VARIATION OF SYMMETRIC, ONE-DIMENSIONAL STOCHASTIC PROCESSES WITH STATIONARY, INDEPENDENT INCREMENTS

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1. Introduction

In this paper we consider symmetric, 1-dimensional stochastic processes with stationary, independent increments, which, in addition, have no Brownian component. The sample functions of such processes can be considered as functions from $[0, t]$ into $(-\infty, +\infty)$. We shall do this and answer certain variational questions about such functions.

2. Notation and standard facts

A text such as [2] *or* [6] is an appropriate reference for this section. As is usual we let X be a real-valued function on $[0, \infty) \times \Omega$ where Ω is some probability space with a probability measure P. Moreover, for each ω, we assume, as is usual, that $X(0, \omega) = 0$, that $X(\cdot, \omega)$ has left limits everywhere, and that $X(\cdot, \omega)$ is right continuous everywhere. We assume that X is a process as described in the introduction. It is well known that there is a one-to-one corrbspondence between such processes and so-called Levy measures ν on $(-\infty, +\infty) - \{0\}$ which are symmetirc and which have the property that

$$\int_{-\infty}^{+\infty} y^2(1 + y^2)^{-1}\nu\,(dy) < \infty.$$

If $F(t, \cdot)$ is the distribution function of $X(t, \cdot)$, this correspondence is expressed through the formula

$$\int_{-\infty}^{+\infty} e^{iux}\, d_2 F(t, x) = \exp\left\{-t \int_{-\infty}^{\infty} (1 - \cos uy)\nu\,(dy)\right\}$$

$$= \exp\left\{-2t \int_{0}^{\infty} (1 - \cos uy)\nu\,(dy)\right\}.$$

Symmetry and the inversion formula imply that

$$F(t, x) - \frac{1}{2} = \frac{1}{\pi}\int_0^\infty \frac{1}{u}\,[\sin ux]\exp\left\{-2t\int_0^\infty (1 - \cos uy)\nu\,(dy)\right\} du.$$

Let

$$J(t, \omega) = X(t, \omega) - X(t-, \omega).$$

If A is a Borel subset of $[0, \infty) \times [(-\infty, \infty) - \{0\}]$, we let $N(A, \omega)$ equal the number of t such that $(t, J(t, \omega)) \in A$. If $\{A_\alpha\}$ is a family of disjoint subsets

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of $[0, \infty) \times [(-\infty, \infty) - \{0\}]$, then $\{N(A_\alpha, \cdot)\}$ is a family of independent random variables. Furthermore, $N(A, \cdot)$ is Poisson distributed with expectation $(\lambda \times \nu)(A)$ [possibly $+\infty$] where λ is Lebesgue measure. Finally

$$X(t, \cdot) = \lim_{n\to\infty} \int_{|y|>1/n} yN([0, t] \times dy, \cdot) \tag{1}$$

with probability one.

The comments in the last paragraph were stated for symmetric processes; however many of them are also true for subordinators; i.e. for increasing processes with stationary independent increments having no deterministic linear component. The differences can be summarized briefly. The measure ν should be concentrated on $(0, \infty)$ and should satisfy the condition

$$\int_0^\infty y(1 + y)^{-1}\nu\,(dy) < \infty.$$

Also,

$$\int_0^\infty e^{-ux}\, d_2F(t, x) = \exp\left\{-t\int_0^\infty (1 - e^{-uy})\nu\,(dy)\right\}, \quad \operatorname{Re} u \geq 0.$$

Of course, the inversion formula is more complicated.

Let h be a monotone increasing function from $[0, \infty)$ into $[0, \infty)$ with $h(0) = 0$. We define

$$X_h(t, \omega) = \int_{-\infty}^{\infty} h(|y|)N([0, t] \times dy, \omega) = \textstyle\sum_{\tau\leq t} h(|J(\tau, \omega)|).$$

If $\int_0^1 h(y)\nu(dy) = \infty$, then the argument of [3, p. 32] shows that $X_h(t, \cdot) = \infty$ with probability 1 if $t > 0$. On the other hand, if $\int_0^1 h(y)\nu(dy) < \infty$, then from (1) it follows that X_h is the subordinator determined by ν_h where $\nu_h(B) = 2\nu(h^{-1}(B))$ [B a Borel subset of $(0, \infty)$].

