A VARIATION OF THE TCHEBICHEFF QUADRATURE PROBLEM

BY

A. MEIR AND A. SHARMA

1. Introduction

The original Tchebicheff problem of quadrature was to find a formula of the form

$$\int_{-1}^{1} f(x)\,dx = B^{(n)} \sum_{i=1}^{n} f(x_i^{(n)}) \tag{1.1}$$

with real $B^{(n)}$, and real nodes $x_i^{(n)}$ in $[-1, 1]$ such that (1.1) is valid for polynomials of degree $\leq n$. It is well known [4] that Bernstein proved that the problem has a negative solution if $n \geq 10$. On the other hand the Gauss quadrature formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} A_i^{(n)} f(\xi_i^{(n)}) \tag{1.2}$$

is known to be valid for polynomials of degree $\leq 2n - 1$, with $A_i^{(n)} > 0$, $-1 < \xi_i^{(n)} < 1$ for all i. In a recent paper, Erdös and Sharma [2] have shown that an "intermediate" formula of the form

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{k} A_i^{(n)} f(y_i^{(n)}) + B^{(n)} \sum_{j=1}^{n-k} f(x_j^{(n)}) \tag{1.3}$$

with fixed k, real $y_i^{(n)}$, $x_j^{(n)}$, $A_i^{(n)}$ and $B^{(n)}$ cannot be valid in general for polynomials of degree $n + k$, if n is sufficiently large. It was also shown that if the degree of exactness of a formula of the form (1.3) is N (i.e. there exists a formula of the form (1.3) valid for polynomials of degree $\leq N = N(n)$) then $N(n) \leq C_k \sqrt{n}$, where C_k is independent of n.

In this paper we consider the problem of the validity (for polynomials) of a formula of the form

$$\int_{-1}^{1} f(x)\,p(x)\,dx = \sum_{i=1}^{k} A_i^{(n)} f(y_i^{(n)}) + B^{(n)} \sum_{j=1}^{n-k} f(x_j^{(n)}) \tag{1.4}$$

where $p(x)$ is a non-negative weight function. Although we give a complete solution of the problem only when $p(x) = (1 - x^2)^\alpha$, $\alpha > -1$, some of our results hold for more general weight functions. A special case, when $k = 0$ has been treated recently by L. Gatteschi [4] who proved that there exists a constant $n_0(\alpha)$ such that if $n > n_0(\alpha)$ then for the degree of exactness N of the formula

$$\int_{-1}^{1} f(x)(1 - x^2)^\alpha\,dx = B^{(n)} \sum_{j=1}^{n} f(x_j^{(n)}) \tag{1.5}$$

Received May 16, 1966.

we have $N < n$. We shall obtain here an estimate in terms of n for the degree of exactness of the more general formula of type (1.4) with $p(x) = (1 - x^2)^\alpha$, $\alpha > -1$.

It is of interest to observe that for $\alpha = -\frac{1}{2}$, a formula of type (1.5) with degree of exactness $2n - 1$ does exist:

$$\int_{-1}^{1} f(x)(1 - x^2)^{-1/2}\, dx = \frac{\pi}{n} \sum_{j=1}^{n} f\left(\cos \frac{2j - 1}{2n} \pi\right). \tag{1.6}$$

This is a special case of the general Gauss quadrature formula also valid for polynomials of degree $\leq 2n - 1$:

$$\int_{-1}^{1} f(x)(1 - x^2)^\alpha\, dx = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} f(\xi_\nu^{(n)}) \tag{1.7}$$

where $\lambda_\nu^{(n)} > 0$ are the so-called Cotes numbers and $\xi_\nu^{(n)}$ are the zeros of the Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$.

2. The main theorem

We shall prove the following:

THEOREM 1. *Let k be a fixed non-negative integer, $p(x) = (1 - x^2)^\alpha$, $\alpha > -1$. Then for the degree of exactness $N = N(n, k, \alpha)$ of a formula of type (1.4), we have*

$$N < C \,.\, n^{1/(2\alpha+2)} \tag{2.1}$$

where $C = C(k, \alpha)$ is independent of n.

As a consequence we can formulate the

COROLLARY. *If $\alpha > -\frac{1}{2}$, then $N = o(n)$ as $n \to \infty$. In particular, if $N = n + k$, then there exists an integer $n_0 = n_0(\alpha, k)$ such that $n < n_0$.*

When $k = 0$ and $\alpha = 0$, it is known that $n_0 = 9$. The determination of n_0 as a function of k and α seems to be more complicated.

