ON THE ORDER TOPOLOGY IN A LATTICE

BY
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Introduction

Order convergence in a lattice S may fail to be pretopological; when pretopological, it may fail to be topological; when topological, S may fail to be a topological lattice under its order topology.

We first state a condition c_1 on the local ideal structure of S which is necessary and sufficient to make order convergence pretopological. Under c_1, the neighborhood filter at each point x in S is generated by a certain family of closed intervals. Next we state conditions c_2 and c_3 each of which, when combined with c_1, suffices to make order convergence topological. The order topology obtained under c_2 differs in a significant way from that obtained under c_3. In both cases, however, S is a topological lattice under its order topology, and, in each case, the order topology has an open subbase of ideals and dual ideals reminiscent of Frink's *ideal topology* [3] in a lattice.

1. Notation

For a subset $A \subset S$, A^* will denote the set of all upper bounds of A, and A^+ the set of lower bounds of A. $\{x\}^*$ and $\{x\}^+$ will be written x^* and x^+ respectively. The "closed interval" notation, $[x, y] = x^* \cap y^+$ $(x \leq y)$ will be employed.

If A, B are subsets of S, we define

$$A \vee B = \{x \vee y : x \in A,\ y \in B\}; \qquad A \wedge B = \{x \wedge y : x \in A,\ y \in B\}.$$

$\mathfrak{F}$ and $\mathfrak{G}$ will be the usual notation for filters on S. Let

$$\mathfrak{F}^+ = (\bigcup F^+ : F \in \mathfrak{F}) \quad \text{and} \quad \mathfrak{F}^* = (\bigcup F^* : F \in \mathfrak{F}).$$

One can show that the families

$$\{F \vee G : F \in \mathfrak{F}, G \in \mathfrak{G}\} \quad \text{and} \quad \{F \wedge G : F \in \mathfrak{F}, G \in \mathfrak{G}\}$$

are filter bases on S; the filter generated by the first is called $\mathfrak{F} \vee \mathfrak{G}$, and by the second, $\mathfrak{F} \wedge \mathfrak{G}$.

The following relationships are easily verified:

(a) $(\mathfrak{F} \vee \mathfrak{G})^* = \mathfrak{F}^* \vee \mathfrak{G}^*$;
(b) $(\mathfrak{F} \wedge \mathfrak{G})^+ = \mathfrak{F}^+ \wedge \mathfrak{G}^+$;
(c) $(\mathfrak{F} \wedge \mathfrak{G})^* \supset \mathfrak{F}^* \wedge \mathfrak{G}^*$;
(d) $(\mathfrak{F} \vee \mathfrak{G})^+ \supset \mathfrak{F}^+ \vee \mathfrak{G}^+$.

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2. Preliminaries

DEFINITION 1. $\mathfrak{F}$ order converges to x (also written $\mathfrak{F} \to x$) if and only if $x = \inf \mathfrak{F}^* = \sup \mathfrak{F}^+$.

The intersection of all filters which order converge to x is denoted $\mathcal{V}(x)$. It is clear that $\mathfrak{F} \to x$ implies $\mathfrak{F} \geq \mathcal{V}(x)$; the converse of this statement is false in general, as one can easily see by examining order convergence in the lattice obtained by placing in parallel two replicas of the open interval $(0, 1)$ of the real line and adding a greatest and a least element.

DEFINITION 2. Order convergence is pretopological if and only if $\mathcal{V}(x) \to x$.

For each $A \subset S$, let $A^0 = \{x \in A : A \in \mathcal{V}(x)\}$. The operation thus defined on subsets of S resembles the interior operator in the topological sense, but, in general, $(A^0)^0 \neq A^0$.

DEFINITION 3. The order topology on S is generated by taking as a basis for open sets the family $\{A^0 : A \subset S\}$.

DEFINITION 4. Order convergence is topological on S if and only if it coincides with convergence under the order topology.

DEFINITION 5. S is a topological lattice with respect to a topology τ if and only if whenever $\mathfrak{F}$ and $\mathcal{G}$ τ-converge to x and y, respectively, then $\mathfrak{F} \vee \mathcal{G}$ and $\mathfrak{F} \wedge \mathcal{G}$ τ-converge to $x \vee y$ and $x \wedge y$, respectively.

A discussion of order convergence from the standpoint of nets is found in [1] and [5]. We use a form of the definition adapted for filters by Ward [6]; our definition of the order topology is taken from the same source. The concept of pretopological convergence was introduced by Choquet [2], and developed further by the author [4].

It is easy to see that if order convergence coincides with convergence under any topology τ, then τ is necessarily the order topology. Another topology (which we shall call λ) related to the order convergence is obtained by identifying as open those sets A such that $A = A^0$. In general, the convergence described by the topology λ is coarser than order convergence, in the sense that a filter which order converges to x also converges to x with respect to λ. These matters, considered under more general circumstances, are discussed in [4].

