INVARIANT SUBSPACES

BY

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1. Introduction

The invariant subspaces of L^p $(1 \leq p < \infty)$ of the unit circle are those closed subspaces $\mathfrak{M}$ (weak*-closed when $p = \infty$) for which $e^{ix}f$ is in $\mathfrak{M}$ for each f in $\mathfrak{M}$. If $\mathfrak{M}$ is invariant and also $e^{-ix}f$ is in $\mathfrak{M}$ for every f in $\mathfrak{M}$, $\mathfrak{M}$ is called doubly invariant; $\mathfrak{M}$ is called simply invariant if it is invariant but not doubly invariant. The structure of these subspaces is known. In their fundamental paper [4], Helson and Lowdenslager used an elegant Hilbert space argument to characterize the simply invariant subspaces of L^2. Forelli [2] extended their result to L^1 by a factoring process that depended on the L^2 case. In the remaining cases [3, p. 26], the structure of the simply invariant subspace $\mathfrak{M}$ results from an analysis of $\mathfrak{M} \cap L^2$ $(1 < p < 2)$ and of the annihilator of $\mathfrak{M}$ in $(L^p)^*$ $(2 < p \leq \infty)$. Thus the structure of the simply invariant subspaces of L^p unfolds only after initial success in the L^2 setting. Much the same situation holds for doubly invariant subspaces, and for invariant subspaces defined in certain abstract spaces [8], [9], [10].

The primary purpose of this paper is to obtain these invariant subspace structure theorems by methods that are free of special Hilbert space techniques. We are successful in all cases except one—the simply invariant subspaces in L^1. In §2 we characterize the simply invariant subspaces of $L^p(dm)$ $(1 \leq p < \infty)$ of a Dirichlet algebra by a method that depends on the reflexivity of the overlying function space $(1 < p < \infty)$ and on a double extremal technique developed by Rogosinski and Shapiro [7]. The same method is implicit in an abstract of E. Bishop [Notices, vol. 12 (1965), p. 123]. In §3 we use a Zorn's lemma argument to characterize the doubly invariant subspaces in $L^p(d\mu)$ $(1 \leq p \leq \infty)$ of a certain measure space. Finally, in §4, we give new proofs and expand on some results of Srinivasan and Hasumi [10] concerning weak*-density of subalgebras of $L^\infty(d\mu)$. Although most of the paper is devoted to new proofs of known results, we believe that Theorem 1 $(1 < p < 2, 2 < p < \infty)$ is new.

As one would expect, the technique described in §2 does not apply to the simply invariant subspaces on the line; neither does it seems to offer great promise in the study of invariant subspaces in the spaces of functions from the unit circle into a Hilbert space [3, Lectures V, VI].

2. Simply invariant subspaces

Let X be a compact Hausdorff space, A a uniformly closed algebra of complex, continuous functions on X that contains the constant functions, separates the points of X and satisfies the additional condition that Re A is uniformly

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dense in $C_R(X)$, the space of real continuous functions on X. Such an algebra is called a Dirichlet algebra on X. In this section A will be a fixed Dirichlet Dirichlet algebra on X, m will be a nonnegative finite Borel measure on X such that

$$\hat{f}(0) = \int f\, dm \qquad (f \in A)$$

defines a multiplicative linear functional on A. (All integrals extend over X.) Define A_0 to be the set

$$A_0 = \{f \in A : \hat{f}(0) = 0\}.$$

For $1 \leq p < \infty$, $H^p(dm)$ will be the closure of A in $L^p(dm)$, and $H^\infty(dm)$ will be the essentially bounded functions in $H^1(dm)$. The norm of f in $L^p(dm)$ will be written $\| f \|_p$. For $p \geq 1$, $H^p(dm)$ may be characterized as the collection of those functions f in $L^p(dm)$ such that $\int gf\, dm = 0$ for all g in A_0. This follows for $1 < p < \infty$ from the fact that there is a version of the M. Riesz conjugate function theorem valid in the context of Dirichlet algebras [1, Theorem 9]. Presumably the same characterization of $H^p(d\mathrm{m})$ holds in the more general setting of logmodular algebras, although only the cases $p = 1$ and $p = 2$ are explicitly mentioned in [5]. For results concerning Dirichlet algebras used in this section, the reader is referred to [5], [11].

