and q_2 are given explicitly, there can be only a finite number of exceptions to what will be called

The natural situation. $B(H_1) \cap B(H_2)$ consists of characters which are non-exceptional for both H_1 and H_2.

By examining the possible intersections, the following is easily verified.

PROPOSITION 5.4. *Suppose that the natural situation occurs. If either q^1 or q_2 does not exceed 3, $B(H_1) \cap B(H_2)$ consists of the principal character and one other, say ζ_1. For $g_i \in S_{\pi_i}(x_i)$, $x_i \in H_i^\#$, $i = 1, 2$,*

$$\zeta_1(g_1) = -\zeta_1(g_2).$$

If one of q_1 or q_2 is 4, and the other not less, $B(H_1) \cap B(H_2)$ will consist of either two or four characters.

The remaining situations where an exceptional character for one Hall subgroup is non-exceptional for the other will be called *unnatural.* More precisely, we say that an *unnatural situation* occurs between H_1 and H_2 if some exceptional character for H_1 lies in $B(H_2)$. It should be noted that, by Theorem 5.2 and Corollary 5.2.4, this implies that $\mathcal{E}(H_1) \subseteq B(H_2)$, and also that the definition is unsymmetrical in H_1 and H_2. Indeed, the group $PSL(2, 9)$ is an example of where this lack of symmetry is necessary, taking $|H_1| = 5$ and $|H_2| = 9$. The definition does not preclude the possibility that some non-principal non-exceptional character for H_1 lies in $B(H_2)$, either as an exceptional or as a non-exceptional character.

Suppose that an unnatural situation occurs between H_1 and H_2. In general one can say little, except to point out that all the characters in $\mathcal{E}(H_1)$ will take the same value $\delta = \pm 1$ on all π_2-singular elements: hence, for any $g_1 \in H_1^\#$ and $g_2 \in H_2^\#$,

$$\sum_{\chi \in \mathcal{E}(H_1)} \chi(g_1)\chi(g_2) = \pm 1. \tag{5.11}$$

Also, if $|H_2| = (q_2 + 1)$ and $|\mathcal{E}(H_1)| = q_2$, by Theorem 5.2

$$1 + \sum_{\chi \in \mathcal{E}(H_1)} \chi(1)\chi(g_2) = 0, \qquad g_2 \in H_2^\#,$$

which is impossible since all characters in $\mathcal{E}(H_1)$ take the same values on 1 and also on g_2. Thus we have:

PROPOSITION 5.5. *If an unnatural situation occurs between H_1 and H_2, then $|\mathcal{E}(H_1)| < a(H_2)$. In particular, $a(H_2) \geq 3$.*

If a bound or an explicit value is known for q_2, then information can be obtained about H_1 since $|\,\mathcal{E}(H_1)\,| < q_2$. In particular, if a bound for q_1 is known, then $|\,H_1\,|$ will be bounded. More can be said if $q_2 \leq 7$: then by Corollary 5.2.3, H_1 is of prime power order. So far we have used only the existence of characters as given by Theorem 5.2: using the full strength of the hypothesis on $N(H_1)$, it can be seen that with only a finite number of exceptions, $|\,H_1\,|$ is prime. We shall be concerned with small values for both q_1 and q_2, and will need part of the following result.

PROPOSITION 5.6. *With the situation as above, suppose that q_1 and q_2 are at most* 4. *Then $B(H_1) \cap B(H_2) = \mathcal{E}(H_1) \cup \{\zeta_0\}$, except possibly if $q_2 = 4$ and $|\,H_2\,| = 5$, where ζ_0 is the principal character. Also H_1 has prime power order.*

Proof. That H_1 has prime power order is shown above. Using the Brauer-Tuan formula (5.10) and (5.11), it can be seen that if $\mathcal{E}(H_2) \neq \emptyset$, then $\mathcal{E}\,(H_2) \cap B(H_1) = \emptyset$ since H_1 and H_2 are of coprime orders. Thus if

$$B(H_1) \cap B(H_2) \neq \mathcal{E}(H_1) \cup \{\zeta_0\},$$

the other characters in $B(H_1) \cap B(H_2)$ must be non-exceptional for both H_1 and H_2. By (5.10) and (5.11), there must be at least two such characters so that $B(H_2)$ has at least five characters non-exceptional for H_2. Thus $q_2 = 4$ and $|\,H_2\,| = 5$.

