COLLISIONS OF STABLE PROCESSES

BY

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1. Introduction

The primary motivation for this study is a result of Pólya [8] which states that two points starting simultaneously in the plane and performing, independently, simple random walks will meet infinitely often with probability one. Dvoretzky and Erdös [3] remark that three points starting simultaneously in the plane and performing independent simple random walks will not meet infinitely often; however, on the line they will meet with probability one. On the other hand, four points will not meet infinitely often even in R^1. The purpose of this paper is to consider some similar questions about stable processes. First we ask for what values of α and N will two independent stable processes of index α in R^N meet? This question can be answered very easily because if $X(t)$ and $Y(t)$ are the two processes and they start simultaneously, they will meet if and only if the process $X(t) - Y(t)$ returns to the origin. Since $X(t) - Y(t)$ is also a stable process of index α and a stable process of index α in R^N will return to the origin with positive probability (actually with probability one) if and only if $\alpha > 1$ and $N = 1$, this solves the problem of which values of α and N give rise to processes which will collide. It is also easy to verify that three independent stable processes can never have a simultaneous collision.

The next problem is then to find the Hausdorff dimension of the intersection when $\alpha > 1$ and $N = 1$, i.e. the dimension of the set

$$A = \{x : X(t) = Y(t) = x \quad \text{for some} \quad t > 0\} .$$

The time set on which $X(t) = Y(t)$ is the same as the set of zeroes of $X(t) - Y(t)$ so that this time set has Hausdorff dimension $1 - 1/\alpha$ [11]. If this set of times were not random, one could immediately conclude from [1] that the dimension of A is almost surely $\alpha(1 - 1/\alpha) = \alpha - 1$. Although no attempt will be made to carry out this line of attack, this result will be included as a particular case of Theorem 4.1.

These problems will be considered in somewhat more generality than indicated above as we will allow the two stable processes $X(t)$, $Y(t)$, to have different indices α and β. The first step then is to use some potential theory to obtain a comparison theorem between the process $(X(t), Y(t))$ in R^2 and the symmetric stable process $Z(t)$ in R^2 with index $\gamma = 1 + \beta - \beta/\alpha$. This

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is carried out in Section 3. The proof that the dimension of the intersection is almost surely $\beta - \beta/\alpha$ is given in Section 4.

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2. Preliminaries and notation

The characteristic function of a stable process $X(t)$ of index α in R^N has the form $\exp [t \cdot \psi(y)]$, where

$$\psi(y) = i(a, y) - \lambda \mid y \mid^{\alpha} \int_{S_N} w_{\alpha}(y, \theta)\mu(d\theta),$$

with $a \in R^N$, $\lambda > 0$,

$$w_{\alpha}(y, \theta) = [1 - i \operatorname{sgn}(y, \theta) \tan \pi\alpha/2] |(y/| y |, \theta)|^{\alpha}$$

for $\alpha \neq 1$, and μ is a probability measure on the surface of the unit sphere S_N in R^N [6]. $w_1(y, \theta)$ has a different form, but we will not need this in the present paper. The element $a \in R^N$ is taken to be zero. The process is called symmetric when μ is uniform. It is assumed that all the processes considered have been defined so as to have sample functions $X(t)$ which are right continuous and have left limits everywhere. The processes will also have the strong Markov property which will be used without further comment.

For $\alpha > N = 1$, the density function $f(t, x)$ of $X(t)$ is known to be positive, continuous, and bounded in x so that in particular there are positive constants c_1 and c_2 such that

$$c_1 \leq f(1, x) \leq c_2 \quad \text{for} \quad | x | \leq 1. \tag{2.1}$$

There is also a positive constant c_3 such that

$$| x |^{1+\alpha} f(1, x) \leq c_3 \quad \text{for} \quad | x | \geq 1; \tag{2.2}$$

see [9] for a summary of the behavior of the density. The constants c_1, c_2, c_3 will depend on which particular stable process we are discussing, but since there will only be a finite number of processes involved at any one time, we shall not indicate their dependence on the process. The density also satisfies the scaling property

$$f(t, x) = f(rt, r^{1/\alpha}x)r^{N/\alpha}$$

for all $r > 0$, or in terms of the process itself, $X(rt)$ and $r^{1/\alpha}X(t)$ have the same distribution.