Theorem 1 gives an upper bound for the order of exactness $N(n)$ of a formula of type (1.4). However, we do not know if this is in fact the right order. A priori, the possibility of $N(n)$ being bounded by a fixed constant is not ruled out except in the special case $\alpha = -\frac{1}{2}$ where $N(n) \geq 2n - 1$. If we assume some restrictions on the nodes, we are able to show that $N(n)$ is indeed bounded. This is the subject of Theorems 2 and 3 in §5 which might be of some independent interest. Theorem 4 in §5 is a generalization of a lemma of Bernstein.

3. Preliminary lemmas

For the proof of the theorems we shall require a number of lemmas. For typographical reasons, we shall write x_j for $x_j^{(n)}$, y_i for $y_i^{(n)}$ etc. whenever there is no danger of misunderstanding.

LEMMA 1. *If a formula of type* (1.4) *is valid for polynomials of degree* $\leq N$, *then* $B^{(n)} > 0$.

It is easy to see that $N > 2k$ and thus the proof is immediate if we apply (1.4) to

$$f(x) = \prod_{i=1}^{k} (x - y_i)^2.$$

LEMMA 2. *If* $p(x) = p(-x)$ *and a formula of type* (1.4) *is valid for polynomials of degree* $\leq N$ *with certain* $\{A_i\}_1^k$, $\{y_i\}_1^k$, B *and* $\{x_j\}_1^{n-k}$, *then the same formula is valid with* $\{A_i\}_1^k$, $\{-y_i\}_1^k$, B *and* $\{-x_j\}_1^{n-k}$.

It is easy to verify that the lemma holds for $f(x) = x^\nu, 0 \leq \nu \leq N$.

LEMMA 3. *The degree of exactness* $N = N(n, k)$ *of formula* (1.4) *is*
(i) *a non-decreasing function of* n *for* $k \geq 1$ *and*
(ii) *a non-decreasing function of* k *for* $k \geq 0$.

Proof. Suppose a formula of type (1.4) is valid for polynomials of degree $\leq N$. Then defining

$$A_i^{(n+1)} = A_i^{(n)} \quad (2 \leq i \leq k), \qquad A_1^{(n+1)} = A_1^{(n)} - B^{(n)}$$

$$B^{(n+1)} = B^{(n)}, \qquad x_j^{(n+1)} = x_j^{(n)} \quad (1 \leq j \leq n - k),$$

$$x_{n+1-k}^{(n+1)} = y_1^{(n)}, \qquad y_i^{(n+1)} = y_i^{(n)} \quad (1 \leq i \leq k),$$

we see that a formula of type (1.4) with n replaced by $(n + 1)$ is valid for polynomials of degree $\leq N$ and thus $N(n, k)$ is non-decreasing in n; (ii) defining $y_{k+1}^{(n)} = x_{n-k}^{(n)}$, $A_{k+1}^{(n)} = B^{(n)}$ we see that a formula of type (1.4) with k replaced by $(k + 1)$ is valid for polynomials of degree $\leq N$ and thus $N(n, k)$ is non-decreasing in k.

LEMMA 4. *Let* $Q(x)$ *be a polynomial of degree* m *satisfying* $0 \leq Q(x) \leq 1$ *in* $[-1, 1]$. *Let* L, M *be fixed integers and*

$$x_0 = -1 + \lambda M m^{-2}, \qquad 1 \leq \lambda \leq L, \tag{3.1}$$

a point in $[-1, 1]$. *Suppose* $Q(x_0) = 1$. *Then*

$$\int_{-1}^{1} Q(x)(1 - x^2)^\alpha \, dx \geq C_1 M^\alpha m^{-2\alpha-2}, \qquad \alpha > -1, \tag{3.2}$$

where $C_1 = C_1(L, \alpha)$ *is independent of* M *and* m.

Proof. Since $Q(x) = Q(x_0) - \int_x^{x_0} Q'(t)\, dt$, we have from Bernstein's inequality for the derivatives of polynomials

$$Q(x) \geq 1 - m^2 \mid x_0 - x \mid.$$

Now, for large m, $x_0 < 0$ and thus $(1 - x)^\alpha \geq \frac{1}{2}$ for $-1 \leq x \leq x_0$. So, if $-1 \leq x_1 < x_0$

$$\int_{-1}^{1} Q(x)(1 - x^2)^\alpha \, dx \geq \frac{1}{2} \int_{x_1}^{x_0} \{1 - m^2(x_0 - x)\}(1 + x)^\alpha \, dx. \tag{3.3}$$

Choosing $x_1 = x_0 - m^{-2}$ we get from (3.1) and (3.3) after some simplifications

$$\begin{aligned}\int_{-1}^{1} Q(x)(1-x^2)^\alpha\,dx &\ge \frac{1}{2(\alpha+1)m^{2\alpha+2}}\left[(\lambda M)^{\alpha+1} - \frac{(\lambda M)^{\alpha+2}-(\lambda M-1)^{\alpha+2}}{\alpha+2}\right]\\ &\ge \frac{1}{4m^{2\alpha+2}}\left(1-\frac{\theta}{\lambda M}\right)^\alpha (\lambda M)^\alpha, \qquad 0<\theta<1,\\ &\ge \frac{C_1(L,\alpha)M^\alpha}{m^{2\alpha+2}}\end{aligned}$$

which proves the lemma.