A subspace $\mathfrak{M}$ of $L^p(dm)$ $(1 \leq p < \infty)$ is *simply invariant* if and only if $\mathfrak{M}$ is closed and the closure of $A_0 \mathfrak{M}$ in $L^p(dm)$ is a proper subset of $\mathfrak{M}$.

THEOREM 1. *Every simply invariant subspace of $L^p(dm)$ $(1 \leq p < \infty)$ is of the form*

$$\phi H^p(dm), \tag{1}$$

where ϕ is a measurable function such that $| \phi | = 1$ almost everywhere.

It is clear that every subspace of $L^p(dm)$ $(1 \leq p < \infty)$ of the form given in (1) is simply invariant. In order to show that every simply invariant subspace of $L^p(dm)$ has this form, we first treat the case $p = 1$. One would like to have a proof divorced from $H^2(dm)$, but we have not been able to find one. We do have a proof that depends on a geometrical property of $H^1(dm)$ that D. J. Newman [6] has called pseudo-uniform convexity: if f_n, f are in H^1 of the unit circle, $\int | f_n |\, d\theta \to \int | f |\, d\theta$ and $f_n(z) \to f(z)$ uniformly on compact subsets of $| z | < 1$, then $\int |f_n - f|\, d\theta \to 0$. We need the analogue of this result in $H^1(dm)$ only for $f \equiv 1$. The question of whether $H^1(dm)$ is pseudo-uniformly convex is open.

LEMMA. *If f_n is a sequence in $H^1(dm)$ such that*

$$\|f_n\|_1 \leq 1 \quad \textit{and} \quad \int f_n\, dm \to 1, \tag{2}$$

then

$$\int |1 - f_n|\, dm \to 0.$$

Proof. We may assume that $\int f_n \, dm$ is never 0. Each f_n has a factorization

$$f_n = F_n g_n , \tag{3}$$

where F_n is an inner function and g_n is an outer function in $H^1(dm)$. Since the multiplicative property of m on the algebra A extends to each product in (3), and since $| F_n | = 1$ almost everywhere, we obtain the inequality

$$\left| \int f_n \, dm \right| = \left| \int F_n \, dm \right| \left| \int g_n \, dm \right| \leq \left| \int g_n \, dm \right|$$
$$\leq \int | g_n | \, dm = \int | f_n | \, dm \leq 1,$$

which, by (2), implies that $| \int g_n \, dm | \to 1$. Thus, by the definition of an outer function,

$$\int \log | f_n | \, dm = \int \log | g_n | \, dm = \log \left| \int g_n \, dm \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{4}$$

Let G_n be the projection of the constant function 1 onto the closure of A_0 in $L^2(| f_n | \, dm)$. According to the generalized Szegö theorem,

$$\int | 1 - G_n |^2 | f_n | \, dm = k_n = \exp \left[\int \log | f_n | \, dm \right], \tag{5}$$

and from (4), $k_n \to 1$. Now the general theory of Dirichlet algebras tells us that

$$| 1 - G_n |^2 | f_n | = k_n , \qquad (1 - G_n)^{-1} \in H^2(dm), \qquad (1 - G_n) f_n \in H^2(dm), \tag{6}$$

and that

$$f_n = (1 - G_n)^{-1} [(1 - G_n) f_n] \tag{7}$$

is a factorization of f_n into the product of two functions in $H^2(dm)$. Since G_n is a member of the closure of A_0 in $L^2(| f_n | \, dm)$, it follows that $\int G_n f_n \, dm = 0$ and

$$\int (1 - G_n) f_n \, dm = \int f_n \, dm \to 1.$$

The information from (5) and (6) combines to yield the following limits:

$$\int | 1 - G_n |^2 | f_n |^2 \, dm = k_n \int | f_n | \, dm \to 1,$$

$$\int (1 - G_n)^{-1} \, dm = \int f_n \, dm \Big/ \int (1 - G_n) f_n \, dm = 1,$$

$$\int | 1 - G_n |^{-2} \, dm = k_n^{-1} \int | f_n | \, dm \to 1.$$

Hence $\int |1 - (1 - G_n)^{-1}|^2 \, dm \to 0$, $\int |1 - (1 - G_n)f_n|^2 \, dm \to 0$ and from (7),

