ON THE CURVATURES OF RIEMANNIAN MANIFOLDS[1]

BY
JOHN A. THORPE

An open question of long standing in Riemannian geometry is the following: given a compact orientable even-dimensional Riemannian manifold X of positive Riemannian sectional curvature, is the Euler-Poincaré characteristic χ of X necessarily positive? This is known to be the case for example when X is a homogeneous space [5], a 4-dimensional manifold [3], or a 6-dimensional Kahler manifold satisfying a certain additional hypothesis [1]. A tool available for studying this question is the Gauss-Bonnet formula [2] which can be expressed as

$$\chi = \frac{2}{c_n} \int_X K \, dV$$

where K is the Lipschitz-Killing curvature of X, dV is the volume element, and c_n is the volume of the Euclidean unit n-sphere.

The Lipschitz-Killing curvature K has an unwieldy algebraic expression in terms of the curvature tensor of X and it is not as yet well understood. In this note we investigate the geometry of this curvature, continuing work begun in [6], and obtain certain partial results in the direction of the "positive curvature implies positive characteristic" conjecture.

1. The curvatures γ_p

Let X be a Riemannian manifold. Since the Riemannian sectional curvature γ of X completely determines the curvature tensor of X, it determines all the curvature properties of X and in particular it determines the Lipschitz-Killing curvature K. It is the nature of this dependence which occupies our attention here. We would like to know, for example, if $\gamma \geq 0$ everywhere implies $K \geq 0$ everywhere. An affirmative answer to this local question would yield an affirmative answer to the global question considered above.

Now between the Riemannian sectional curvature γ and the Lipschitz-Killing curvature K there is defined a sequence of intermediate curvatures γ_p. The function γ_p, called the p^{th} sectional curvature, is a smooth function on the bundle of p-planes over X and it measures the Lipschitz-Killing curvature of geodesic p-dimensional submanifolds. It is defined for all even integers p between 2 and the dimension of X. $\gamma_2 = \gamma$ is the Riemannian sectional curvature and $\gamma_n = K$ is, for n even, the Lipschitz-Killing curvature.

That these curvatures γ_p lie at the heart of the problem under consideration is well illustrated by the following example. The most definitive result thus far obtained relative to this problem is its solution in dimension 4.

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Milnor showed that if the Riemannian sectional curvature γ_2 of a compact orientable 4-manifold X is either ≥ 0 everywhere or ≤ 0 everywhere then the characteristic χ of X is ≥ 0. A close analysis of the proof of this result (see e.g. [1]) shows that what was in fact proved was that, for arbitrary Riemannian manifolds, $\gamma_2 \geq 0$ everywhere or $\gamma_2 \leq 0$ everywhere implies $\gamma_4 \geq 0$ everywhere.

Now for p and q even integers with $p + q \leq \dim X$, the $(p + q)^{\text{th}}$ sectional curvature γ_{p+q} is completely determined by γ_p and γ_q (cf. the remark in §2 below). We need to know more about this dependence. For example we would like to know if $\gamma_p \geq 0$ everywhere and $\gamma_q \geq 0$ everywhere implies $\gamma_{p+q} \geq 0$ everywhere. The author has shown [6] that this is indeed the case when γ_p and γ_q are both constant. An affirmative answer to this question in general would yield a confirmation of the "positive curvature implies positive characteristic" conjecture.

Our main theorem determines explicitly the dependence of γ_{p+q} on γ_p and γ_q in the case where γ_p (or γ_q) is constant.

THEOREM. *Let X be a Riemannian manifold with constant p^{th} sectional curvature γ_p. Let P be a $(p + q)$-plane tangent to X. Then the value at P of the $(p + q)^{\text{th}}$ sectional curvature γ_{p+q} is equal to the constant value of γ_p multiplied by the average value of γ_q over all q-planes Q contained in P.*

Postponing the proof of this theorem to the next section we derive several corollaries, the first of which is an immediate consequence of the theorem.

COROLLARY 1. *Suppose γ_p is constant and that γ_q keeps constant sign for some p and q with $p + q \leq \dim X$. Then γ_{p+q} keeps constant sign and*

$$\operatorname{sign}(\gamma_{p+q}) = \operatorname{sign}(\gamma_p \gamma_q).$$

Remark. This statement is to be interpreted in the broadest possible sense. Thus, for example, if γ_p is constant > 0 and $\gamma_q < 0$ everywhere then $\gamma_{p+q} < 0$ everywhere, whereas if γ_p is constant > 0 and $\gamma_q \leq 0$ everywhere then $\gamma_{p+q} \leq 0$ everywhere.