LEMMA 5 (Szegö) [6, pp. 166, 236, 351]. *For $\alpha > -1$, let $\xi_1 < \xi_2 < \cdots < \xi_m$ be the zeros of $P_m(x) = P_m^{(\alpha,\alpha)}(x)$, and let $\lambda_\nu^{(m)}$ be the corresponding Cotes numbers. Then*

$$\xi_\nu = \cos(\nu\pi/m + \rho_\nu/m) \tag{3.4}$$

where ρ_ν is uniformly bounded independent of ν and m;

$$|P_m'(\xi_\nu)| = \gamma_\nu \cdot \nu^{-\alpha-3/2} m^{\alpha+2} \tag{3.5}$$

where γ_ν remains between fixed positive bounds independent of ν and m for $1 \le \nu \le m/2$,

$$\lambda_\nu^{(m)} = \delta_\nu\, \nu^{2\alpha+1} m^{-2\alpha-2} \tag{3.6}$$

where δ_ν remains between fixed positive bounds independent of ν and m for $1 \le \nu \le m/2$,

$$\max_{|x|\le 1} |P_m(x)| = \binom{m+\alpha}{m}. \tag{3.7}$$

LEMMA 6. *Let k, K be fixed positive integers and $p(x) = (1-x^2)^\alpha$, $\alpha > -1$. Suppose a formula of type (1.4) is valid for polynomials of degree $\le N = N(n, k)$. Let $m \le N/2$ and suppose there exists an integer $p = p(n)$, $1 \le p \le K$ such that in the interval $I_p \equiv [\xi_{p-1}^{(m)}, \xi_{p+2}^{(m)}]$ there is no y_i of formula (1.4) but there is an x_j in the interval $I_p' = [\xi_p^{(m)}, \xi_{p+1}^{(m)}]$. Then*

$$B^{(n)} < C_2 N^{-2\alpha-2} \tag{3.8}$$

where $C_2 = C_2(k, K, \alpha)$ is independent of n.

Proof. According to a lemma of Erdös and Turán [3] we have

$$l_p(x) + l_{p+1}(x) \ge 1 \qquad \text{for} \quad x \in I_p' \tag{3.9}$$

where

$$l_p(x) = l_p^{(m)}(x) = P_m(x)/(x-\xi_p)P_m'(\xi_p). \tag{3.10}$$

Since by hypothesis one of the x_j's, say x_1, is in I_p', we have by (3.9)

$$\max(|l_p(x_1)|, |l_{p+1}(x_1)| \ge \tfrac{1}{2}. \tag{3.11}$$

To be specific let $|l_p(x_1)| \ge \frac{1}{2}$. Consider the polynomial

$$f_1(x) = l_p^2(x) \prod_{j=1}^{k} (x - y_j)^2 (x - \xi_{p+j})^{-2}. \tag{3.12}$$

Then by Gauss quadrature formula

$$\int_{-1}^{1} f_1(x)(1 - x^2)^\alpha \, dx = \sum_{j=p}^{p+k} \lambda_j^{(m)} f_1(\xi_j), \tag{3.13}$$

and since $2m - 2 < N$, applying formula (1.4) to $f_1(x)$ and observing that $f_1(x) \geq 0$ and $B^{(n)} > 0$

$$\int_{-1}^{1} f_1(x)(1 - x^2)^\alpha \, dx \geq B^{(n)} f_1(x_1). \tag{3.14}$$

From (3.13) and (3.14)

$$B^{(n)} \leq \sum_{j=p}^{p+k} \lambda_j^{(m)} \frac{f_1(\xi_j)}{f_1(x_1)}. \tag{3.15}$$

Now, by (3.10) and (3.12) we have after simplification for $p \leq j \leq p + k$

$$\frac{f_1(\xi_j)}{f_1(x_1)} = \frac{1}{l_p^2(x_1)} \cdot \left(\frac{P_m'(\xi_j)}{P_m'(\xi_p)}\right)^2 \cdot \prod_{\nu=1}^{k} \left(\frac{\xi_j - y_\nu}{x_1 - y_\nu}\right)^2 \cdot \Lambda_j \tag{3.16}$$

where

$$\Lambda_j = \prod_{\nu=p+1}^{p+k} (x_1 - \xi_\nu)^2 \cdot \prod_{\nu=p,\nu\neq j}^{p+k} (\xi_j - \xi_\nu)^{-2}.$$