In the situations to which these techniques will be applied, we shall have a group G with a set of Hall subgroups $H_1, \cdots, H_n$, any pair of which satisfy these conditions, and a bound for $a(H_i)$, $i = 1, \cdots, n$. The Brauer-Tuan formula bounds the number of Hall subgroups which do not enter into unnatural situations as the first partner: the number of Hall subgroups remaining is also bounded since the order of each is bounded.

6. The case (A_2)

Let G be a simple group satisfying the hypothesis (A_2). Then by Lemma 4.2 and its corollary, G has a set of maximal abelian Hall π_i-subgroups $H_1, \cdots, H_n$ of coprime orders with $n \geq 2$, each of which satisfies the hypothesis of Theorem 5.2 with $q = 2$, and to any pair of which the discussion of the last section applies. Then by Proposition 5.5, any pair give rise to a natural situation. Thus, if $i \neq j$, $B(H_i) \cap B(H_j)$ consists precisely of the principal character and a non-principal non-exceptional character in $B(H_i)$ and in $B(H_j)$, uniquely determined except in the case of a Hall subgroup of order 3. This gives a very tight restriction on G.

LEMMA 6.1. *G has exactly two classes of maximal Hall subgroups of odd order.*

Proof. Suppose otherwise, namely that $n \geq 3$. Then we may suppose that $|\,H_1\,| > 3$ and $|\,H_2\,| > 3$. Let $\zeta_0 = 1$ and let ζ_1 be the common non-principal character of $B(H_1)$ and $B(H_2)$. For $x_1 \in H_1^{\#}$ and $x_2 \in H_2^{\#}$,

$$\zeta_1(x_1) = -\zeta_1(x_2) = \pm 1$$

TABLE 1

	1	$g_1 \in S_{\pi_1}(x_1)$; $x_1 \in H_1^{\#}$	$g_2 \in S_{\pi_2}(x_2)$; $x_2 \in H_2^{\#}$	2-element x
ζ_0	1	1	1	1
ζ_1	$\zeta_1(1)$	1	-1	$\zeta_1(x)$
χ_1 $\vdots$ χ_s	$\zeta_1(1) + 1$	$\varphi_i(x_1)$	0	$\zeta_1(x) + 1$
η_1 $\vdots$ η_t	$\zeta_1(1) - 1$	0	$-\psi_j(x_2)$	$\zeta_1(x) - 1$

by Proposition 5.4. If H_3 is a third Hall subgroup,

$$B(H_1) \cap B(H_3) = B(H_2) \cap B(H_3) = \{\zeta_0, \zeta_1\}.$$

Since for $x_3 \in H_3^{\#}$ we must have

$$\zeta_1(x_1) = -\zeta_1(x_3) \quad \text{and} \quad \zeta_1(x_2) = -\zeta_1(x_3),$$

no such third Hall subgroup can exist.

Thus G has two non-conjugate Hall subgroups H_1 and H_2. Again let

$$B(H_1) \cap B(H_2) = \{\zeta_0, \zeta_1\},$$

and let the remaining characters of $B(H_1)$ be $\{\chi_1, \cdots, \chi_s\}$ and of $B(H_2)$ be $\{\eta_1, \cdots, \eta_t\}$. Without loss, it may be assumed that, for $x_1 \in H_1^{\#}$ and $x_2 \in H_2^{\#}$,

$$\zeta_1(x_1) = -\zeta_1(x_2) = +1.$$

Since any odd prime divisor of $|G|$ lies in either π_1 or π_2, any non-identity element of G which is both π_1-regular and π_2-regular must be a 2-element. Then Theorem 5.2 yields the part of the character table of G shown in Table 1. Here $s = \frac{1}{2}(|H_1| - 1)$, $\varphi_1, \cdots, \varphi_s$ are the non-linear characters of $N(H_1)$ that appear in the statement of Theorem 5.2, and t, $\psi_1, \cdots, \psi_t$ are similarly defined for H_2.