Remark. Greub and Tondeur [4] have recently obtained a related result in the case of compact locally symmetric homogeneous spaces. They proved that, for such spaces, $\gamma_p \geq 0$ for all p.

COROLLARY 2. *Let X be a compact orientable Riemannian manifold of even dimension n. Suppose γ_p is constant and that γ_{n-p} keeps constant sign for some p. Then the Euler-Poincaré characteristic of X has the same sign as $\gamma_p \gamma_{n-p}$.*

Proof. By the theorem, the sign of γ_n is everywhere the same as that of $\gamma_p \gamma_{n-p}$. But γ_n is the Lipschitz-Killing curvature, i.e. the integrand in the Gauss-Bonnet formula for the characteristic χ. Thus χ also has this sign.

COROLLARY 3. *Let X be compact orientable and of even dimension n. Assume that γ_{n-2} is constant. Then the Euler-Poincaré characteristic of X is given by the formula*

$$\chi = \frac{2K_{n-2}}{n(n-1)c_n} \int_X \rho \, dV$$

where c_n is the volume of the Euclidean unit n-sphere, K_{n-2} is the constant value of γ_{n-2}, and ρ is the scalar curvature of X.

Proof. By the theorem, the Lipschitz-Killing curvature γ_n at $x \in X$ is equal to K_{n-2} multiplied by the average value of the Riemannian sectional curvature γ_2 over all 2-planes at x. In terms of the components of the curvature tensor R relative to any orthonormal frame at x, this average is given by

$$\frac{(n-2)!\,2!}{n!} \sum_{i<j} R_{ijij} = \frac{1}{n(n-1)} \sum_{i,j} R_{ijij} = \frac{1}{n(n-1)} \rho(x).$$

Inserting this information into the Gauss-Bonnet formula completes the proof.

2. Proof of the theorem

We shall adopt here the notation used in [6] and shall assume the results of that paper. Recall that, from the curvature forms Ω_{ij} of the Riemannian connection of X, we constructed p-forms

$$(1) \qquad \Theta^{(p)}_{i_1 \cdots i_p} = \frac{1}{p!} \sum_{(j)} \delta(i_1, \cdots, i_p; j_1, \cdots, j_p) \Omega_{j_1 j_2} \vee \cdots \vee \Omega_{j_{p-1} j_p}$$

on the orthonormal frame bundle F. These forms are defined for each selection $(i) = (i_1, \cdots i_p)$ from $\{1, \cdots, n\}$. The sum here ranges over all such selections $(j) = (j_1, \cdots, j_p)$. The symbol

$$\delta(i_1, \cdots, i_p; j_1, \cdots, j_p)$$

is zero unless (j) is a permutation of (i) in which case it is equal to the sign of this permutation. These forms are the components of a p-form $\Theta^{(p)}$ on F with values in the p^{th} exterior power $R^n_{[p]}$ of real n-space.

Now the p^{th} sectional curvature γ_p of X is given by the formula

$$(2) \qquad \gamma_p(x, P) = \Theta^{(p)}_{1 \cdots p}(b)(f'_1, \cdots, f'_p)$$

where $b = (x; f_1, \cdots, f_n)$ is any orthonormal frame at x such that $\{f_1, \cdots, f_p\}$ spans P and f'_i is any vector at b which projects onto $f_i (i \in \{1, \cdots, n\})$. Furthermore γ_p is constant if and only if the equations

$$(3) \qquad \Theta^{(p)}_{i_1 \cdots i_p} = K_p \, \omega_{i_1} \vee \cdots \vee \omega_{i_p}$$

are satisfied for all (i), where K_p is the constant value of γ_p and $\omega_1, \cdots, \omega_n$ are the canonical 1-forms on F.

In the proof of the theorem we shall use the following lemma, which was implicit in [6].

LEMMA. *The form* $\Theta^{(p+q)}_{i_1\cdots i_{p+q}}$ *can be expressed as*

$$\Theta^{(p+q)}_{i_1\cdots i_{p+q}} = \frac{p!\,q!}{(p+q)!}\sum_A \Theta^{(p)}_{k_1\cdots k_p} \vee \Theta^{(q)}_{k_{p+1}\cdots k_{p+q}}$$

where the sum ranges over all partitions $A = (A_1, A_2)$ *of*

$$\{i_1, \cdots, i_{p+q}\}$$

into sets A_1 *of* p *elements and* A_2 *of* q *elements, and where* $(k_1, \cdots, k_{p+q})$ *is, for each* A, *an even permutation of* $(i_1, \cdots, i_{p+q})$ *such that*

$$k_1, \cdots, k_p \in A_1 \quad \text{and} \quad k_{p+1}, \cdots, k_{p+q} \in A_2 .$$

The proof of this lemma is contained in the proof of Theorem 6.2 of [6] and will not be repeated here.