$$\int |1 - f_n| \, dm \to 0.$$

Proof for $p = 1$. If $\mathfrak{M}$ is a simply invariant subspace of $L^1(dm)$, then, since the closure of $A_0 \mathfrak{M}$ is proper in $\mathfrak{M}$, there is a function ϕ in $L^\infty(dm)$ such that

$$\int fg\bar{\phi} \, dm = 0 \quad (f \in A_0, g \in \mathfrak{M}), \qquad \| \phi \|_\infty = 1 \tag{8}$$

and

$$\sup \left| \int g\bar{\phi} \, dm \right| = 1, \tag{9}$$

the supremum being taken over all g in $\mathfrak{M}$ such that $\| g \|_1 \leq 1$.

Since the functions $g\bar{\phi}$ $(g \in \mathfrak{M})$ are orthogonal to A_0, $\bar{\phi}\mathfrak{M} \subset H^1(dm)$. Choose a sequence g_n in $\mathfrak{M}$ such that

$$\| g_n \|_1 = 1 \quad \text{and} \quad \int g_n \bar{\phi} \, dm \to 1. \tag{10}$$

It follows from (8) and (10) that

$$\int | g_n \bar{\phi} | \, dm \leq 1, \tag{11}$$

and therefore, by the lemma,

$$\int | 1 - g_n \bar{\phi} | \, dm \to 0. \tag{12}$$

Thus the constant function 1 belongs to the closure of $\bar{\phi}\mathfrak{M}$ in $L^1(dm)$. From the definition of $H^1(dm)$ and the simple invariance of $\mathfrak{M}$, we conclude that the closure of $\bar{\phi}\mathfrak{M}$ in $L^1(dm)$ is $H^1(dm)$. If we show that $| \phi | = 1$ almost everywhere, the proof will be complete, for then $\bar{\phi}\mathfrak{M}$ will be closed, hence $\bar{\phi}\mathfrak{M} = H^1(dm)$ and $\mathfrak{M} = \phi H^1(dm)$ as required.

The limit in (12) implies that

$$\int_E | g_n \bar{\phi} | \, dm \to m(E) \tag{12'}$$

for every Borel subset E of X. But if $| \phi(x) | \leq 1 - c (c > 0)$ on a set E of positive m-measure, (8), (10) and (11) combine to yield

$$\int | g_n \bar{\phi} | \, dm \leq (1 - c) \int_E | g_n | \, dm + \int_{E'} | g_n | \, dm = 1 - c \int_E | g_n | \, dm$$

or

$$c \int_E | g_n | \, dm \leq 1 - \int | g_n \bar{\phi} | \, dm.$$

Since the right member of this inequality tends to 0 as $n \to \infty$,

$$\int_E |g_n| \, dm \to 0 \quad \text{and} \quad \int_E |g_n \bar{\phi}| \, dm \to 0 \neq m(E),$$

contrary to (12′). Hence $|\phi| = 1$ almost everywhere.

Proof for $1 < p < \infty$. Let $\mathfrak{M}$ be a simply invariant subspace of $L^p(dm)$. Since the closure of $A_0 \mathfrak{M}$ is a proper subset of $\mathfrak{M}$, there exists a function h in $L^q(dm)$ $(1/p + 1/q = 1)$ such that

$$\|h\|_q = 1, \qquad \int fgh \, dm = 0 \qquad (f \in A_0, g \in \mathfrak{M}), \tag{13}$$

and

$$\sup \left| \int gh \, dm \right| = 1, \tag{14}$$

the supremum being taken over all g in $\mathfrak{M}$ such that $\|g\|_p \leq 1$.

According to (14) there exists a sequence g_n in $\mathfrak{M}$ such that

$$\|g_n\|_p = 1 \quad \text{and} \quad \int g_n h \, dm \to 1. \tag{15}$$

Now a closed subspace of a Banach space is weakly closed, and in a reflexive Banach space the closed unit sphere is weakly compact. Hence the g_n have a weak limit point ϕ in $\mathfrak{M}$ which, by (13) and (15), satisfies

$$\|\phi\|_p \leq 1, \qquad \int \phi h \, dm = 1 \quad \text{and} \quad \int f\phi h \, dm = 0 \qquad (f \in A_0). \tag{16}$$