A process $X(t)$ in R^N is called point-recurrent if for any x and y in R^N,

$$P^x[X(t) = y \quad \text{for some} \quad t > 0] = 1,$$

while it is called neighborhood-recurrent if for any x in R^N and any open sphere $G \subset R^N$,

$$P^x[X(t) \in G \quad \text{for some} \quad t > 0] = 1.$$

A Borel set B in R^N is said to be polar for X if

$$P^x[X(t) \in B \quad \text{for some} \quad t > 0] = 0$$

for all x in R^N.

Let $X_\alpha(t)$ and $Y_\beta(t)$ be two independent stable processes in R^1 with indices α and β respectively, where we assume for now that α and β are not both 2. (Each of these processes will have another parameter corresponding to the measure μ in the representation for the characteristic function, but this other parameter will be suppressed throughout the paper.) Let

$$U_{\alpha\beta}(x) = \int_0^\infty f_\alpha(t, x_1) f_\beta(t, x_2)\, dt \tag{2.3}$$

where $x = (x_1, x_2) \in R^2$ and f_α, f_β are the densities of X_α, Y_β. The integral converges for all $x \neq 0$ as we shall see from Lemma 3.1 below. The process $(X_\alpha(t), Y_\beta(t))$ in R^2 has a continuous density and the potential kernel of this process will have a density with respect to Lebesgue measure in R^2 given by $u(x, y) = U_{\alpha\beta}(y - x)$. It follows from the general theory of Hunt [5] (see also [2]) that if we let

$$W_{\alpha\beta}\, \mu(x) = \int U_{\alpha\beta}(y - x) \mu(dy) \tag{2.4}$$

be the potential of a measure μ, then a Borel set B is polar for $(X_\alpha(t), Y_\beta(t))$ if and only if $W_{\alpha\beta}\, \mu$ is unbounded for all finite non-zero measures with compact support contained in B. The same theory also applies to the symmetric stable process of index γ (<2) in R^2, where the potential of a measure μ will be

$$W_\gamma\, \mu(x) = \int \frac{c_4}{|y - x|^{2-\gamma}}\, \mu(dy). \tag{2.5}$$

3. A comparison theorem

$X_\alpha(t)$ and $Y_\beta(t)$ will denote independent, one-dimensional stable processes with indices α and β respectively. We will consider the process $(X_\alpha(t), Y_\beta(t))$ in R^2. When $\alpha \neq \beta$, this process is not stable, but it is similar in certain respects to a stable process and the main purpose of this section is to prove the relevant similarity.

THEOREM 3.1. *Let* $1 < \beta \leq \alpha \leq 2$, *and let* $X_\alpha(t)$, $Y_\beta(t)$ *be independent stable processes in* R^1. *Then the process* $(X_\alpha(t), Y_\beta(t))$ *has the same Borel polar subsets of the line* $y = x$ *in* R^2 *as does the symmetric stable process* $Z_\gamma(t)$ *in* R^2 *with index* $\gamma = 1 + \beta - \beta/\alpha$.

The following lemmas will be needed in the proof.

