BOUNDARY BEHAVIOUR OF TEMPERATURES, PART 2

BY

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In his paper, Stein [2] defined the concept of "determining polynomials" for harmonic functions. He then developed a sufficient condition on a set of polynomials to guarantee that they are determining. We will develop analogous results for temperatures on

$$E_{n+1}^{+}, \; E_{n+1}^{+} = \{(x, t) : x \in E_n , t > 0\}.$$

Let D^i be the operator of partial differentiation with respect to x_i for $i = 1, \cdots, n$, and with respect to t for $i = 0$. Let α be an $(n + 1)$-dimensional multi-index, i.e., an $(n + 1)$-tuple, each of whose entries is a non-negative integer. Let D^α be the product of the operators D^i each raised the respective power α_i, $i = 0, 1, \cdots, n$. We will say that D^α has "index"

$$r = 2\alpha_0 + \textstyle\sum_{i=1,n} \alpha_i .$$

Let $M(D)$ be the differential operator D^α of index r. Then, there exists a differential operator $M^*(D)$, which is homogeneous of order r in $D^1, \cdots, D^n$ such that, for T a temperature, $M^*(D)T = M(D)T$. We obtain $M^*(D)$ from $M(D)$ by replacing D^0 with $\sum_{i=1,n}(D^i)^2$.

Let $m(x)$ be the polynomial obtained from $M^*(D)$ by replacing D^i by x_i. We note that m is homogeneous of degree r where r is the "index" of M. m will be called the "polynomial associated with M".

In a differential polynomial in $D^0, D^1, \cdots, D^n$ has the property that each of its terms has the same "index" r, then we will say that the differential polynomial is "homogeneous of index r".

In our previous work [1] we defined the concept of parabolic limits on a set E in E_n for functions on E_{n+1}^{+}.

For U, V, two vector-valued temperatures of respective dimensions k and s, we proved there (theorem 4) the following:

THEOREM A. *If* (a) *V has parabolic limits on a set $E \subset E_n$, and* (b) *$(D^0)^q U = P(D)V$, where $P(D)$ is a $k \times s$ matrix each of whose entries is a homogeneous differential polynomial in $D^1, \cdots, D^n$ of order $2q$, then U has parabolic limits a.e. on E.*

In view of the discussion above, we see that hypothesis (b) can be weakened to $P(D)$ is a $k \times s$ matrix, each of whose entries is a differential polynomial in $D^0, D^1, \cdots, D^n$ homogeneous of index $2q$. It is in this form that we will use the theorem in the following.

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THEOREM 1. (a) *Suppose that* $(D^0)^q T$, T *a temperature on* E_{n+1}^+, *has parabolic limits on a set* $E \subset E_n$. *Then every derivative of index* $2q$ *of* T *has parabolic limits a.e. on* E.

(b) *Suppose that* $\{D^i(D^0)^q T, i = 1, \cdots, n\}$ *have parabolic limits on a set* $E \subset E_n$. *Then every derivative of index* $2q + 1$ *of* T *has parabolic limits a.e. on* E.

Proof. It is sufficient to prove each part of the theorem for monomial derivatives of the appropriate index of T since if each term of a differential polynomial applied to T has parabolic limits on a set so does the polynomial.

(a) Let M be a monomial of index $2q$. In the theorem stated above, let $U = M(D)T$, $V = (D^0)^q T$, and $P(D) = M(D)$. Then $(D^0)^q U = P(D)V$, and (a) follows from the theorem.

(b) Let $M(D) = D^\alpha$ be a monomial of index $2q + 1$. Then $\alpha_i \neq 0$ for some $i = 1, \cdots, n$. Let $M^1(D) = D^\beta$, $\beta_j = \alpha_j$, for $j \neq i$, $\beta_i = \alpha_i - 1$. Let $U = M(D)T$, $V = D^i(D^0)^q T$, and $P(D) = M^1(D)$. Then, $(D^0)^q U = P(D)V$ and the index of $P(D)$ is $2q$, so the theorem gives us part (b).

Suppose we have a set of differential polynomials $P_1(D), \cdots, P_k(D)$, all homogeneous of the same index r. If the existence of parabolic limits on a set $E \subset E_n$ for the functions obtained by the application of these polynomials to a temperature T implies their existence a.e. on E for the function obtained by application of any differential polynomial homogeneous of index r to T, then we will say that the set of polynomials $P_1, \cdots, P_k$ is "determining."

Theorem 1 shows that for even index r, the single operator $(D^0)^{(1/2)r}$ is determining, and for odd index r, the set $\{D^i(D^0)^{(1/2)r-1/2}\}_{i=1,n}$ is determining. This suggests the question: Under what conditions will a given set of polynomials be determining? A sufficient condition is given by Theorem 2 of this paper.