In the two examples that follow, we vindicate some remarks about order convergence given in the Introduction.

Example 1. Let S_1 be the subset of the Euclidian plane defined as follows:

$$S_1 = \{(1, 0)\} \cup \{(x, y) : 0 \leq x < 1, 0 \leq y \leq 1\} \cup \{(1, 1)\}.$$

(Here $\leq$ refers to the usual order relation among real numbers.) The set S_1 is thus the closed unit square with the right boundary, minus endpoints, removed. This set becomes a lattice when the order relation $\prec$ is introduced as follows:

$$(x_1, y_1) \prec (x_2, y_2) \quad \text{means} \quad x_1 \leq x_2 \quad \text{and} \quad y_1 \leq y_2 .$$

By examining order convergence at the point $(1, 0)$, one can draw the following conclusions: (1) The set

$$A = \{(x, y) : \tfrac{1}{2} \leq x \leq 1, y = 0\}$$

is in $\mathcal{V}((1, 0))$, but $A^0 = \{(1, 0)\}$ is not in $\mathcal{V}((1, 0))$; (2) Convergence with respect to the order topology is strictly finer than order convergence; (3) Order convergence is strictly finer than convergence with respect to the topology λ; (4) Order convergence is pretopological, but not topological.

Example 2. The following sets are constructed in the Euclidian plane:

$$H = \{(x, y) : 0 \leq x \leq 5, 0 \leq y \leq 2\};$$

$$K = \{(x, y) : 0 \leq y \leq 5, 0 \leq x \leq 2\};$$

$$L = \{(4, 4)\} \cup \{(5, 5)\}.$$

Let $S_2 = H \cup K \cup L$. S_2 is a lattice under the order described in Example 1; furthermore, order convergence on S_2 is topological. However, by considering order convergence at $(4, 1)$ from the lower right, at $(1, 4)$ from the upper left, and at $(4, 4)$, we see that S_2 is not a topological lattice in its order topology.

3. The condition c_1

Let I be an ideal in S such that $\sup I = x$; let D be a dual ideal in S such that $\inf D = x$. Then the family

$$\{[y, z] : y \in I, z \in D\}$$

is a filter base; if $\mathfrak{F}$ is the filter generated thereby, then $\mathfrak{F}^+ = I$ $\mathfrak{F}^* = D$, and it follows that $\mathfrak{F} \to x$. On the other hand, for any filter $\mathcal{G}$, $\mathcal{G}^+$ is an ideal and $\mathcal{G}^*$ is a dual ideal.

Let $L(x)$ designate the intersection of all ideals I such that $\sup I = x$; let $U(x)$ denote the intersection of all dual ideals D such that $\inf D = x$.

Lemma 1. (1) $L(x) = (\mathcal{V}(x))^+$; (2) $U(x) = (\mathcal{V}(x))^*$.

Proof. By the remarks preceding the lemma,

$$L(x) = \bigcap\{\mathfrak{F}^+ : \mathfrak{F} \to x\}.$$

If $y \in \bigcap\{\mathfrak{F}^+ : \mathfrak{F} \to x\}$, then for each filter $\mathfrak{F}$ order converging to x there is a set $F \in \mathfrak{F}$ such that $y \in F^+$. But y is a lower bound of the union of such sets, and hence $y \in (\mathcal{V}(x))^+$. On the other hand, if $y \in (\mathcal{V}(x))^+$, then $y \in V^+$ for some $V \in \mathcal{V}(x)$; but $\mathfrak{F} \to x$ implies $V \in \mathfrak{F}$, and thus $y \in \mathfrak{F}^+$. The proof of (2) is similar.

THEOREM 1. *Order convergence is pretopological on* S *if and only if* $x = \sup L(x) = \inf U(x)$ *for each* $x \in S$. *When order convergence is pretopological, the family*

$$\mathcal{S} = \{[y, z] : y \in L(x), z \in U(x)\}$$

is a filter base for $\mathcal{V}(x)$.

Proof. The first assertion is an immediate consequence of Lemma 1. Let $\mathfrak{F}$ be the filter generated by $\mathcal{S}$. Then $\mathfrak{F} \to x$ implies $\mathfrak{F} \geq \mathcal{V}(x)$. On the other hand, if $[y, z] \in \mathcal{S}$, then there is $V_1 \in \mathcal{V}(x)$ such that $y \in V_1^+$, $V_2 \in \mathcal{V}(x)$ such that $z \in V_2^*$, and it follows that $V_1 \cap V_2 \subset [y, z]$. Thus $\mathfrak{F} = \mathcal{V}(x)$.