From the inequalities

$$1 = \int \phi h \, dm \leq \int |\phi h| \, dm \leq \|\phi\|_p \|h\|_q = \|\phi\|_p \leq 1,$$

we conclude that

$$\|\phi\|_p = 1, \qquad \int \phi h \, dm = \int |\phi h| \, dm \quad \text{and} \quad \int |\phi h| \, dm = \|\phi\|_p \|h\|_q.$$

The second equality implies that $\phi h = |\phi h|$; hence

$$\phi h = 1, \tag{17}$$

because a real function in $L^p(dm)$ orthogonal to A_0 (see (16)) is constant. The third equality is precisely an assertion of equality in Hölder's inequality, which holds only in case real numbers α and β exist such that

$$\alpha |\phi|^p = \beta |h|^q. \tag{18}$$

Combining (17) and (18), we obtain

$$1 = \phi h = |\phi h| = (\beta/\alpha)^{1/p} |h|^q$$

which implies that $|h|$ is constant almost everywhere. Now, by (17), $\phi = |h|^{-2}\bar{h}$ is in $\mathfrak{M}$ and, by (13), the functions $\bar{\phi}g$ $(g \in \mathfrak{M})$ are orthogonal to A_0, and therefore belong to $H^p(dm)$. Hence $\bar{\phi}\mathfrak{M}$ is a simply invariant subspace of $H^p(dm)$ that contains the constants. Consequently $\bar{\phi}\mathfrak{M} = H^p(dm)$ and $\mathfrak{M} = \phi H^p(dm)$ as required.

In the next section we treat doubly invariant subspaces in a more general context and by another procedure. We think it worth-while to point out here that the above procedure for the case $1 < p < \infty$ applies equally well to doubly invariant subspaces of $L^p(dm)$ $(1 < p < \infty)$, where by a doubly invariant subspace of $L^p(dm)$ we mean a closed subspace $\mathfrak{M}$ such that gf is in $\mathfrak{M}$ for all f in $\mathfrak{M}$ and all g in $A \cup \bar{A}$. The only additional piece of information required is that $A \cup \bar{A}$ is weak*-dense in $L^\infty(dm)$.

3. Doubly invariant subspaces

Let X be a locally compact Hausdorff space that is σ-compact, μ a nonnegative Radon measure on X, and $\mathfrak{A}$ a subalgebra of $L^\infty(d\mu)$ that is weak*-dense in $L^\infty(d\mu)$.

We say that a subspace $\mathfrak{M}$ of $L^p(d\mu)$ is *doubly invariant* ($\mathfrak{A}$-invariant) provided that [10, p. 525]

(i) $\mathfrak{M}$ is closed in $L^p(d\mu)$ $(1 \le p < \infty)$ and weak*-closed if $p = \infty$,

(ii) the product ϕf is in $\mathfrak{M}$ for every ϕ in $\mathfrak{A}$ and f in $\mathfrak{M}$.

On the unit circle, $\mathfrak{A}$ is the algebra generated by e^{ix} and e^{-ix}, and in the case of a Dirichlet algebra A, $\mathfrak{A}$ is the smallest subalgebra of $L^\infty(dm)$ that contains $A_0 \cup \bar{A}_0$. The genesis of $\mathfrak{A}$ is obliterated in the setting of this section.

THEOREM 2. *The doubly invariant subspaces of $L^p(d\mu)$ $(1 \le p < \infty)$ are of the form*

$$C_E L^p(d\mu), \tag{19}$$

where C_E is the characteristic function of a measurable set E.

Proof. It is clear that any subspace of $L^p(d\mu)$ of the form (19) is doubly invariant.

If $\mathfrak{M}$ is doubly invariant, then the product ϕf $(\phi \in L^\infty(d\mu), f \in \mathfrak{M})$, is always in $\mathfrak{M}$. Otherwise, for some ϕ and f, there would exist an h in

$$L^q(d\mu)\ (1/p + 1/q = 1)$$

such that

$$\int gh\, d\mu = 0 \quad (g \in \mathfrak{M}) \quad \text{and} \quad \int \phi fh\, d\mu \ne 0. \tag{20}$$

Since $\mathfrak{M}$ is doubly invariant and $\mathfrak{A}$ is weak*-dense in $L^\infty(d\mu)$, it follows that $fh = 0$ almost everywhere. Hence the second integral in (20) must also be 0.