By (3.5), for $p \leq j \leq p + k$

$$| P_m'(\xi_j)/P_m'(\xi_p)| \leq c_1(k, K, \alpha).$$

Also, since $y_\nu \notin I_p$, one obtains easily on using (3.4)

$$\left| \frac{\xi_j - y_\nu}{x_1 - y_\nu} \right| \leq 1 + \left| \frac{\xi_j - x_1}{x_1 - y_\nu} \right| \leq c_2(k, K, \alpha)$$

an similarly

$$\Lambda_j \leq c_3(k, K, \alpha).$$

Combining the above inequalities and (3.11) we get

$$f_1(\xi_j)/f_1(x_1) \leq c_4(k, K, \alpha),$$

whence from (3.6) and (3.15)

$$B^{(n)} \leq c_5(k, K, \alpha) m^{-2\alpha-2}.$$

Since $m \leq N/2$, the statement of the lemma is immediate.

LEMMA 7. *Let $p(x) = (1 - x^2)^\alpha$, $\alpha > -1$ and let k be a fixed non-negative integer, $k > \frac{3}{2}\alpha$. Suppose a formula of type* (1.4) *is valid for polynomials of degree $\leq N$ and let $m \leq N/4$. Then there exists a fixed positive integer μ independent of n and an integer m_0 such that for $m \geq m_0$ in each interval $I_{p,\mu}^{(m)} = [\xi_{p-\mu}^{(m)}, \xi_{p+\mu}^{(m)}]$, $\mu \leq p \leq 6k\mu$, there is an x_i or a y_i of formula* (1.4).

Proof. Suppose the lemma is false. Then there exists an infinite sequence of integers μ_r, with $\mu_r \to \infty$ and for each r a sequence $m_{r,s}$ with $m_{r,s} \to \infty$ such that in the corresponding intervals $I_{p_{r,s},\mu_r}^{(m_{r,s})}$ there is no x_i or y_j of formula (1.4). For simplicity of printing we shall omit the subscripts r and s in the rest of the proof.

Consider now the polynomials of degree $\leq 2m - 2$

$$(3.17)\qquad \begin{aligned} f_2(x) &= l_p^2(x) \prod_{j=1}^{k} \left\{ \frac{(x - y_j)(\xi_p - \xi_{p+j})}{(x - \xi_{p+j})(\xi_p - y_j)} \right\}^2 \\ &\equiv l_p^2(x) \cdot \prod_{j=1}^{k} h_j^2(x). \end{aligned}$$

It is easy to see that

$$(3.18)\qquad | h_j(x) | \leq \left| \frac{\xi_{p+j} - \xi_p}{x - \xi_{p+j}} \right| + \left| \frac{\xi_{p+j} - \xi_p}{\xi_p - y_j} \right| + \frac{(\xi_{p+j} - \xi_p)^2}{| x - \xi_{p+j} || \xi_p - y_j |}.$$

Then by (3.4) for $1 \leq j \leq k$ we have

$$| \xi_{p+j} - \xi_p | \leq c_6(k, \alpha) p \cdot m^{-2}$$

and if $x \notin I_{p,\mu}$ then for $0 \leq j \leq k$ we have

$$| x - \xi_{p+j} | \geq c_7(k, \alpha) p\mu \cdot m^{-2},$$

and thus by (3.18)

$$(3.19)\qquad | h_j(x)| \leq c_8(k, \alpha)\mu^{-1}, \qquad x \notin I_{p,\mu}.$$

Also by (3.4), (3.5) and (3.7) for $x \notin I_{p,\mu}$

$$(3.20)\qquad l_p^2(x) \leq c_9(k, \alpha) p^{2\alpha+1} \mu^{-2} \leq c_{10}(k, \alpha) \mu^{2\alpha-1}$$

where the last inequality follows from $p \leq 6k\mu$. From (3.17), (3.19) and (3.20)

$$(3.21)\qquad f_2(x) \leq c_{11}(k, \alpha) \mu^{-2k+2\alpha-1}, \qquad x \notin I_{p,\mu}$$

Now we observe that the function $f_2(x)$ is non-negative on $[-1, 1]$ and that for $\nu < p$ and $\nu > p + k$

$$(3.22)\qquad f_2(\xi_\nu) = 0.$$

On the other hand from (3.4) and (3.5) it follows easily as before

$$(3.23)\qquad f_2(\xi_{p+j}) \leq c_{12}(k, \alpha), \qquad 0 \leq j \leq k.$$

By (3.21) $f_2(x) < 1$ for $x \notin I_{p,\mu}$ if μ is sufficiently large. Since $f_2(\xi_p) = 1$, the maximum of $f_2(x)$ in $[-1, 1]$ is attained at some point $x_0 \in I_{p,\mu}$. Obviously $f_2(x_0) \geq 1$ and $x_0 = -1 + \lambda\mu m^{-2}$ with some λ, $1 \leq \lambda \leq k + 1$.