Let $|G| = 2^a \cdot |H_1| \cdot |H_2|$.

LEMMA 6.2. *For some positive integer n, $\zeta_1(1) = 2^n$, $|H_1| = 2^n - 1$, and $|H_2| = 2^n + 1$.*

Proof. By Corollary 5.2.2,

$$\zeta_1(1) \equiv 1 \pmod{|H_1|} \quad \text{and} \quad \zeta_1(1) \equiv -1 \pmod{|H_2|}:$$

since $\zeta_1(1) \mid |G|$, it follows that $\zeta_1(1) = 2^n$ for some positive integer $n \leq a$.

For each $i = 1, \cdots, s$, the character χ_i vanishes on all π_2-singular elements: thus

$$(1_{H_2}, \chi_i|_{H_2})_{H_2} = \chi_i(1)/|H_2|.$$

Hence $\chi_i(1)$ is divisible by $|H_2|$. On the other hand, since $\chi_i(1) = \zeta_1(1) + 1$, $\chi_i(1)$ is odd. Also, by Corollary 5.2.2,

$$\chi_i(1) \equiv 2 \pmod{|H_1|}.$$

Thus

$$\chi_i(1) = |H_2| = 2^n + 1.$$

Similarly, $|H_1|$ divides $\eta_j(1)$, $j = 1, \cdots, t$, and

$$\eta_j(1) \equiv -2 \pmod{|H_2|}:$$

hence

$$\eta_j(1) = |H_1| = 2^n - 1.$$

LEMMA 6.3. *G is a CIT-group.*

Proof. Since G has an irreducible character of degree $|H_1|$, $C_G(H_1)$ has odd order by Proposition 2.3. Thus, by Lemma 4.2, $N_G(H_1) = H_1\langle x\rangle$ for some involution x, and $C_{H_1}(x) = 1$. Also, if g is a non-identity element of H_1 of prime power order, $C^*(g) = O_2(C(g))\cdot H_1\langle x\rangle$. By Proposition 2.4, all involutions in $C^*(g)$ lying outside $C(g)$ $(= O_2(C(g))\cdot H_1)$ are conjugate. Any involution of G is conjugate to such an involution by Corollary 5.2.5: thus all involutions in G are conjugate.

Let y be any involution in G. Then $y \in Z(S)$ for some Sylow 2-subgroup S of G. Let $h_y = |G|/|C(y)|$. Then $h_y \mid (2^n + 1)(2^n - 1)$. Since $h_y\zeta_1(y)/\zeta_1(1)$ must be an algebraic integer, $2^n \mid \zeta_1(y)$. Thus, as G is simple, $\zeta_1(y) = 0$. The same observation applied to the characters χ_1 and η_1 now shows that $(2^n + 1) \mid h_y$ and that $(2^n - 1) \mid h_y$. Thus $|C(y)| = 2^a$.

The additional information that is available now permits a short proof of the isomorphism of G with $SL(2, 2^a)$.

LEMMA 6.4. $|G| = 2^a(2^a + 1)(2^a - 1)$.

Proof. For any $g \in H_1^\#$, we now have $C_G(g) = H_1$. Since

$$\zeta_0(g)^2 + \zeta_1(g)^2 + \textstyle\sum_{i=1}^{s}|\chi_i(g)|^2 = |H_1|,$$

all other characters of G vanish on g. Thus we may compute the number of ways in which g may be expressed as the product of two involutions to obtain the equation

$$\frac{|G|}{|C(y)|^2}\left(1 + \sum_{i=1}^{s}\frac{\varphi_i(g)}{2^n + 1}\right) = |C(g)| = 2^n - 1,$$

where y is an involution, since the previous lemma showed that $\zeta_1(y) = 0$. Thus

$$2^a(2^n - 1)(2^n + 1)(1 - 1/(2^n + 1)) = 2^{2a}(2^n - 1):$$

hence

$$2^a = 2^n.$$

COROLLARY. *A Sylow 2-subgroup S of G is elementary abelian, and*

$$|N(S)| = 2^a(2^a - 1).$$

Proof. Since

$$1 + \zeta_1(1)^2 + \sum_{i=1}^{s} \chi_i(1)^2 + \sum_{j=1}^{t} \eta_j(1)^2 = 2^a(2^{2a} - 1),$$

G can have no further characters, and so can have only one class of 2-elements. Thus S is elementary abelian, all involutions must be conjugate in $N(S)$, and so $|N(S)| = 2^a(2^a - 1)$.