Due to the σ-compactness of X, we can construct a nonnegative function u in $L^p(d\mu) \cap L^1(d\mu) \cap L^\infty(d\mu)$ that does not vanish on a set of positive μ-measure.

(In case X is compact, let u be identically 1.) For each f in $\mathfrak{M}$ define

$$\begin{aligned} f_n(x) &= 1/f(x) \quad \text{if} \quad |f(x)| \geq 1/n \\ &= 0 \qquad\quad \text{if} \quad |f(x)| < 1/n \end{aligned}$$

$$E(f) = \{x \in X : f(x) \neq 0\},$$

$$C(f) = C_{E(f)}\,.$$

The products $f_n\,u$ are bounded; hence $f_n\,uf$ is a sequence in $\mathfrak{M}$ that converges pointwise to $C(f)u$, and $C(f)u \in \mathfrak{M}$ in consequence of the bounded convergence theorem. The result follows quite easily once we show there is a maximal set among the sets $E(f)$ $(f \in \mathfrak{M})$.

Define a partial ordering on the family $\mathfrak{N} = \{E(f) : f \in \mathfrak{M}\}$ by $E(f) < E(g)$ if and only if $E(f) \subset E(g)$ and $\int C(f)u\,d\mu < \int C(g)u\,d\mu$. Suppose that T is a totally ordered subset of $\mathfrak{N}$ and set

$$k = \sup \int C(f)u\,d\mu,$$

the supremum being over $E(f)$ in T. Choose $E(f_n)$ in T such that $\int C(f_n)u\,d\mu \to k$. The function $C_D\,u$ $(D = \bigcup_{n\geq 1} E(f_n))$ is the limit in $L^p(d\mu)$ of $C(f_n)u$ $(n = 1, 2, 3, \cdots)$. Thus $C_D\,u \in \mathfrak{M}$, and the set D in $\mathfrak{N}$ is clearly an upper bound for T. By Zorn's lemma, $\mathfrak{N}$ has a maximal element E. If $C_E\,g \notin \mathfrak{M}$ for some g in $L^p(d\mu)$, there would exist an h in $L^q(d\mu)$ such that

$$\int fh\,d\mu = 0 \quad (f \in \mathfrak{M}) \quad \text{and} \quad \int C_E\,gh\,d\mu \neq 0.$$

Since $C_E\,u \in \mathfrak{M}$, it follows from the double invariance of $\mathfrak{M}$ and the first integral that $C_E\,uh = 0$ almost everywhere. However, u does not vanish on a set of positive μ-measure, hence $C_E\,h = 0$ almost everywhere and the second integral is 0. This shows that $C_E\,L^p(d\mu) \subset \mathfrak{M}$. On the other hand, if $f \in \mathfrak{M}$, the function

$$C(f)u + C_E\,u - C_{E(f)\cap E}\,u = C_{E(f)\cup E}\,u$$

is a function in $\mathfrak{M}$ whose non-zero set is $E(f) \cup E$. From the maximality of E, we conclude that $f = C_E\,f$. Thus $\mathfrak{M} \subset C_E\,L^p(d\mu)$ and the proof is complete.

We note that the obvious result for doubly invariant subspaces of $L^\infty(d\mu)$ follows from the observation that the set of all $f \in L^1(d\mu)$ such that

$$\int fg\,d\mu = 0 \qquad\qquad (g \in \mathfrak{M})$$

is a doubly invariant subspace of $L^1(d\mu)$ whose annihilator in $L^\infty(d\mu)$ is precisely $\mathfrak{M}$.

4. Weak*-density of certain subalgebras of $L^\infty(dm)$

In [10], Srinivasan and Hasumi proved a doubly invariant subspace theorem for $p = 1, 2$ under the assumption $\mathfrak{A}$ is dense in $L^2(dm)$, then through the invariant subspace theory concluded that $\mathfrak{A}$ is weak*-dense in $L^\infty(dm)$. In this section we establish their result without recourse to invariant subspace theory. In fact our technique enables us to show that if $\mathfrak{A}$ is a weak*-dense subalgebra of $L^\infty(dm)$ of a finite measure space (X, m), then every bounded function on X is the pointwise limit of a uniformly bounded sequence from $\mathfrak{A}$. This generalizes the situation for the unit circle, where the standard theorem asserts that every bounded and measurable function is the pointwise limit of a subsequence of the Cesàro means of its Fourier series.