Let f be a random function. Then we make the following definition of the h-variation of f through time t.

Definition 1. For each n let $0 = t_{n,0} < t_{n,1} < \cdots < t_{n,k(n)} = t$ be a subdivision of $[0, t]$. If $\delta(n)$ is the norm of this subdivision, assume that $\lim_{n\to\infty} \delta(n) = 0$. Then we define, if it exists (possibly infinite),

$$(v_h f)(t) = \mathrm{P}\lim_{n\to\infty} \textstyle\sum_{i=1}^{k(n)} h(|f(t_{n,i}) - f(t_{n,i-1})|).$$

Actually $(v_h f)(t)$ depends on the sequence of subdivisions used, but we suppress this dependence in our notation.

We shall sometimes omit ω from our notation. Finally, for our theorem we shall need more restrictions on h than those mentioned above. Therefore, we have

Definition 2. We let M be the class of all functions h from $[0, \infty)$ into $[0, \infty)$ such that $h(0) = 0$, $[h'(y)/y] \geq 0$, $[h'(y)/y]' \leq 0$, $[h'(y)/y]'' \geq 0$, and $[h'(y)/y]''' \leq 0$.

3. Proof that $v_h X = X_h$

THEOREM. *If $h \in M$ and if X is symmetric, then*

$$P\{(v_h X)(t) = X_h(t)\} = 1.$$

Proof. The theorem follows immediately from two facts:

$$X_h(t) \le \liminf_{n\to\infty} \sum_{i=1}^{k(n)} h(|X(t_{n,i}) - X(t_{n,i-1})|); \tag{2}$$

$$\sum_{i=1}^{k(n)} h(|X(t_{n,i}) - X(t_{n,i-1})|) \to X_h(t) \tag{3}$$

in distribution as $n \to \infty$, if $\int_0^1 h(y)\nu(dy) < \infty$. The first of these two facts is obvious so we content ourselves with proving the second. We should note that $\int_0^1 h(y)\nu(dy) < \infty$ is equivalent to $\int_0^1 \nu[y, \infty)h'(y)\,dy < \infty$.

We use the central convergence criterion on page 311 of [6]. A few easy manipulations show us that we have only to prove that

$$\sum_{i=1}^{k(n)} [F(t_{n,i} - t_{n,i-1}, -h^{-1}(x))] + [1 - F(t_{n,i} - t_{n,i-1}, h^{-1}(x))] \tag{4}$$
$$\to t\nu_h(x, \infty) \quad \text{as} \quad n \to \infty \quad \text{if} \quad x > 0 \quad \text{and} \quad \nu_h\{x\} = 0;$$

and

$$\sum_{i=1}^{k(n)} \int_{-c}^{c} h(|x|)\, d_2 F(t_{n,i} - t_{n,i-1}, x) \to t \int_0^c x\nu_h\,(dx) \tag{5}$$
$$\text{as} \quad n \to \infty \quad \text{for some} \quad c > 0 \quad \text{such that} \quad \nu_h\{c\} = 0.$$

We know that $X(t) = \sum_{i=1}^{k(n)} X(t_{n,i}) - X(t_{n,i-1})$.

From this fact and the central convergence criterion we conclude that

$$\sum_{i=1}^{k(n)} F(t_{n,i} - t_{n,i-1}, x) \to t\nu(-\infty, x)$$
$$\text{as} \quad n \to \infty \quad \text{if} \quad x < 0 \quad \text{and} \quad \nu\{x\} = 0;$$

and

$$\sum_{i=1}^{k(n)} [1 - F(t_{n,i} - t_{n,i-1}, x)] \to t\nu(x, +\infty)$$
$$\text{as} \quad n \to \infty \quad \text{if} \quad x > 0 \quad \text{and} \quad \nu\{x\} = 0.$$

Clearly, (4) now follows.