For a differential polynomial $P(D)$ homogeneous of index r, we define $P^*(D)$ and the associated polynomial $P(x)$ in the same way that we defined $M^*(D)$ and $m(x)$ for the monomial $M(D)$.

THEOREM 2. *A sufficient condition that the* k *differential polynomials* $P_1(D), \cdots, P_k(D)$, *homogeneous of index* r, *be determining is that the common complex zeros* $z^1, \cdots, z^m$, $m \leq r$ *of the associated polynomials* $p_1(x), \cdots, p_k(x)$ *satisfy*

$$(x^{j,1})^2 + (x^{j,2})^2 + \cdots + (x^{j,n})^2 = 0, \qquad j = 1, \cdots, m,$$

where $x^{j,i}$ *is the i-th component of* z^j, $i = 1, \cdots, n$.

The technique used in the proof of this theorem is that used by Stein [2]. It is interesting to note that in some ways the arguments are simpler here than in the harmonic functions case.

LEMMA. *Let* I *be an ideal in the ring of all polynomials in* $x_1, \cdots, x_n$. *For a given polynomial* $f(x)$, *a sufficient condition that there exist an integer* N *such that* $[f(x)]^N \in I$ *is that* f *be zero at all of the common zeros of* I.

This is the Hilbert "Nullstellensatz" for the complex number field. See [3, §79].

Proof of theorem. Let I be the ideal generated by $p_1(x), \cdots, p_k(x)$. Let $f(x) = \sum_{i=1,n} x_i^2$. Then, by our hypothesis on the common zeros of $p_i(x), \cdots, p_k(x)$ and the above lemma, there exists an integer N, and polynomials $q_1(x), \cdots, q_k(x)$ such that

$$(*) \qquad (\textstyle\sum_{i=1,n} x_1^2)^N = \sum_{j=1,k} q_j(x)p_j(x).$$

Since the polynomials p_j are homogeneous of degree r, we may assume that the q_j have degree less than or equal to $2N - r$. Let us now write $q_j(x) = \sum_e q_{j,e}(x)$, where $q_{j,e}(x)$ is homogeneous of degree e.

For r even, we may assume e to be even. Then,

$$(**) \qquad (\textstyle\sum_i x_i^2)^N = \sum_j (\sum_e q_{j,e}(x)p_j(x)).$$

Replacing x_j by $\eta^{-1/2}x_j$, we obtain

$$\eta^{-N}(\textstyle\sum_i x_i^2)^N = \sum_j (\sum_e q_{j,e}(x)\eta^{-(1/2)e}p_j(x)\eta^{-(1/2)r}),$$

or

$$\eta^{(1/2)r}(\textstyle\sum_i x_i^2)^N = \sum_{j,e} q_{j,e}(x)p_j(x)\eta^{N-(1/2)e}.$$

Now, since $2N - r \geq e$, $2N - e \geq r \geq 1$, whence $N - \frac{1}{2}e \geq 0$. Thus, for a temperature T, replacing x_j by D^j, and η by D^0, we have

$$(D^0)^N[(D^0)^{(1/2)r}T] = \textstyle\sum_j Q_j(D)P_j^*(D)T,$$

where

$$Q_j(D) = \textstyle\sum_e q_{j,e}(D)(D^0)^{N-(1/2)e}.$$

Thus, the $Q_j(D)$ have index $2N$. By Theorem A, and by part (a) of Theorem 1, all polynomial derivatives of index r of T have parabolic limits a.e. on a set $E \subset E_n$ if $P_1(D)T, \cdots, P_k(D)T$ have parabolic limits on E. This shows the theorem for r even.

For r odd, we obtain $(*)$ and $(**)$ as above. Now, however, we may assume that e is odd. Multiplying $(**)$ by x_i, we have

$$x_i(\textstyle\sum_j x_j^2)^N = \sum_j(\sum_e x_i\, q_{j,e}(x)p_j(x)).$$

Substituting as before,

$$\eta^{-N-1/2}x_i(\textstyle\sum_j x_j^2)^N = \sum_j(\sum_e x_i\, q_{j,e}(x)p_j(x)\eta^{-(1/2)e-1/2-(1/2)}),$$

$$\eta^{(1/2)r-1/2}x_i(\textstyle\sum_j x_j^2)^N = \sum_j(\sum_e x_i\, q_{j,e}(x)\eta^{N-(1/2)e-1/2}p_j(x)).$$

Since $2N - r \geq e$, we have $2N - e - 1 \geq 0$. Thus, for a temperature T, as before, we obtain

$$(D^0)^N[D^i(D^0)^{(1/2)r-1/2}T] = \textstyle\sum_j Q_j(D)P_j^*(D)T,$$

where $Q_j(D) = \sum_e x_i\, q_{j,e}(D)(D^0)^{N-(1/2)e-1/2}$. Using Theorem A, and part (b) of Theorem 1, the proof is completed as above.

BIBLIOGRAPHY

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