LEMMA 3.1. *Let* $x = (x_1, x_2)$ *and* $1 < \beta \leq \alpha \leq 2$, $\beta < 2$. *Then there exist positive constants* c_5, c_6 *such that*

$$\begin{aligned} U_{\alpha\beta}(x) &\leq c_5 |x_1|^{\alpha-1-\alpha/\beta} \quad \text{if} \quad |x_2|^\beta \leq |x_1|^\alpha \\ &\leq c_6 |x_2|^{\beta-1-\beta/\alpha} \quad \text{if} \quad |x_1|^\alpha \leq |x_2|^\beta. \end{aligned}$$

Proof. If $x = 0$, the inequality is trivial. Suppose $|x_2|^\beta \le |x_1|^\alpha$; we may assume $|x_1| > 0$. Using the scaling property, we have

$$\begin{aligned} U_{\alpha\beta}(x) &= \int_0^\infty f_\alpha(t, x_1)f_\beta(t, x_2)\,dt \\ &= \int_0^\infty f_\alpha(1, t^{-1/\alpha}x_1)f_\beta(1, t^{-1/\beta}x_2)t^{-1/\alpha-1/\beta}\,dt \\ &= \int_0^{|x_2|^\beta} \text{———} + \int_{|x_2|^\beta}^{|x_1|^\alpha} \text{———} + \int_{|x_1|^\alpha}^\infty \text{———}. \end{aligned}$$

Now we use the inequalities (2.1) and (2.2) on the densities to obtain

$$\begin{aligned} U_{\alpha\beta}(x) &\le c_3^2\,|x_1|^{-\alpha-1}\,|x_2|^{-\beta-1}\int_0^{|x_2|^\beta} t^2\,dt + c_3\,c_2\,|x_1|^{-\alpha-1}\int_{|x_2|^\beta}^{|x_1|^\alpha} t^{1-1/\beta}\,dt \\ &\qquad + c_2^2\int_{|x_1|^\alpha}^\infty t^{-1/\alpha-1/\beta}\,dt \\ &\le c_7\,|x_1|^{-\alpha-1}\,|x_2|^{2\beta-1} + c_8\,|x_1|^{\alpha-1-\alpha/\beta}. \end{aligned}$$

Since $|x_2|^{2\beta-1} = (|x_2|^\beta)^{2-1/\beta} \le (|x_1|^\alpha)^{2-1/\beta} = |x_1|^{2\alpha-\alpha/\beta}$ under the given inequality relating x_1 and x_2, this completes the proof in this case. The other estimate is obtained similarly or by simply interchanging the roles of x_1 and x_2, α and β.

LEMMA 3.2. *Let $1 < \beta \le \alpha \le 2$, $\beta < 2$. There exists a positive constant c_9 such that*

$$U_{\alpha\beta}(y - x) \le c_9\,|y - \bar{x}|^{\beta-1-\beta/\alpha},$$

where $y = (y_1, y_1)$, $x = (x_1, x_2)$, and $\bar{x} = (x_2, x_2)$.

Proof. If $|y_1 - x_1|^\alpha \le |y_1 - x_2|^\beta$, the inequality follows immediately from Lemma 3.1. If, on the other hand, $|y_1 - x_2|^\beta \le |y_1 - x_1|^\alpha$, then

$$\begin{aligned} U_{\alpha\beta}(y - x) &\le c_5\,|y_1 - x_1|^{\alpha-1-\alpha/\beta} = c_5(|y_1 - x_1|^\alpha)^{1-1/\alpha-1/\beta} \\ &\le c_5(|y_1 - x_2|^\beta)^{1-1/\alpha-1/\beta} = c_5\,|y_1 - x_2|^{\beta-1-\beta/\alpha}. \end{aligned}$$

LEMMA 3.3. *Let $1 < \beta \le \alpha \le 2$, $\beta < 2$. Then there is a positive constant c_{10} such that*

$$U_{\alpha\beta}(x) \ge c_{10}\,|x|^{\beta-1-\beta/\alpha} \quad \text{for} \quad |x| \le 1.$$

Proof. Using the scaling property and (2.1)

$$\begin{aligned} U_{\alpha\beta}(x) &\ge \int_{|x|^\beta}^\infty f_\alpha(1, t^{-1/\alpha}x_1)f_\beta(1, t^{-1/\beta}x_2)t^{-1/\alpha-1/\beta}\,dt \\ &\ge c_1^2\int_{|x|^\beta}^\infty t^{-1/\alpha-1/\beta}\,dt \end{aligned}$$

which gives the result. (Note that $|x_1|^\alpha \le |x|^\beta$ since $|x| \le 1$.)