We now obtain the isomorphism by letting G act as a permutation group on the conjugates of S (cf. [8]). Since there are $(2^a + 1)$ conjugates, G must be doubly transitive since clearly no two Sylow 2-subgroups can intersect non-trivially. The corresponding permutation character must be $(\zeta_0 + \zeta_1)$ so that no non-identity element fixes more than two points. Thus G is a Zassenhaus group. Furthermore, the group order shows that G must be sharply triply transitive, and the fundamental classification of such groups by Zassenhaus [16] establishes the isomorphism.

7. The case $a(H) = 3$ for some H

Let $H_1, \cdots, H_n$ be representatives of the classes of maximal abelian Hall subgroups of odd order as given by Lemma 4.2. Then, in view of the last section, we may assume that $a(H) = 3$ for one of them. Thus another, H', has order divisible by 3 so that $a(H') = 2$. Thus this condition holds for at least one of the Hall subgroups $H_1, \cdots, H_n$. For the remainder of the proof, all references will be to this set of n subgroups.

LEMMA 7.1. *For no pair of Hall subgroups can an unnatural situation arise.*

Proof. Suppose that one does, and that H_1 and H_2 are such that $\mathcal{E}(H_1) \subseteq B(H_2)$. By Proposition 5.5, $a(H_2) = 3$ and $|\mathcal{E}(H_1)| = 2$. Thus either $|H_1| = 5$ or $|H_1| = 7$, so that 3 divides neither $|H_1|$ nor $|H_2|$. Let H_3 be the Hall subgroup whose order is divisible by 3. Then $a(H_3) = 2$, and natural situations must occur both between H_1 and H_3, and between H_2 and H_3. Now some non-principal non-exceptional character in $B(H_2)$ lies in $B(H_3)$: however, this character lies in $\mathcal{E}(H_1)$, contradicting the assertion of a natural situation between H_1 and H_3.

LEMMA 7.2. *Precisely one of the Hall subgroups satisfies $a(H) = 2$.*

Proof. Certainly there is at least one, namely that of order divisible by 3. Let this be H_1, and suppose that H_2 is a second Hall subgroup with $a(H_2) = 2$, and H_3 one with $a(H_3) = 3$. Then $|H_2| \neq 3$ so that $B(H_2)$ contains a unique non-principal non-exceptional character ζ_1. Since no unnatural situations occur,

$$\zeta_1 \in B(H_1) \cap B(H_2) \quad \text{and} \quad \zeta_1 \in B(H_2) \cap B(H_3).$$

Furthermore, by Proposition 5.4,

$$B(H_1) \cap B(H_2) = B(H_2) \cap B(H_3) = B(H_3) \cap B(H_1) = \{\zeta_0, \zeta_1\}$$

where $\zeta_0 = 1$, and for $x_i \in H_i^{\#}$, $i = 1, 2, 3$,

$$\zeta_1(x_1) = -\zeta_1(x_2) = \pm 1,$$
$$\zeta_1(x_2) = -\zeta_1(x_3) = \pm 1,$$
$$\zeta_1(x_3) = -\zeta_1(x_1) = \pm 1,$$

a set of equations impossible to satisfy. Thus H_1 is the only Hall subgroup with $a(H) = 2$.

For the remainder of the proof, H_1 will denote this Hall subgroup.

