ON THE ZEROS OF RIESZ' FUNCTION IN THE ANALYTIC THEORY OF NUMBERS

BY

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In a classical paper [1] M. Riesz introduced the entire function

$$F(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{(n-1)!\zeta(2n)} \tag{1}$$

and showed that a necessary and sufficient condition for the truth of Riemann's hypothesis is that for each $\varepsilon > 0$

$$F(x) = O(x^{1/4+\varepsilon}) \qquad (x \to +\infty). \tag{2}$$

Riesz also showed that $F(z)$ is of order one, type one, genus one, has infinitely many zeros off the real axis, at least one on the real axis, has none in the left half-plane and satisfies

$$\sum_{n=1}^{\infty} F(z/n^2) = ze^{-z}, \tag{3}$$

$$F(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} ze^{-z/n^2} \tag{4}$$

for all z.

In this note we prove certain additional properties of the set of zeros of $F(z)$. Let $\{r_n e^{i\theta_n}\}_1^{\infty}$ denote some arrangement of these zeros in nondecreasing order of modulus, let $x_1, x_2, \cdots$ denote the subsequence of positive real zeros of $F(z)$, and let $h(r, \delta)$ denote the number of zeros in the sector

$$|z| \leqq r, \qquad |\arg z| \leqq \tfrac{1}{2}\pi - \delta \qquad (\delta > 0).$$

Then we show that

$$r_n \sim n\pi \qquad (n \to \infty), \tag{5}$$

$$h(r, \delta) = o(r) \qquad (r \to \infty), \tag{6}$$

$$\sum_{n=1}^{\infty} x_n^{-1} < \infty. \tag{7}$$

(8) There are infinitely many x_n and in fact

$$\sum_{x_n < x} 1 = \Omega(\log x) \qquad (x \to \infty).$$

The relations (5)–(7) depend hardly at all on the nature of the coefficients $\mu(n)$ in (4) whereas (8) depends on very specific properties of these coefficients.

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To prove these assertions we consider an entire function which is represented by a Dirichlet series with bounded exponents,

(9) $$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}$$

where

(10) $$-\infty < \lambda = \inf \{\lambda_n\} \leqq \sup \{\lambda_n\} = \Lambda < \infty$$

and

(11) $$\sum | a_n | < \infty.$$

We adopt the convention that $a_n \neq 0$ $(n = 0, 1, 2, \cdots)$. The Borel transform of $f(z)$ is

$$\Phi(\zeta) = \int_0^{\infty} e^{-\zeta t} f(t)\, dt = \sum_{n=1}^{\infty} a_n/(\zeta - \lambda_n).$$

The indicator diagram of $f(z)$ is therefore the interval $[\lambda, \Lambda]$ of the real axis and its indicator function is

(12) $$h(\theta) = \max (\lambda \cos \theta, \Lambda \cos \theta).$$

Further it is clear that $f(z)$ is bounded on the imaginary axis. It follows then from a theorem of Cartwright [2, p. 87] that

(13) $$\sum_{r_n < r} 1 \sim \frac{\Lambda - \lambda}{\pi} r \qquad (r \to \infty),$$

(14) $$\sum_{n=1}^{\infty} \frac{\cos \theta_n}{r_n} < \infty.$$

We remark that if both the numbers λ, Λ actually appear among the λ_n then it is known from the theory of almost periodic functions [3] that the zeros of $f(z)$ lie in a vertical strip of finite width and so (13) could be replaced by an estimate in terms of the ordinates instead of the moduli of the zeros.

In the present case $f(z) = F(z)/z$, $\lambda_n = -n^{-2}$, $\lambda = -1$, $\Lambda = 0$ and (5) follows from (13) while (6) and (7) follow from (14).

The assertion (8) is an easy consequence of a beautiful theorem of Pólya [4] who proved (sharpening an earlier result of Landau) that if the function $\Phi(s)$ represented by the integral

$$\Phi(s) = \int_1^{\infty} \omega(u) u^{-s}\, du$$

is regular in $\operatorname{Re} s > \Theta$ say, but in no half-plane $\operatorname{Re} s \geqq \Theta - \varepsilon$ and is meromorphic in $\operatorname{Re} s \geqq \Theta - b$ for some $b > 0$, then

$$\limsup_{x \to \infty} W(x)/\log x \geqq \gamma/\pi$$

where $W(x)$ is the number of changes of sign of $\omega(u)$ on $(1, x)$ and γ is the ordinate of the singularity of $\Phi(s)$ on the line $\operatorname{Re} s = \Theta$ of smallest imaginary part (or $+\infty$ if no such singularity exists).

In the case of the Riesz function we have

$$\int_0^\infty F(x)x^{-s}\,dx = -\Gamma(2-s)/\zeta(2s-2)$$

and so

$$\Phi(s) = \int_1^\infty F(x)x^{-s}\,dx = -\Gamma(2-s)/\zeta(2s-2) + R(s).$$

The trivial estimate (see [5, page 260, ex. 4] or [1])

$$F(x) = O(x^{1/2+\varepsilon}) \qquad (x \to +\infty)$$

shows that $\Phi(s)$ is regular for $\operatorname{Re} s > \frac{3}{2}$, and since $R(s)$ is regular for $\operatorname{Re} s < 2$, $\Phi(s)$ is meromorphic in the plane with singularities only at the zeros of $\zeta(2s-2)$.

On the Riemann hypothesis we could take $\Theta = \frac{5}{4}$ in Pólya's theorem and 2γ the ordinate of the first zero of $\zeta(s)$ on the critical line. Without any hypothesis we know that $\frac{5}{4} \leqq \Theta \leqq \frac{3}{2}$ and whatever the true value of Θ, $\Phi(s)$ has no singularity on the real axis at $s = \Theta$, proving (8).

References

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