Proof of Theorem 3.1. Note first that if $\alpha = \beta = 2$, the conclusion follows since all fully two dimensional processes with index 2 have the same polar sets. Hence we can assume that $1 < \beta \leq \alpha \leq 2$, $\beta < 2$. By the remarks between (2.4) and (2.5), it is enough to show that $W_{\alpha\beta}\,\mu(x)$ is bounded for some finite non-zero measure with compact support contained in B if and only if $W_\gamma\,\mu(x)$ is bounded. By Lemma 3.2, $W_{\alpha\beta}\,\mu(x) \leq c_9 c_4^{-1} W_\gamma\,\mu(\bar{x})$, so that if $W_\gamma\,\mu$ is bounded then so is $W_{\alpha\beta}\,\mu$. On the other hand, by Lemma 3.3,

$$\begin{aligned} W_\gamma\mu(x) &= \int_{|y-x|\leq 1} \frac{c_4}{|\,y-x\,|^{2-\gamma}}\,\mu(dy) + \int_{|y-x|>1} \frac{c_4}{|\,y-x\,|^{2-\gamma}}\,\mu(dy) \\ &\leq c_4 c_{10}^{-1} \int_{|y-x|\leq 1} U_{\alpha\beta}(y-x)\mu(dy) + c_4\mu(B) \\ &\leq c_4 c_{10}^{-1} W_{\alpha\beta}\mu(x) + c_4\mu(B). \end{aligned}$$

4. The dimension of the intersection

Let $1 < \beta \leq \alpha \leq 2$, and $X_\alpha(t)$, $Y_\beta(t)$ be independent stable processes in R^1 defined on a probability space $(\Omega, \mathfrak{F}, P)$. For $\omega \in \Omega$, we define

$$A(\omega) = \{x : X_\alpha(t, \omega) = Y_\beta(t, \omega) = x \quad \text{for some} \quad t > 0\}.$$

The principal result of the paper is

THEOREM 4.1. *If $1 < \beta \leq \alpha \leq 2$, then the Hausdorff dimension of the set $A(\omega)$ is almost surely equal to $\beta - \beta/\alpha$.*

In particular, if $\alpha = \beta$ then $\dim A(\omega)$ is $\alpha - 1$ as mentioned in the introduction.

Before starting the proof of the theorem, we shall need some lemmas.

LEMMA 4.1. *The process $X_\alpha(t) - Y_\beta(t)$ is point-recurrent if $1 < \beta \leq \alpha$.*

Proof. Since $1 < \beta \leq \alpha$, both $X_\alpha(t)$ and $Y_\beta(t)$ are point-recurrent; in particular, they are neighborhood-recurrent. Hence each process satisfies the Chung-Fuch's criterion for recurrence (see, e.g., Feller, p. 578 [4]). Using this fact it is easily checked that the process $X_\alpha(t) - Y_\beta(t)$ also satisfies this recurrence criterion. Hence the process $X_\alpha(t) - Y_\beta(t)$ is neighborhood-recurrent. Let

$$\Phi(x, y) = P^x[X_\alpha(t) - Y_\beta(t) = y \quad \text{for some} \quad t > 0].$$