LEMMA 7.3. *Precisely one of the Hall subgroups satisfies $a(H) = 3$.*

Proof. Suppose that there is more than one such Hall subgroup. If $|H_1| > 3$, there is a unique non-principal non-exceptional character in $B(H_1)$, and a contradiction will occur exactly as in the previous lemma. Thus we may assume that $|H_1| = 3$. Let $B(H_1) = \{\zeta_0, \zeta_1, \zeta_2\}$ where $\zeta_0 = 1$. The same argument shows that if H_i and H_j are distinct Hall subgroups, different from H_1,

$$B(H_1) \cap B(H_i) \cap B(H_j) = \{\zeta_0\}.$$

Thus there are at most two Hall subgroups other than H_1. Suppose that there are two, H_2 and H_3, that

$$B(H_1) \cap B(H_2) = \{\zeta_0, \zeta_1\}, \qquad B(H_1) \cap B(H_3) = \{\zeta_0, \zeta_2\},$$

and that ζ_3 is the non-principal character in $B(H_2) \cap B(H_3)$. Then Theorem 5.2 yields the part of the character table of G shown in Table 2. Here, $\alpha, \beta, \delta, \varepsilon = \pm 1$, α and β so as to make the degrees positive,

$$\{\chi_1, \cdots, \chi_s\} = \mathcal{E}(H_2) \quad \text{and} \quad \{\eta_1, \cdots, \eta_t\} = \mathcal{E}(H_3),$$

and $\{\varphi_i\}$ and $\{\psi_j\}$ are the corresponding characters of $N(H_2)$ and $N(H_3)$ respectively.

By Corollary 5.2.2, $\delta\zeta_1(1) \equiv -\delta\zeta_2(1) \equiv -1 \pmod 3$: thus 3 divides all of $\zeta_3(1)$, $\{\chi_i(1)\}$ and $\{\eta_j(1)\}$, or none. Suppose none: then by Corollaries 4.2.1 and 5.2.4, all the characters $\{\chi_i\}$ lie in the same 3-block, as do all the characters $\{\eta_j\}$. Since either $s > 3$ or $t > 3$ but $9 \nmid |G|$, this is impossible since a 3-block of defect 1 will contain precisely three characters [1, Lemma 6 and Theorem 2]. Thus all the characters ζ_3, $\{\chi_i\}$ and $\{\eta_j\}$ have degree divisible by 3. Furthermore, Corollary 5.2.2 and similar computations yield the following congruences, where $h_2 = |H_2|$ and $h_3 = |H_3|$:

TABLE 2

	1	$g_1 \in S_3(x_1)$; $x_1 \in H_1^{\#}$	$g_2 \in S_{\pi_2}(x_2)$; $x_2 \in H_2^{\#}$	$g_3 \in S_{\pi_3}(x_3)$; $x_3 \in H_3^{\#}$
ζ_0	1	1	1	1
ζ_1	$\zeta_1(1)$	$-\delta$	δ	0
ζ_2	$\zeta_2(1)$	δ	0	$-\delta$
ζ_3	$\zeta_3(1)$	$\zeta_3(g_1)$	ε	$-\varepsilon$
χ_1 $\vdots$ χ_s	$\alpha(1 + \delta\zeta_1(1) + \varepsilon\zeta_3(1))$	$\alpha\varepsilon\zeta_3(g_1)$	$\alpha\varphi_i(x_2)$	0
η_1 $\vdots$ η_t	$\beta(1 - \delta\zeta_2(1) - \varepsilon\zeta_3(1))$	$-\beta\varepsilon\zeta_3(g_1)$	0	$\beta\psi_j(x_3)$

$$\begin{aligned}
\zeta_1(1) &\equiv -\delta \ (3), \quad \delta \ (h_2), \quad 0 \ (h_3);\\
\zeta_2(1) &\equiv \delta \ (3), \quad 0 \ (h_2), \quad -\delta \ (h_3);\\
\zeta_3(1) &\equiv 0 \ (3), \quad \varepsilon \ (h_2), \quad -\varepsilon \ (h_3);\\
\chi_i(1) &\equiv 0 \ (3), \quad 3\alpha \ (h_2), \quad 0 \ (h_3);\\
\eta_j(1) &\equiv 0 \ (3), \quad 0 \ (h_2), \quad 3\beta \ (h_3).
\end{aligned}$$