THEOREM 3. *Let (X, m) be a finite measure space. If $\mathfrak{A}$ is a conjugate closed subalgebra of $L^\infty(dm)$ and $1 \leq p < \infty$, the following are equivalent:*
(i) *$\mathfrak{A}$ is dense in $L^p(dm)$.*
(ii) *$\mathfrak{A}$ is weak*-dense in $L^\infty(dm)$.*

Proof. That (ii) implies (i) is obvious. To see that (i) implies (ii), we start with a ϕ in $L^\infty(dm)$ such that $-1 \leq \phi \leq 1$ and define $u = \arcsin \phi$. Since $u \in L^p(dm)$, there exists a sequence v_n in $\mathfrak{A}$ such that

$$\int |v_n - u|^p \, dm \to 0.$$

Because $\mathfrak{A}$ is conjugate closed, we may take the functions v_n to be real. By passing to a subsequence if necessary, we have $v_n(x) \to u(x)$ almost everywhere. The functions $\sin v_n$ belong to the weak*-closure of $\mathfrak{A}$, $|\sin v_n| \leq 1$ and

$$\sin v_n(x) \to \sin u(x) = \phi(x)$$

almost everywhere. If we apply this argument to the real and imaginary parts of an arbitrary ϕ in $L^\infty(dm)$, there emerges a sequence w_n in $\mathfrak{A}$, such that

$$\limsup \|w_n\|_\infty \leq \sqrt{2}\, \|\phi\|_\infty \quad \text{and} \quad w_n(x) \to \phi(x) \tag{21}$$

almost everywhere. In particular, ϕ belongs to the weak*-closure of $\mathfrak{A}$.

COROLLARY. *Let (X, m) be a finite measure space. If $\mathfrak{A}$ is a conjugate closed and weak*-dense subalgebra of $L^\infty(dm)$, then, for each ϕ in $L^\infty(dm)$, there exists a sequence w_n in $\mathfrak{A}$ for which (21) holds.*

THEOREM 4. *Let X be a locally compact Hausdorff space, μ a nonnegative Radon measure on X, and suppose that the Banach conjugate of $L^1(d\mu)$ is (isometric to) $L^\infty(d\mu)$. If $1 \leq p < \infty$ and $\mathfrak{A}$ is a conjugate closed subalgebra of $L^\infty(d\mu)$ such that $\mathfrak{A} \mid K$ is dense in $L^p(d\mu \mid K)$ for every compact subset K of X, then $\mathfrak{A}$ is weak*-dense in $L^\infty(d\mu)$.*

Proof. Let ϕ be in $L^{\infty}(d\mu)$ and K a compact subset of X. Since $\mathfrak{A} \mid K$ is dense in $L^{p}(d\mu \mid K)$, we may apply Theorem 3 to establish the existence of a sequence g_n in $\mathfrak{A}$ such that $\limsup \| g_n \|_{\infty} \leq \sqrt{2} \| \phi \|_{\infty}$ and $g_n(x) \to \phi(x)$ on K. Since every uniformly bounded sequence in $L^{\infty}(d\mu)$ has a weak*-cluster point, there exists a function u_K in the weak*-closure of $\mathfrak{A}$ such that

$$\| u_K \|_{\infty} \leq \sqrt{2} \| \phi \|_{\infty} \quad \text{and} \quad u_K = \phi \quad \text{on} \quad K.$$

Now suppose that $f \in L^{1}(d\mu)$ is such that $\int gf \, d\mu = 0$ for all $g \in \mathfrak{A}$. For each compact set K in X there exists a function u_K in the weak*-closure of $\mathfrak{A}$ such that $u_K = \bar{f}/|f|$ on K and $\| u_K \|_{\infty} \leq \sqrt{2}$. Consequently,

$$\int_K |f| \, d\mu + \int_{K'} u_K f \, d\mu = 0$$

or

$$\int_K |f| \, d\mu \leq \sqrt{2} \int_{K'} |f| \, d\mu.$$

Since $|f| \, d\mu$ is a regular measure, and this last inequality holds for every compact set, we conclude that

$$\int |f| \, d\mu = 0.$$

Hence $f = 0$ almost everywhere, proving that $\mathfrak{A}$ is weak*-dense in $L^{\infty}(d\mu)$.

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