To show that $X_\alpha(t) - Y_\beta(t)$ is point-recurrent we must demonstrate that $\Phi(x, y) = 1$ for all x and y in R^1. We first show that $\Phi(y, y) = 1$ for all y in R^1. We use Lemma 3.1 of [7] from which it follows directly that if $\exp[-t\psi(\xi)]$ is the characteristic function of $X_\alpha(t) - Y_\beta(t)$ and if $[\lambda + \operatorname{Re}\psi(\xi)]^{-1}$ is integrable for some $\lambda > 0$, then y is regular for $\{y\}$ for this process, and this implies $\Phi(y, y) = 1$. But we have

$$\int_{-\infty}^{\infty} \frac{d\xi}{\lambda + \operatorname{Re} \psi(\xi)} = \int_{-\infty}^{\infty} \frac{d\xi}{\lambda + a \mid \xi \mid^{\alpha} + b \mid \xi \mid^{\beta}},$$

where λ, a, b are positive constants and α, β are greater than 1, so the integral does converge. Therefore $\Phi(y, y) = 1$. Now $\Phi(\cdot, y)$ for y fixed is an excessive function with respect to the process $X_\alpha(t) - Y_\beta(t)$ (see [2] for the relevant definitions.) Furthermore, since the density of $X_\alpha(t) - Y_\beta(t)$ is continuous, it follows by an application of Fatou's Lemma that any excessive function is lower semi-continuous. Hence $\Phi(x, y)$ is lower semi-continuous as a function of x. We now show that $\Phi(x, y) = 1$ for all x and y. For any y in R^1 and $\varepsilon > 0$, there is a neighborhood G of y such that $\Phi(z, y) > 1 - \varepsilon$ for all $z \in G$ by the lower semicontinuity. Thus starting from any x in R^1, $X_\alpha(t) - Y_\beta(t)$ will enter a neighborhood G_1 of y, whose closure is contained in G, with probability one and then hit y at some later time with probability at least $1 - \varepsilon$, so that $\Phi(x, y) \geq 1 - \varepsilon$. This completes the proof of the lemma.

We remark here that point-recurrence of the process $X_\alpha(t) - Y_\beta(t)$ implies that the two dimensional process $(X_\alpha(t), Y_\beta(t))$ hits the line $y = x$ in R^2 with probability one, no matter where it starts. This fact will be used in the proof of Theorem 4.1.

Following Taylor [10], let $X_{\theta,N}(t)$ denote a symmetric stable process of index θ in R^N. For an analytic set A in R^N, let

$$\Phi_{\theta,N}(x, A) = P^x[X_{\theta,N}(t) \in A \quad \text{for some} \quad t > 0].$$

We shall need Theorem 4 of [10], which we state here as

LEMMA 4.2. *Suppose A is an analytic subset of R^1 or R^2. Then, for any x,*

$$\text{if} \quad A \subset R^1, \qquad \dim A = 1 - \inf\{\theta : \Phi_{\theta,1}(x, A) > 0\};$$

$$\text{if} \quad A \subset R^2, \qquad \dim A = 2 - \inf\{\theta : \Phi_{\theta,2}(x, A) > 0\}.$$

Proof of Theorem 4.1. We first show that $\dim A(\omega) \leq \beta - \beta/\alpha$ almost surely. Since the set $A(\omega)$ is linear, the case $\alpha = \beta = 2$ is trivial. By Lemma 4.2, it will suffice to show that for any positive $\theta < 1 - \beta + \beta/\alpha$, an independent symmetric stable process $X_{\theta,1}(t, \omega')$ running on the diagonal in R^2 hits $A(\omega)$ with probability zero. (Here we adopt for convenience the convention that $A(\omega)$ also refers to the diagonal of the set $A(\omega) \times A(\omega)$ in R^2.) If $X_{\theta,1}(t, \omega')$ is defined on the probability space $(\Omega', \mathfrak{F}', P')$, then $X_{\theta,1}(t, \omega')$, $X_\alpha(t, \omega)$, $Y_\beta(t, \omega)$ are all defined on the product space $(\Omega \times \Omega', \mathfrak{F} \times \mathfrak{F}', P \times P')$. We need to show that