Let

$$\psi_n(y) = t\nu[y, \infty) - \sum_{i=1}^{k(n)} [1 - F(t_{n,i} - t_{n,i-1}, y)].$$

Integration by parts shows us that (5) is equivalent to

$$\lim_{n\to\infty} \int_0^c \psi_n(y)h'(y)\,dy = 0.$$

Fatou's lemma implies that

$$\limsup_{n\to\infty} \int_0^c \psi_n(y)h'(y)\,dy \le 0. \tag{6}$$

Thus, we want to show that

$$\liminf_{n\to\infty} \int_0^c \psi_n(y)h'(y)\, dy \geq 0. \tag{7}$$

The key to the proof is to notice that the function $2a - 3 \sin a + a \cos a$ is non-negative if $a \geq 0$, that this function and its first three derivatives are zero at zero, and that the third derivative is $a \sin a$. The fact that we arranged so that the third derivative turns out to equal $a \sin a$ rather than the first or second is only crucial because we need a non-negative function with which to work. The fact that the third derivative is involved is the reason for the appearance of triple integrals in the following calculation:

$$\sum_{i=1}^{k(n)} \int_0^x \int_0^w \int_0^c y[1 - F(t_{n,i} - t_{n,i-1}, y)]\, dy\, dc\, dw$$

$$= \sum_{i=1}^{k(n)} \left\{ \frac{x^4}{48} - \frac{1}{\pi} \int_0^\infty \frac{1}{u^5} (2ux - 3 \sin ux + ux \cos ux) \cdot \exp\left[-2(t_{n,i} - t_{n,i-1}) \int_0^\infty (1 - \cos uz)\nu\, (dz) \right] du \right\}$$

$$\leq \sum_{i=1}^{k(n)} \left\{ \frac{x^4}{48} - \frac{1}{\pi} \int_0^\infty \frac{1}{u^5} (2ux - 3 \sin ux + ux \cos ux) \cdot \left[1 - 2(t_{n,i} - t_{n,i-1}) \int_0^\infty (1 - \cos uz)\nu\, (dz) \right] du \right\}$$

$$= t \int_0^x \int_0^w \int_0^c z\nu[z, \infty)\, dz\, dc\, dw;$$

where the two equalities both result in a straightforward manner from several tedious integrations by parts; and, in the case of the second equality, one easy application of Fubini's theorem. Thus, we have shown that $\phi(x) \geq 0$, where

$$\int_0^x \int_0^w \int_0^c y\psi_n(y)\, dy\, dc\, dw = \phi(x').$$

We now perform a large number of integrations by parts. The facts that $h \in M$ and $\int_0^1 h'(y)\nu[y, \infty)\, dy < \infty$ enable us to conclude that terms evaluated at the lower limits are zero We obtain, after these long but straightforward calculation, the formula

$$\int_0^x \int_0^w \int_0^c h'(y)\psi_n(y)\, dy\, dc\, dw = [h'(x)/x]\phi(x) - 3 \int_0^x [h'(w)/w]'\phi(w)\, dw$$

$$+ 3 \int_0^x \int_0^w [h'(c)/c]''\phi(c)\, dc\, dw - \int_0^x \int_0^w \int_0^c [h'(y)/y]'''\phi(y)\, dy\, dc\, dw$$

which is non-negative since $h \in M$. Hence, by Fatou's lemma applied to a sequence of functions bounded above by an integrable function we conclude that

$$\lim_{n\to\infty} \int_0^x \int_0^w \int_0^c \psi_n(y)h'(y)\, dy\, dc\, dw = 0. \tag{8}$$

If (7) were not true, then it would not be true for c in a set of positive Lebesgue measure. Then it would be possible to introduce two more integrations and maintain a strict inequality, contradicting (8)—the fact that we maintain a strict inequality follows from (6) and still another application of Fatou's lemma.

Remark 1. If $h(y) = y^2$, then $\int_0^1 h(y)\nu(dy) < \infty$ and $h \in M$. If $h(y) = y^r$, $0 \leq r \leq 2$, then $h \in M$.

Remark 2. The situation for Brownian motion is discussed beginning on page 205 of [5].

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