Also, from Theorem 5.2,

$$1 - \delta\zeta_1(1) + \delta\zeta_2(1) = 0. \tag{7.1}$$

Thus one of ζ_1 and ζ_2 has odd degree, and the other even: we may suppose that $\zeta_1(1)$ is odd. Since H_1, H_2 and H_3 are a complete set of non-conjugate odd Hall subgroups, $|G| = 2^l \cdot 3 \cdot h_2 h_3$. The above congruences now give the following degrees since each must divide $|G|$:

$$\zeta_1(1) = h_3, \qquad \zeta_2(1) = 2^m h_2, \qquad \zeta_3(1) = 2^n \cdot 3,$$
$$\chi_i(1) = 2^a \cdot 3 \cdot h_3 \quad \text{and} \quad \eta_j(1) = 2^b \cdot 3 \cdot h_2,$$

where m, n, a, b are non-negative integers, and $m \geq 1$. Furthermore, since $a(H_3) = 3$, $h_3 \equiv 1 \ (3)$ so that $\delta = -1$.

Suppose first that $n \geq 1$. Then χ_i has even degree and η_j odd, so that $a \geq 1$ and $b = 0$. Thus

$$\alpha(1 - h_3 + \varepsilon \cdot 2^n \cdot 3) = 2^a \cdot 3 \cdot h_3, \qquad a, n \geq 1,$$

and

$$\beta(1 + 2^m h_2 - \varepsilon \cdot 2^n \cdot 3) = 3h_2, \qquad m, n \geq 1.$$

Also, from (7.1),

$$1 + h_3 - 2^m h_2 = 0:$$

in particular,

(7.2) $$h_3 > h_2 .$$

On the other hand, these three equations yield

$$1 - h_3 + \varepsilon \cdot 2^n \cdot 3 = \alpha \cdot 2^a \cdot 3 \cdot h_3 \quad \text{and} \quad 2 + h_3 - \varepsilon \cdot 2^n \cdot 3 = \beta \cdot 3 h_2 :$$

thus

$$3 = 3(\beta h_2 + \alpha \cdot 2^a h_3).$$

Hence

$$\beta h_2 + \alpha \cdot 2^a h_3 = 1,$$

so that, as $a \geq 1$, $h_2 > h_3$, contrary to (7.2).

So we may assume that $\zeta_3(1) = 3$. This time χ_i has odd degree and η_j even so that the degree equation for χ_i is

$$\alpha(1 - h_3 + 3\varepsilon) = 3h_3 .$$

Hence

$$(3 + \alpha)h_3 = \alpha(1 + 3\varepsilon).$$

Since this implies that $h_3 \leq 2$ which is clearly impossible, the lemma is established.

Let H_2 be the Hall subgroup with $a(H_2) = 3$. Let $h_1 = | H_1 |$ and $h_2 = | H_2 |$. Then $| G | = 2^a h_1 h_2$ for some positive integer a. Since no unnatural situation occurs, $B(H_1)$ and $B(H_2)$ give rise to the portion of the character table of G indicated in Table 3. In this table, α, δ, $\varepsilon = \pm 1$ with α such that $\chi_i(1)$ is positive, $\{\chi_1 , \cdots , \chi_s\} = \mathcal{E}(H_2)$, and $\{\theta_1 , \cdots , \theta_r\}$ are the exceptional characters for H_1 if they exist: otherwise $r = 1$ and $h_1 = 3$. The sets

TABLE 3

	1	$g_1 \in S_{\pi_1}(x_1);\ x_1 \in H_1^{\#}$	$g_2 \in S_{\pi_2}(x_2);\ x_2 \in H_2^{\#}$
ζ_0	1	1	1
ζ_1	$\zeta_1(1)$	δ	$-\delta$
ζ_2	$\zeta_2(1)$	$\zeta_2(g_1)$	ε
χ_1 $\vdots$ χ_s	$\alpha(1 - \delta\zeta_1(1) + \varepsilon\zeta_2(1))$	$\alpha\varepsilon\zeta_2(g_1)$	$\alpha\varphi_i(x_2)$
θ_1 $\vdots$ θ_r	$\zeta_1(1) + \delta$	$\delta\omega_j(x_1)$	0

$\{\varphi_i\}$ and $\{\omega_j\}$ are the appropriate non-linear characters of $N(H_2)$ and $N(H_1)$ respectively. It should be noted that the characters ζ_2 and $\{\chi_i\}$ are not necessarily constant on π_1-sections since they do not lie in $B_0(p)$ for any $p \in \pi$.