Remark. It follows from this lemma that γ_{p+q} is completely determined by γ_p and γ_q. For in fact γ_{p+q} is, according to formula (2), determined by $\Theta^{(p+q)}$ which by the lemma is determined by $\Theta^{(p)}$ and $\Theta^{(q)}$. But $\Theta^{(p)}$ is completely determined by γ_p (and similarly for $\Theta^{(q)}$). For if there existed two such p-forms $\Theta^{(p)}$ and $\Theta^{(p)\prime}$ (horizontal equivariant $R^n_{[p]}$-valued p-forms on F whose components satisfy the Bianchi type identity of Lemma 4.4 (i) in [6]) such that γ_p was obtained from each by formula (2), then replacing the Φ in the proof of Theorem 5.1 in [6] by $\Theta^{(p)} - \Theta^{(p)\prime}$ and applying that proof implies that $\Theta^{(p)} - \Theta^{(p)\prime} \equiv 0$.

Proof of the theorem. From the lemma,

$$\Theta^{(p+q)}_{1\cdots p+q} = \frac{p!\,q!}{(p+q)!}\sum_A \Theta^{(p)}_{i_1\cdots i_p} \vee \Theta^{(q)}_{i_{p+1}\cdots i_{p+q}}$$

where $(i) = (i_1, \cdots, i_{p+q})$ is, for each partition $A = (A_1, A_2)$ of $\{1, \cdots, p+q\}$, an even permutation of $(1, \cdots, p+q)$ such that

$$(i_1, \cdots, i_p) \in A_1 \quad \text{and} \quad i_{p+1}, \cdots, i_{p+q} \in A_2 .$$

Since γ_p is constant, equations (3) imply that

$$\Theta^{(p+q)}_{1\cdots p+q} = \frac{p!\,q!}{(p+q)!}\sum_A K_p\,\omega_{i_1} \vee \cdots \vee \omega_{i_p} \vee \Theta^{(q)}_{i_{p+1}\cdots i_{p+q}} \tag{4}$$

where K_p is the constant value of γ_p.

Now $\Theta^{(q)}_{i_{p+1}\cdots i_{p+q}}$ is a horizontal q-form on F. Since the set

$$\{\omega_{j_1} \vee \cdots \vee \omega_{j_q} \mid 1 \leq j_1 < \cdots < j_q \leq n\}$$

is a basis for the horizontal q-forms at each point of F, there exist functions

$S(i_{p+1} \cdots i_{p+q}\,; j_1 \cdots j_q)$ on F, defined for each selection $(j_1, \cdots, j_q)$ from $\{1, \cdots, n\}$ and completely alternating in the j's, such that

$$\Theta^{(q)}_{i_{p+1}\cdots i_{p+q}} = \sum\nolimits_{(j)} S(i_{p+1} \cdots i_{p+q}\,; j_1 \cdots j_q)\omega_{j_1} \vee \cdots \vee \omega_{j_q}$$

where the sum ranges over all (j) with $j_1 < \cdots < j_q$. Putting this into (4), we obtain

$$\Theta^{(p+q)}_{1\cdots p+q} = \frac{p!\,q!}{(p+q)!} K_p \sum_A \Big[\sum_{(j)} S(i_{p+1} \cdots i_{p+q}\,; j_1 \cdots j_q) \cdot\omega_{i_1} \vee \cdots \vee \omega_{i_p} \vee \omega_{j_1} \vee \cdots \vee \omega_{j_q}\Big]. \tag{5}$$

For (x, P) a $(p + q)$-plane on X, let $b = (x; f_1, \cdots, f_n) \in F$ be a frame at x such that $\{f_1, \cdots, f_{p+q}\}$ spans P. By (2), the sectional curvature γ_{p+q} at (x, P) is obtained by evaluating the left hand side of (5):

$$\gamma_{p+q}(x, P) = \Theta^{(p+q)}_{1\cdots p+q}(b)(f_1', \cdots, f_{p+q}').$$

But, for each A, the only terms in the brackets on the right hand side of (5) which are non-zero upon such evaluation are those where

$$(i_1, \cdots, i_p\,; j_1, \cdots, j_q)$$

is a permutation of $(1, \cdots, p+q)$, i.e. those where $(j_1, \cdots, j_q)$ is a permutation of $(i_{p+1}, \cdots, i_{p+q})$. In this case, let σ denote this permutation:

$$\sigma = \begin{pmatrix} i_{p+1} & \cdots & i_{p+q} \\ j_1 & \cdots & j_q \end{pmatrix}.$$