The condition "$x = \sup L(x) = \inf U(x)$ for all $x \in S$" will be referred to henceforth as c_1.

THEOREM 2. *Let* S *conform to* c_1.

(1) *If* $U(x \vee y) = U(x) \vee U(y)$, *then* $\mathfrak{F} \to x$ *and* $\mathcal{G} \to y$ *implies*

$$\mathfrak{F} \vee \mathcal{G} \to x \vee y.$$

(2) *If* $L(x \wedge y) = L(x) \wedge L(y)$, *then* $\mathfrak{F} \to x$ *and* $\mathcal{G} \to y$ *implies*

$$\mathfrak{F} \wedge \mathcal{G} \to x \wedge y.$$

Proof. It suffices to prove (1). We note that

$$(\mathfrak{F} \vee \mathcal{G})^* = \mathfrak{F}^* \vee \mathcal{G}^* \supset U(x) \vee U(y) = U(x \vee y).$$

If z is a lower bound of $(\mathfrak{F} \vee \mathcal{G})^*$, then

$$x \vee y = \inf U(x \vee y) \quad \text{implies} \quad z \leq x \vee y.$$

But x is a lower bound of $\mathfrak{F}^*$, y a lower bound of $\mathcal{G}^*$, and thus

$$x \vee y = \inf (\mathfrak{F} \vee \mathcal{G})^*.$$

Furthermore, since all elements of $(\mathfrak{F} \vee \mathcal{G})^*$ are upper bounds of $(\mathfrak{F} \vee \mathcal{G})^+$, $x \vee y$ is an upper bound of $(\mathfrak{F} \vee \mathcal{G})^+$. If z is an upper bound of $(\mathfrak{F} \vee \mathcal{G})^+$, then z is an upper bound of $\mathfrak{F}^+ \vee \mathcal{G}^+$, and it follows that $z \geq x \vee y$. Thus

$$x \vee y = \sup (\mathfrak{F} \vee \mathcal{G})^+.$$

4. The condition c_2

We now seek conditions on S under which order convergence will be topological.

c_2. If $x \in L(y)$ then

$$U(x) \cap L(y) \neq \emptyset,$$

and if $x \in U(y)$ then

$$L(x) \cap U(y) \neq \emptyset.$$

Each of the lattices considered in Examples 1 and 2 fails to satisfy c_2.

The next two lemmas pertain to a lattice S which has the properties c_1 and c_2. The notation $U(x) \cap L(y)$ will be shortened to $\langle x, y\rangle$.

LEMMA 2. *$y \in U(x)$ if and only if $x \in L(y)$.*

Proof. If $y \in U(x)$, then by c_2 there is $z \in \langle x, y\rangle$, and $x \in z^+ \subset L(y)$. The converse is similar.

LEMMA 3. *Sets of the form $L(x)$ and $U(x)$ are open in the topology λ.*

Proof. It must be shown that $y \in U(x)$ implies $U(x) \in \mathfrak{V}(y)$. If $y \in U(x)$, choose $z \in \langle x, y\rangle$. Then $z \in L(y)$ implies that there is $V \in \mathfrak{V}(y)$ such that $V \subset z^* \subset U(x)$, whence $U(x) \in \mathfrak{V}(y)$ The proof that $L(x)$ is λ-open is similar.

THEOREM 3. *Under conditions c_1 and c_2, order convergence is topological on S, and the order topology has an open subbase consisting of ideals and dual ideals of the form $L(x)$ and $U(x)$ respectively.*

Proof. Let $\mathfrak{U}(x)$ be the filter generated by sets of the form $\langle y, z\rangle$ which contain x. By Lemma 3, and Theorem 1, Section 1, [4], $\mathfrak{V}(x) \geq \mathfrak{U}(x)$. If $y \in L(x)$ and $z \in U(x)$ (so that $[y, z]$ is a typical basis element of $\mathfrak{V}(x)$), we can choose $y' \in \langle y, x\rangle$ and $z' \in \langle x, z\rangle$; it follows that

$$x \in [y', z'] \subset \langle y, z\rangle \subset [y, z].$$

Since $[y', z'] \in \mathfrak{V}(x)$, it follows that $\langle y, z\rangle \in \mathfrak{V}(x)$, and $\mathfrak{U}(x) \geq \mathfrak{V}(x)$. Thus $\mathfrak{U}(x)$ is the neighborhood filter at x for the topology λ. From the remarks preceding Example 1, it follows that the order convergence is topological on S. Since $\mathfrak{U}(x)$ is the neighborhood filter at x for the order topology, the last assertion of the theorem is evident.