By Corollary 5.2.2, the following congruences hold:

$$\begin{aligned} \zeta_1(1) &\equiv \delta\ (h_1), \quad -\delta\ (h_2); \\ \zeta_2(1) &\equiv \varepsilon\ (h_2); \\ \chi_i(1) &\equiv 3\alpha\ (h_2); \\ \theta_j(1) &\equiv 2\delta\ (h_1), \quad 0\ (h_2). \end{aligned} \tag{7.3}$$

Thus

$$\zeta_1(1) = 2^n \quad \text{for some positive integer } n, \tag{7.4}$$

and

$$\theta_j(1) = \zeta_1(1) + \delta = h_2 . \tag{7.5}$$

LEMMA 7.4. *Either ζ_2 or all the characters $\{\chi_i\}$ have degree dividing h_1.*

Proof. $\chi_i(1) = \alpha(1 - \delta \cdot 2^n + \varepsilon\zeta_2(1))$: thus one of ζ_2 and χ_i has odd degree. Since $3 \nmid h_2$, both have degree coprime to h_2 by the congruences (7.3) so that one of them satisfies the conclusion of the lemma.

LEMMA 7.5. *All involutions in G are conjugate. G is a CIT-group, and $\zeta_1(1) = 2^a$.*

Proof. By Lemma 7.4 and Proposition 2.3, no involution can centralise H_1 : thus, as in the proof of Lemma 6.3, all involutions in G are conjugate.

We claim that for some element $g \in H_1$, $C_G(g) = H_1$. For suppose otherwise. Then every element of H_1 has centraliser of even order so that an involution y of G centralises a conjugate of every cyclic subgroup of H_1. Each such cyclic subgroup is normal in $N(H_1)$: thus, by Proposition 2.2, y centralises some conjugate of H_1, contrary to the assertion at the beginning of this proof. Now, for such an element g,

$$1 + \delta^2 + \textstyle\sum_{j=1}^{r} (\omega_j(g))^2 = h_1 = |\, C_G(g)\, |,$$

so that any character of G outside $B(H_1)$ vanishes on g. Thus we may compute the number of ways in which g may be written as the product of two involutions since this is just h_1, as $C^*(g) = N(H_1)$, to obtain the equation

$$\frac{|\,G\,|}{|\,C(y)\,|^2} \cdot \left(1 + \frac{\delta\zeta_1(y)^2}{\zeta_1(1)} + \frac{(\zeta_1(y) + \delta)^2}{\zeta_1(1) + \delta} \sum_{j=1}^{r} \omega_j(g)\right) = h_1,$$

where y is any involution. Putting $|\, C(y)\, | = 2^a h_1^* h_2^*$ and using (7.4) and (7.5), this equation reduces to

$$\frac{2^a h_1 h_2}{2^{2a} h_1^{*2} h_2^{*2}} \cdot \frac{(2^n - \zeta_1(y))^2}{2^n h_2} = h_1,$$

or

$$(2^n - \zeta_1(y))^2 = 2^{a+n} h_1^{*2} h_2^{*2}. \tag{7.6}$$

Since G is simple and ζ_1 is non-linear, $| \zeta_1(y) | < 2^n$. Hence $| 2^n - \zeta_1(y) | < 2^{n+1}$ and, in particular, 2^{2n+1} does not divide $(2^n - \zeta_1(y))^2$. Thus the only solution to equation (7.6), since $a \geq n$, is given by

$$\zeta_1(y) = 0, \qquad a = n, \qquad h_1^* = h_2^* = 1:$$

in particular, G is a CIT-group, and $\zeta_1(1) = 2^a$.