$$P'\{\omega' : X_{\theta,1}(t, \omega') \in A(\omega) \quad \text{for some} \quad t > 0\} = 0 \tag{4.1}$$

for almost all ω. Let

$$\Gamma = \{(\omega, \omega') : X_{\theta,1}(t, \omega') \in A(\omega) \quad \text{for} \quad \text{some} \quad t > 0\};$$

then $\Gamma \in \mathfrak{F} \times \mathfrak{F}'$. Let $B(\omega')$ denote the range of $X_{\theta,1}(\cdot, \omega')$. Then Γ equals

$$\Gamma_1 = \{(\omega, \omega') : (X_\alpha(t, \omega), Y_\beta(t, \omega)) \in B(\omega') \quad \text{for some} \quad t > 0\}.$$

Since dim $B(\omega') = \theta$ a.s. (P') by [1], $B(\omega')$ is a.s. (P') polar for the symmetric process $X_{1+\beta-\beta/\alpha,2}(t)$ by Lemma 4.2. Now it is a consequence of Theorem 3.1 that $B(\omega')$ is polar for $(X_\alpha(t), Y_\beta(t))$, for almost all ω'. An application of Fubini's theorem gives that $P \times P'(\Gamma) = P \times P'(\Gamma_1) = 0$ and then (4.1).

We now show that dim $A(\omega) \geq \beta - \beta/\alpha$ almost surely. Choose θ so that $1 - \beta + \beta/\alpha < \theta < 2$ and consider an independent symmetric stable process $X_{\theta,1}(t, \omega')$ running on the diagonal. As in the other part, we find that the symmetric stable process $X_{1+\beta-\beta/\alpha,2}(t)$ will hit $B(\omega')$ with positive probability, and so the process $(X_\alpha(t), Y_\beta(t))$ will also by Theorem 3.1. (Note that since the density of the process is positive, $U_{\alpha\beta}(x) > 0$ for all x, consequently when a set is not polar for $(X_\alpha(t), Y_\beta(t))$ it will be hit with positive probability from any starting point.) Hence $P \times P'(\Gamma_1) > 0$, and so there is a T such that

$$P \times P'\{(\omega, \omega') : (X_\alpha(t, \omega), Y_\beta(t, \omega) \in B(\omega') \quad \text{for some} \quad t \in (0, T)\} > 0.$$

By Fubini's theorem there is a set $\wedge \in \mathfrak{F}$ with $P(\wedge) > 0$ such that if $\omega \in \wedge$ then

$$P'\{\omega' : X_{\theta,1}(t, \omega') \in A_T(\omega) \quad \text{for some} \quad t > 0\} > 0,$$

where

$$A_T(\omega) = \{(x, x) : X_\alpha(t, \omega) = Y_\beta(t, \omega) = x \quad \text{for some} \quad t \in (0, T)\}.$$

It then follows from Lemma 4.2 that dim $A_T(\omega) \geq 1 - \theta$ for $\omega \in \wedge$. In order to see that this is actually the case for almost all ω, let $\tau_0(\omega) = 0$ and for $n \geq 1$,

$$\tau_n(\omega) = \inf\{t > \tau_{n-1} + T : X_\alpha(t) = Y_\beta(t)\}.$$

The τ_n are all finite almost surely by Lemma 4.1. Define

$$G_n(\omega) = \dim\{(x, x) : X_\alpha(t) = Y_\beta(t) = x \quad \text{for some} \quad t \in (\tau_{n-1}, \tau_{n-1} + T)\};$$

the G_n are independent, identically distributed random variables with $G_1 =$ dim A_T. Also dim $A(\omega) \geq \sup_n G_n(\omega)$ so that

$$P[\dim A < 1 - \theta] \leq [1 - P(\wedge)]^n$$

for all n. Therefore dim $A(\omega) \geq 1 - \theta$ almost surely which concludes the proof of the theorem since θ was arbitrarily close to $1 - \beta + \beta/\alpha$.

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