Then

$$S(i_{p+1} \cdots i_{p+q}\,; j_1 \cdots j_q) = (\operatorname{sgn}\sigma) S(i_{p+1} \cdots i_{p+q}\,; i_{p+1} \cdots i_{p+q})$$

and

$$\omega_{j_1} \vee \cdots \vee \omega_{j_q} = (\operatorname{sgn}\sigma)\omega_{i_{p+1}} \vee \cdots \vee \omega_{i_{p+q}}$$

so, since $(i_1, \cdots, i_{p+q})$ is an even permutation of $(1, \cdots, p+q)$,

$$\begin{aligned}\gamma_{p+q}(x, P) &= \frac{p!\,q!}{(p+q)!} K_p \sum_A (\operatorname{sgn}\sigma)^2 S(i_{p+1} \cdots i_{p+q}\,; i_{p+1} \cdots i_{p+q}) \\ &\qquad \cdot\omega_{i_1} \vee \cdots \vee \omega_{i_{p+q}}(f_1', \cdots, f_{p+q}') \\ &= \frac{p!\,q!}{(p+q)!} K_p \sum_A S(i_{p+1} \cdots i_{p+q}\,; i_{p+1} \cdots i_{p+q}) \\ &\qquad \cdot\omega_1 \vee \cdots \vee \omega_{p+q}(f_1', \cdots, f_{p+q}')\end{aligned}$$

or

$$\gamma_{p+q}(x, P) = \frac{p!\,q!}{(p+q)!} K_p \sum_A S(i_{p+1} \cdots i_{p+q}\,; i_{p+1} \cdots i_{p+q}). \tag{6}$$

It remains only to express the terms in this summation in terms of γ_q. For this, let $Q(A)$ be the q-plane spanned by

$$\{f_{i_{p+1}}, \cdots, f_{i_{p+q}}\}.$$

Let $b_1 = (x; e_1, \cdots, e_n) \in F$ be such that $e_k = f_{i_{p+k}}$ for $k \in \{1, \cdots, q\}$. Then

$b_1 = bg$ for some $g = [g_{ij}] \in O(n)$. Note that the first q columns of g are determined by our requirement on b_1. Let $e'_k = R_{g*} f'_{i_{p+k}}$ for $k \in \{1, \cdots, q\}$ where R_{g*} denotes the differential of right translation by g on F. Then e'_k projects onto e_k and, by (2),

$$\begin{aligned} \gamma_q(x, Q(A)) &= \Theta^{(q)}_{1\cdots q}(b_1)(e'_1, \cdots, e'_q) \\ &= \Theta^{(q)}_{1\cdots q}(bg)(R_{g*} f'_{i_{p+1}}, \cdots, R_{g*} f'_{i_{p+q}}) \\ &= R^*_g\, \Theta^{(q)}_{1\cdots q}(b)(f'_{i_{p+1}}, \cdots, f'_{i_{p+q}}) \end{aligned}$$

where R^*_g is the map on differential forms induced by right translation by g. But, by equivariance of $\Theta^{(q)}$ (cf. [6, §4]),

$$\begin{aligned} R^*_g\, \Theta^{(q)}_{1\cdots q} &= \textstyle\sum_{(j)} g_{j_1 1} \cdots g_{j_q q}\, \Theta^{(q)}_{j_1\cdots j_q} \\ &= \Theta^{(q)}_{i_{p+1}\cdots i_{p+q}}. \end{aligned}$$

We have used here our knowledge of the first q columns of g. Putting this into the above expression for $\gamma_q(x, Q(A))$ we obtain

$$\gamma_q(x, Q(A)) = \Theta^{(p)}_{i_{p+1}\cdots i_{p+q}}(b)(f'_{i_{p+1}}, \cdots, f'_{i_{p+q}}).$$

Thus, from (6),

$$\gamma_{p+q}(x, P) = \frac{p!\, q!}{(p+q)!} K_p \sum_A \gamma_q(x, Q(A)). \tag{7}$$

Since there are exactly $(p+q)!/p!\,q!$ partitions A of $\{1, \cdots, p+q\}$,

$$\frac{p!\, q!}{(p+q)!} \sum_A \gamma_q(x, Q(A))$$

is the average value of γ_q over all q-planes Q which can be obtained as the span of q vectors in the chosen basis $f_1, \cdots, f_{p+q}$ of P. But from (7) it is clear that this value is independent of the basis $f_1, \cdots, f_{p+q}$ chosen. Hence this value is in fact equal to the average value of γ_q over all q-planes Q contained in P.

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Massachusetts Institute of Technology
Cambridge, Massachusetts