As in Section 6, an appeal to Suzuki's classification would complete the proof, but we already have a very tight arithmetic grip that allows an easy conclusion. As an immediate consequence of the last lemma, we have

COROLLARY. *ζ_1 is a character of 2-defect zero.*

Thus ζ_1 vanishes on all elements of G not conjugate to an element of either H_1 or H_2. Using the orthogonality relation $\sum_{g \in G} | \zeta_1(g) |^2 = | G |$, it follows that

$$2^{2a} + (h_1 - 1) | G |/2h_1 + (h_2 - 1) | G |/3h_2 = | G |.$$

Hence

$$2^{2a} + 2^{a-1} h_2(h_1 - 1) + 2^a h_1(h_2 - 1)/3 = 2^a h_1 h_2,$$

or

$$2^{a+1} - h_2 - \tfrac{2}{3} h_1 = \tfrac{1}{3} h_1 h_2 .$$

Since $h_2 = 2^a + \delta$ by (7.5),

$$2^a - \delta = \tfrac{1}{3} h_1(2^a + \delta + 2).$$

Thus

$$\delta = -1 \quad \text{and} \quad h_1 = 3. \tag{7.7}$$

From the congruences (7.3) and Lemma 7.4, it follows that $\chi_i(1) = 3$ and $\alpha = 1$ so that the degree equation for χ_i becomes

$$1 + 2^a + \varepsilon \zeta_2(1) = 3.$$

Thus $\varepsilon = -1$. Again by (7.3),

$$2^a + 1 = \zeta_1(1) - \delta \equiv 0 \pmod 3:$$

hence $\zeta_2(1) = 2^m \cdot 3$ for some positive integer m. Then

$$2^a + 1 = 3(2^m + 1).$$

Thus $a = 3$ so that $| G | = 168$, and hence G is isomorphic to $PSL(2, 7)$.

REFERENCES

1. R. BRAUER, *On groups whose order is divisible by prime number to the first power I*, Amer. J. Math., vol. 64 (1942), pp. 401–420.

2. ———, *Some applications of the theory of blocks of characters of finite groups II*, J. Algebra, vol. 1 (1964), pp. 307–334.
3. ———, *Some applications of the theory of blocks of characters of finite groups III*, J. Algebra, vol. 3 (1966), pp. 225–255.
4. R. BRAUER AND H. F. TUAN, *On simple groups of finite order I*, Bull. Amer. Math. Soc., vol. 51 (1945), pp. 756–766.
5. C. W. CURTIS AND I. REINER, *Representation theory of finite groups and associative algebras*, Interscience, New York, 1962.
6. E. C. DADE, *Lifting group characters*, Ann. of Math., vol. 79 (1964), pp. 590–596.
7. ———, *Blocks with cyclic defect groups*, Ann. of Math., vol. 84 (1966), pp. 20–48.
8. W. FEIT, *A characterization of the simple groups $SL(2,2^a)$*, Amer. J. Math., vol. 82 (1960), pp. 281–300.
9. ———, *Characters of finite groups*, Benjamin, New York, 1967.
10. D. GORENSTEIN, *Finite groups*, Harper and Row, New York, 1968.
11. I. SCHUR, *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math., vol. 132 (1907), pp. 85–137.
12. M. SUZUKI, *Finite groups with nilpotent centralizers*, Trans. Amer. Math. Soc., vol. 99 (1961), pp. 425–470.
13. ———, *On a class of doubly transitive groups*, Ann. of Math., vol. 75 (1962), pp. 105–145.
14. H. WIELANDT, *Zum Satz von Sylow*, Math. Zeitschr., vol. 60 (1954), pp. 407–408.
15. W. J. WONG, *Exceptional character theory and the theory of blocks*, Math. Zeitschr., vol. 91 (1966), pp. 363–379.
16. H. ZASSENHAUS, *Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen*, Abh. Math. Sem. Univ. Hamburg, vol. 11 (1936), pp. 17–40.

UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE,
CHICAGO, ILLINOIS
UNIVERSITY COLLEGE
OXFORD, ENGLAND