Let $f_3(x) = (f_2(x))^2 (f_2(x_0))^{-2}$. Then $f_3(x_0) = 1$ and thus by Lemma 4,

$$(3.24)\qquad \int_{-1}^{1} f_3(x)(1 - x^2)^\alpha \, dx \geq c_{13}(k, \alpha) \mu^\alpha \cdot m^{-2\alpha-2}.$$

On the other hand, since we may apply formula (1.4) to $f_3(x)$, we have

$$\begin{aligned} \int_{-1}^{1} f_3(x)(1-x^2)^\alpha\,dx &= B^{(n)} \sum_{j=1}^{n-k} f_3(x_j) \\ &\le B^{(n)} \sum_{j=1}^{n-k} (f_2(x_j))^2 \qquad (3.25) \\ &\le c_{11}(k,\alpha)\mu^{-2k+2\alpha-1} \cdot B^{(n)} \sum_{j=1}^{n-k} f_2(x_j) \end{aligned}$$

where the last inequality follows from (3.21) on using the fact that $x_j \notin I_{p,\mu}$.

Applying formula (1.4) and Gauss quadrature formula to $f_2(x)$, using (3.6), (3.22) and (3.23)

$$\begin{aligned} \int_{-1}^{1} f_2(x)(1-x^2)^\alpha\,dx &= B^{(n)} \sum_{j=1}^{n-k} f_2(x_j) = \sum_{\nu=0}^{k} \lambda_{p+\nu}^{(m)} f_2(\xi_{p+\nu}) \qquad (3.26) \\ &\le c_{14}(k,\alpha)\mu^{2\alpha+1} m^{-2\alpha-2}. \end{aligned}$$

Hence from (3.24), (3.25) and (3.26) we have

$$c_{15}(k,\alpha) \le \mu^{-2k+3\alpha}.$$

Since $2k > 3\alpha$ by hypothesis, the last inequality is impossible for sufficiently large μ. This contradiction proves the lemma.

4. Proof of Theorem 1

We shall prove the theorem for $k \ge 1$. The statement of the theorem for the case $k = 0$ is an immediate consequence of formulas (11) and (13) of Gatteschi's paper [4], and of formula (3.6) of this work.

If $k \ge 1$, by Lemma 3(ii) we may assume that $k > \frac{3}{2}\alpha$. Also, by Lemma 3(i) it is sufficient to prove the theorem when $\lim_{n\to\infty} N(n) = \infty$.

Let n be a sufficiently large integer and let $N(n)$ be the degree of exactness of a formula of type (1.4). Let μ be the integer whose existence was proved in Lemma 7, and $m \le N/4$. Consider the $k+1$ intervals

$$I_{(2\nu+1)(\mu+1),\mu+1}^{(m)} \quad (\nu = 0, 1, \cdots, k) \quad \text{where} \quad I_{i,j}^{(m)} = [\xi_{i-j}^{(m)}, \xi_{i+j}^{(m)}].$$

Then clearly at least one of these intervals is free of the y_i's of formula (1.4) under consideration. Suppose this happens for $\nu = \nu_1$. Then consider the subinterval $I_{(2\nu_1+1)(\mu+1),\mu}^{(m)}$ which is a fortiori free of y_i's and thus by Lemma 7, must include at least one of the x_j's, say x_1.

But then for x_1 all the conditions of Lemma 6 are satisfied with $K = (2k+2)(\mu+1)$. Hence by (3.8)

$$B^{(n)} \le C_3(k,\alpha)N^{-2\alpha-2}. \qquad (4.1)$$

If β_k is the coefficient of x^{2k} in $P_{2k}(x) \equiv P_{2k}^{(\alpha,\alpha)}(x)$ then from the minimization property of these polynomials, we have on using formula (1.4)

$$\begin{aligned}(4.2)\qquad \beta_k^{-1}\int_{-1}^{1} P_{2k}(x)(1-x^2)^\alpha\,dx &\le \int_{-1}^{1}\prod_{i=1}^{k}(x-y_i)^2(1-x^2)^\alpha\,dx\\ &\le B^{(n)}(n-k)2^{2k}.\end{aligned}$$

From (4.1) and (4.2)

$$(n-k)N^{-2\alpha-2} \ge C_4(k,\alpha)$$

whence

$$N \le C_5(k,\alpha)n^{1/(2\alpha+2)}$$

which concludes the proof of Theorem 1.