THEOREM 4. *Under the conditions c_1 and c_2, S is a topological lattice in its order topology.*

Proof. By Theorem 2, it suffices to show that

$$U(x \vee y) = U(x) \vee U(y) \quad \text{and} \quad L(x \wedge y) = L(x) \wedge L(y).$$

We verify only the first of these propositions. It is clear that $z \in U(x) \vee U(y)$ implies $z \in U(x)$ and $z \in U(y)$, and by Lemma 2, $x \in L(z)$ and $y \in L(z)$. Thus

$$x \vee y \in L(z) \quad \text{and} \quad z \in U(x \vee y).$$

This reasoning is reversible.

Let T be a finite set. We define the finite product lattice $\prod_T S$ to be the set of all functions mapping T into S, ordered as follows: $f \leq g$ means $f(t) \leq g(t)$ for all $t \in T$. If S satisfies c_1 and c_2, then so does $\prod_T S$. In particular, the properties c_1 and c_2 extend to n-dimensional Euclidian space with its usual ordering.

Note. Definitions 1–4 are valid for partially ordered sets as well as for lattices. If the statement of Lemma 1 is regarded as the definition of $L(x)$ and $U(x)$, then the proofs of Lemmas 2 and 3 and of Theorems 1 and 3 are valid without alteration. Thus partially ordered sets which meet the requirements c_1 and c_2 have the order topology described in Theorem 3. An interesting non-lattice that meets these conditions is the set of all "open" disks in the Euclidian plane, ordered by inclusion.

5. The condition c_3

The results of the last section can be essentially duplicated by imposing a different condition on S.

c_3. If $x \in L(y)$ and $x \leq z$ then $x \in L(z)$, and if $x \in U(y)$ and $x \geq z$ then $x \in U(z)$.

THEOREM 5. *Let S be subject to c_1 and c_3. Then the following statements are true:*

(1) *If $y \in L(x)$ and $z \in U(x)$ then the set $[y, z]$ is open in the topology λ.*

(2) *The order convergence is topological.*

(3) *S is a topological lattice under its order topology.*

Proof. (1) For given $x \in S$, let $y \in L(x)$, $z \in U(x)$, $w \in [y, z]$. If $x \geq y$ then $y \in L(w)$, $w \leq z$ implies $z \in U(w)$; thus by Theorem 1, $[y, z] \in \mathcal{V}(w)$. It follows that $[y, z] = [y, z]^0$, and $[y, z]$ is open in the topology λ.

(2) From (1) we deduce, as in the proof of Theorem 3, that order convergence and λ-convergence coincide; hence order convergence is topological.

(3) It suffices to show that $U(x \vee y) = U(x) \vee U(y)$. If

$$z \in U(x \vee y),$$

then $z \geq x \vee y$ implies $z \in U(x)$ and $z \in U(y)$. If

$$z \in U(x) \vee U(y),$$

then $z \in U(x)$, $z \in U(y)$, and $x \vee y \leq z$. Thus $z \in U(x \vee y)$.

An example of a lattice satisfying c_1 and c_3 is the set of all subsets of any non-void set Q, ordered by inclusion. If x is a subset of Q, $L(x)$ consists of those finite subsets of Q which are included in x; $U(x)$ is composed of the complements of those finite subsets of Q which have no points in common with x.

A finite product of lattices satisfying c_1 and c_3 is again a lattice with these properties.

It is not difficult to show that c_1 and c_3 can be extended to partially ordered sets if $L(x)$ and $U(x)$ are defined, for each x, in accordance with the note at the end of the preceding section. Parts (1) and (2) of Theorem 5 extend to partially ordered sets that meet these conditions.

A final theorem shows that conditions c_2 and c_3 are, in a sense, independent.

THEOREM 6. *If the order convergence is topological on S, then the order topology is discrete if and only if S satisfies* c_2 *and* c_3 *simultaneously.*

Proof. If the order topology is discrete then, for each x, $L(x) = x^+$ and $U(x) = x^*$, and c_2 and c_3 follow immediately. Conversely, let $y \in L(x)$. By Lemma 2, $x \in U(y)$, and x^+ is open by Theorem 5. A similar argument establishes that x^* is open, and hence that $\{x\}$ is open.

Concluding Remarks. When order convergence is topological, "closed intervals" are closed in the order topology, and from Theorem 1 we deduce that the order topology is regular as well as Hausdorff; hence it is metrizable whenever it is second countable. The set of all open disks in the plane, partially ordered by inclusion, is metrizable in its order topology. For a disk x, $L(x)$ consists of those open disks which are included in x and not tangent to x; $U(x)$ consists of the open disks which include x and are not tangent to x. The order topology is also metrizable in the complete lattice of all subsets of a countable set S ordered by inclusion. If x is a subset of S, then $L(x)$ is the collection of finite subsets of x and $U(x)$ is the family of all complements of finite subsets of S not in x.

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