5. Distribution of nodes

We now turn to the result briefly indicated in §2 that if we assume certain restrictions on the nodes of a formula of type (1.4), then N is bounded. More precisely we shall prove

THEOREM 2. *Let $\delta > 0$ be fixed and let I_δ be any subinterval of* $[-1, 1]$ *of length δ. Suppose $p(x)$ of formula* (1.4) *satisfies*

$$(5.1)\qquad \int_{I_\delta} p(x)\,dx \ge \gamma > 0$$

and suppose that I_δ is free of the nodes x_j's and y_i's of a formula of type (1.4). *Then there exists an integer $N_0 = N_0(k, \delta)$ such that the formula* (1.4) *fails to be valid in general for polynomials of degree $\ge N_0$.*

Proof. Let x_0 be the midpoint of I_δ and denote the zeros of the Legendre polynomials $P_N(x)$ of order N by η_k $(1 \le k \le N)$, $\eta_0 = -1$, $\eta_{N+1} = 1$.

Suppose the formula (1.4) is valid for polynomials of degree $4N + 4k + 2$ and that N is so large that

$$\eta_{p+1} - \eta_p \le \delta/6 \quad \text{where} \quad \eta_p \le x_0 < \eta_{p+1}\,.$$

Set

$$(5.2)\qquad F(x) = \omega(x)\sum\nolimits_{|\eta_\nu - x|\le \delta/3} r_\nu^{(N)}(x)$$

where

$$\omega(x) = \prod_{i=1}^{k}(x-y_i)^2$$

$$(5.3)\qquad r_\nu^{(N)}(x) = \frac{1-x^2}{1-\eta_\nu^2}\left\{\frac{P_N(x)}{(x-\eta_\nu)P_N'(\eta_\nu)}\right\}^2, \qquad 1\le\nu\le N$$

$$r_0^{(N)}(x) = \frac{1+x}{2}P_N^2(x), \qquad r_{N+1}^{(N)}(x) = \frac{1-x}{2}P_N^2(x).$$

The polynomials $r_\nu^{(N)}(x)$ have been defined by Egerváry and Turán in the study of the stability problem of an interpolation process. From one of their results in [1] we obtain

$$\sum\nolimits_{|x-\eta_\nu|>\delta/6} r_\nu^{(N)}(x) < c_{16}(\delta)N^{-1} \tag{5.4}$$

uniformily in $-1 \leq x \leq 1$, and

$$\sum\nolimits_{\nu=0}^{N+1} r_\nu^{(N)}(x) \equiv 1. \tag{5.5}$$

Since $|x - x_0| \leq \delta/6$ and $|x - \eta_\nu| \leq \delta/6$ together imply $|x_0 - \eta_\nu| \leq \delta/3$, we have from (5.4) and (5.5)

$$F(x) \geq \omega(x) \sum\nolimits_{|x-\eta_\nu|\leq\delta/6} r_\nu^{(N)}(x) \geq (\delta/3)^{2k}(1 - c_{16}(\delta)N^{-1}). \tag{5.6}$$

On the other hand, since $|x - x_0| > \delta/2$ and $|x_0 - \eta_\nu| \leq \delta/3$ together imply $|x - \eta_\nu| \geq \delta/6$, we have again from (5.4) and (5.5) that for $|x - x_0| > \delta/2$,

$$F(x) < \omega(x) \sum\nolimits_{|x-\eta_\nu|>\delta/6} r_\nu^{(N)}(x) \leq 2^{2k}c_{16}(\delta)N^{-1}. \tag{5.7}$$

Also by (5.5), we have

$$F(x) \leq 2^{2k} \tag{5.8}$$

uniformly in $-1 \leq x \leq 1$. Applying the formula (1.4) to $F(x)$, we get from (5.8),

$$2^{2k}\int_{-1}^{1} p(x)\,dx \geq \int_{-1}^{1} F(x)p(x)\,dx = B^{(n)}\sum_{j=1}^{n-k} F(x_j). \tag{5.9}$$

Since, by hypothesis, $|x_j - x_0| > \delta/2$, $1 \leq j \leq n - k$, it follows from (5.7) on applying (1.4) to $F^2(x)$ that

$$\begin{aligned}\int_{-1}^{1} F^2(x)p(x)\,dx &= B^{(n)}\sum_{j=1}^{n-k} F^2(x_j)\\ &\leq 2^{2k}c_{16}(\delta)N^{-1}B^{(n)}\sum_{j=1}^{n-k} F(x_j).\end{aligned} \tag{5.10}$$

Combining (5.9) and (5.10), we get

$$\int_{-1}^{1} F^2(x)p(x)\,dx \leq c_{17}(\delta, k)N^{-1}. \tag{5.11}$$

On the other hand it follows from (5.6) that

$$\begin{aligned}\int_{-1}^{1} F^2(x)p(x)\,dx &\geq \int_{x_0-\delta/6}^{x_0+\delta/6} F^2(x)p(x)\,dx\\ &\geq \gamma\cdot(\delta/3)^{4k}(1 - c_{16}(\delta)N^{-1})^2\end{aligned} \tag{5.12}$$

so that from (5.11) and (5.12), we have

$$c_{17}(\delta, k)N^{-1} \geq \gamma\cdot(\delta/3)^{2k}(1 - c_{16}(\delta)N^{-1})^2$$

which is obviously impossible if N is sufficiently large. This contradiction proves the theorem.

Let $\rho > 0$ be fixed and let $I_{2\rho}$ be any closed subinterval of length 2ρ in $[-1, 1]$. Let I' be a subinterval of $I_{2\rho}$ having the same midpoint and of length $2\delta/3$, $\delta \leq \rho$. We can now formulate

THEOREM 3. *Suppose* $p(x) \geq 0, \delta^{-1} \int_{I'} p(x)\,dx \leq c_{17}$. *Suppose a formula of form* (1.4) *holds for polynomials of degree* $2N + 2k + 1$, $N = N(n)$ *and denote by* $\mu(\delta)$ *the number of* x_j*'s in* I'. *If* $y_i \notin I_{2\rho}$ $(1 \leq i \leq k)$ *then there exist constants* $c(k, \rho)$ *and* $N_0(k, \delta)$ *such that*

$$\mu(\delta)/n\delta \geq c(k, \rho) \quad \textit{implies} \quad N(n) \leq N_0(k, \delta). \tag{5.13}$$

Proof. Let x_0 be the midpoint of $I_{2\rho}$ and therefore also of I'. Applying formula (1.4) to the polynomial $F(x)$ given by (5.2), we have

$$\begin{aligned} \int_{-1}^{1} F(x)p(x)\,dx &= B^{(n)} \cdot \sum_{j=1}^{n-k} F(x_j) \\ &\geq B^{(n)} \cdot \sum_{|x_j - x_0| \leq \delta/6} F(x_j) \\ &\geq B^{(n)} \cdot (5\rho/6)^{2k} \cdot (1 - c_{16}(\delta)N^{-1})\mu(\delta) \end{aligned} \tag{5.14}$$

where the last inequality follows from the inequality $\rho \geq \delta$ and from (5.4) and (5.5).

On the other hand, from (5.7) and (5.8) we have

$$\begin{aligned} \int_{-1}^{1} F(x)p(x)\,dx &= \int_{|x_0 - x| \leq \delta/2} + \int_{|x_0 - x| > \delta/2} F(x)p(x)\,dx \\ &\leq 2^{2k} \int_{|x_0 - x| \leq \delta/2} p(x)\,dx + 2^{2k} c_{16}(\delta)N^{-1} \\ &\leq 2^{2k}(c_{17}\delta + c_{16}(\delta)N^{-1}). \end{aligned} \tag{5.15}$$

Also it is easy to see directly (or from the theory of orthogonal polynomials) that there exists a constant $c_{18}(k)$ such that

$$\int_{-1}^{1} \omega(x)p(x)\,dx \geq c_{18}(k).$$

Hence from formula (1.5)

$$c_{18}(k) \leq B^{(n)} \textstyle\sum_{j=1}^{n-k} \omega(x_j) \leq B^{(n)} \cdot (n - k) \cdot 2^{2k}. \tag{5.16}$$

Combining (5.14), (5.15) and (5.16) we see easily that if N is sufficiently large then $\mu(\delta)/n\delta < c(k, \rho)$, which completes the proof of the theorem.

Remark. The number $3\mu(\delta)/2n\delta$ is, in a way, the relative density of the x_j's in I'. This theorem shows that if the relative density of the x_j's is too high in a certain interval then a formula of the type (1.4) cannot be valid for polynomials of very high degree. In this sense Theorem 2 and 3 are complementary.

Bernstein's proof of the impossibility of Tchebicheff quadrature for $n \geq 10$ is based on a lemma which shows that the smallest node $x_1^{(n)}$ is smaller than the smallest zero of the Legendre polynomial of order m, $m = [(n-1)/2]$. An analogous result holds for the more general formula (1.4) and is formulated in

THEOREM 4. *Let $p(x) \geq \nu > 0$ and suppose a formula of type* (1.4) *is valid for polynomials of degree $\leq 2m+1$. Then for $m \geq m_0$, $m_0 = m_0(k)$ we have*

$$\min\,(x_1^{(n)}, -x_{n-k}^{(n)}) < \zeta_k^{(m)} \tag{5.17}$$

where $-1 < \zeta_1^{(m)} < \zeta_2^{(m)} < \cdots < \zeta_m^{(m)} < 1$ are the zeros of the m^{th} orthogonal polynomial $Q_m(x)$ with weight function $p(x)$ on $[-1, 1]$.

Proof. We shall omit the superscripts of $x_j^{(n)}$, $y_i^{(n)}$ and $\zeta_k^{(m)}$ where the intention is clear.

By Lemma 2, we may assume without loss of generality that

$$y_i \geq 0 \qquad \text{for} \quad i \geq [k/2]+1. \tag{5.18}$$

In this case we shall show that $x_1 < \zeta_k$. Set

$$\begin{aligned} R(x) &= (x-\zeta_k)Q_m^2(x)\textstyle\prod_{i=1}^{k}(x-y_i)^2(x-\zeta_i)^{-2} \\ S(x) &= R(x) - \{(x-\zeta_k) + 2\textstyle\sum_{i=1}^{k}(\zeta_i - y_i)\}Q_m^2(x). \end{aligned} \tag{5.19}$$

Then $S(x)$ is a polynomial of degree $2m-1$, so that by Gauss quadrature formula and by (5.19),

$$\int_{-1}^{1} S(x)p(x)\,dx = \sum_{i=1}^{k-1} \Lambda_i^{(m)} R(\zeta_i), \qquad \Lambda_i^{(m)} > 0$$

since $S(\zeta_i) = R(\zeta_i)$ for $1 \leq i \leq m$. Also from (5.19) it is easily seen that $R(\zeta_i) < 0$, $1 \leq i \leq k-1$, and since $\Lambda_i^{(m)} > 0$ for all i, we have

$$\int_{-1}^{1} S(x)p(x)\,dx < 0\,. \tag{5.20}$$

By a theorem of Erdös and Turán [6, Theorem (6.11.1) p. 111], it follows that $\lim_{m\to\infty}\zeta_\nu^{(m)} = -1$ for fixed ν. Then if $m \geq m_0 = m_0(k)$, we have $\zeta_i^{(m)} < -2/3$ for $1 \leq i \leq k$. Thus from (5.18), we get for $k \geq 1$,

$$-\zeta_k + 2\textstyle\sum_{i=1}^{k}(\zeta_i - y_i) < -\frac{2}{3}(2k-1) + 2[k/2] \leq 0. \tag{5.21}$$

Observing that

$$\int_{-1}^{1} xQ_m^2(x)p(x)\,dx = 0$$

we obtain from (5.19), (5.20) and (5.21) that

$$\int_{-1}^{1} R(x)p(x)\,dx < 0\,.$$

Hence using formula (1.5),

$$\int_{-1}^{1} R(x)p(x)\,dx = B^{(n)} \sum_{j=1}^{n-k} R(x_j) < 0\,.$$

Since by Lemma 1, $B^{(n)} > 0$ we have for some j, $R(x_j) < 0$. But $R(x) > 0$, if $x > \zeta_k$, so that $x_j < \zeta_k$ and a fortiori $x_1 < \zeta_k$, which completes the proof of the theorem.

6. Remarks

The above method can be modified to show that not all the x_j's and y_i's can be real if the quadrature formula of form (1.4) is to hold. Analogous problems for an infinite interval with $k = 0$ have been investigated by Ullman [7] and Wilf [8]. An extension of their results for a general k remains open. A further possible extension of our result could be the case when $k = k(n)$ tends to infinity at a certain rate. To these and related problems we propose to return later.

REFERENCES

1. E. EGERVÁRY AND P. TURÁN, *Notes on interpolation V (On the stability of interpolation)*, Acta Math. Acad. Sci. Hungar., vol. 9 (1958), pp. 259–267.
2. P. ERDÖS AND A. SHARMA, *On Tchebycheff quadrature*, Canad. J. Math., vol. 17 (1965), pp. 652–658.
3. P. ERDÖS, AND P. TURÁN, *On interpolation III (Interpolatory theory of polynomials)*, Ann. of Math., vol. 41 (1960), pp. 510–553.
4. L. GATTESCHI, *Su di un problema conuesso alle formula di quadratura di Tschebyscheff*, Univ. e Politec. Torino Rend. Sem. Mat., vol. 23 (1963/64), pp. 75–87.
5. I. P. NATANSON, *Konstruktive Funktionentheorie*, Akademie Verlag, Berlin, 1955, pp. 446–479.
6. G. SZEGÖ, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1959, revised edition.
7. J. L. ULLMAN, *Tchebycheff quadrature on the infinite interval*, Trans. Amer. Math. Soc., vol. 107 (1963), pp. 291–299.
8. H. S. WILF, *The possibility of Tschebycheff quadrature on infinite intervals*, Proc. Nat Acad. Sci. U.S.A., vol. 47 (1961), pp. 209–213.

UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA