ON THE EXISTENCE AND REPRESENTATION OF INTEGRALS

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1. Introduction

Suppose that Ω is a set, R is a non-empty collection of subsets of Ω, and D is the collection of finite non-empty subsets of R to which M belongs only in case M^*, the union of all the members of M, is in R and the members of M are relatively prime in R, i.e., if A and B are in M then there is no non-empty member of R which is contained in both A and B. We will assume that each non-empty A in R contains a point x such that if M is in D and A is in M then no other member of M contains x.

Let $B(\Omega, R)$ denote the closure in the space of functions from Ω to the number-plane which have bounded final sets of the linear space spanned by the characteristic functions of members of R with respect to the supremum norm $|\cdot|$. We will assume that $B(\Omega, R)$ is an algebra. An *integral* on $B(\Omega, R) \times R$ is a function K from $B(\Omega, R) \times R$ to the number-plane such that (1) for each (f, A) in $B(\Omega, R) \times R$, $K[\ \ , A]$ is a linear functional on $B(\Omega, R)$ and $K[f, \ \ ]$ is *additive* on R, i.e., $K(f, M^*) = \sum_{H \text{ in } M} K(f, H)$ for each M in D, and (2) there is an additive function λ from R to the non-negative numbers such that $|K(f, A)| \leq |1_A f| \lambda(A)$, for each (f, A) in $B(\Omega, R) \times R$. This paper is concerned with the existence and representation of integrals on $B(\Omega, R) \times R$.

2. Bounded variation

A finite subset M of R is said to *partition* a member A of R provided $M^* = A$. If each of M_1 and M_2 is a finite subset of R then M_2 is said to *refine* M_1 provided that $M_1^* = M_2^*$ and each member of M_2 is contained in some member of M_1. If (A, B) is in $R \times R$ then $[A, B]$ will denote the collection of non-empty members of R which are contained in both A and B. A subset A of Ω is said to be *R-measurable* if for each B in R there is a partition M of B in D such that each H in M is either contained in A or $[H, A] = \emptyset$ and if $[A, B] \neq \emptyset$ then the common part of A and B is the union of those members of M contained in A.

THEOREM 2.1. *If each member of R is R-measurable, each of M_1 and M_2 is in D, and $M_1^* = M_2^*$, then there is a member M of D which refines each of M_1 and M_2 such that each A in M_1 is the union of those members of M contained in A.*

Proof. Let $\{B_p\}_1^n$, be a reversible sequence with final set M_2. There is a sequence $\{N_p\}_0^n$ with values in D such that $N_0 = M_1$ and, for each integer p in $[1, n]$,

(1) N_p is a refinement of N_{p-1} such that each A in N_{p-1} is the union of those members of N_p contained in A, and

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(2) if A is in N_p then either A is contained in B_p or $[A, B_p] = \emptyset$.

N_n is a refinement of M_1 such that each A in M_1 is the union of those members of N_n contained in A. Suppose that A is in N_n and x is a member of A which is not in $(M - \{A\})^*$, for any M in D which contains A. There is a member of M_2 which contains x and hence a member of M_2 which contains A. Therefore N_n refines M_2.

COROLLARY. *If each of M_1 and M_2 is in D and M_2 refines M_1, then each A in M_1 is the union of those members of M_2 contained in A.*

Proof. Let M be a member of D which refines each of M_1 and M_2 such that each A in M_1 is the union of those members of M contained in A. If A is in M_1 then, since each member of M is contained in a member of M_2, A is the union of those members of M_2 contained in A.

We will assume from this point that each member of R is R-measurable and if (A, B) is in $R \times R$ then there is a member of D which contains a partition of each of A and B.

A function W from R to the plane is said to have *bounded variation* on a member A of R provided there is a number k such that $\sum_{H \text{ in } M} |W(H)| \leq k$, for each member M of D which partitions A. If W has bounded variation on A then we will denote the least such number by $\int_A |W|$. Let BV denote the set of additive functions from R to the plane to which W belongs only in case W has bounded variation on each member of R.

THEOREM 2.2. *If W is in BV then the set of ordered pairs λ to which (A, k) belongs only in case A is in R and $k = \int_A |W|$ is an additive function from R to the non-negative numbers.*

Proof. Suppose that M is in D and N is a function from M into D such that, for each H in M, $N(H)$ partitions H. Then

$$\sum_{H \text{ in } M} \sum_{G \text{ in } N(H)} |W(G)| \leq \int_{M^*} |W|.$$

Hence $\sum_{H \text{ in } M} \lambda(H) \leq \lambda(M^*)$. Suppose that M' is a member of D which partitions M^*. There is a member M'' of D which refines each of M and M'.

$$\begin{aligned} \sum_{H \text{ in } M'} |W(G)| &\leq \sum_{H \text{ in } M'} \sum_{G \text{ in } M'', G \subseteq H} |W(G)| \\ &= \sum_{H \text{ in } M} \sum_{G \text{ in } M'', G \subseteq H} |W(G)| \\ &\leq \sum_{H \text{ in } M} \lambda(H). \end{aligned}$$

Hence $\lambda(M^*) \leq \sum_{H \text{ in } M} \lambda(H)$ and so λ is additive.

3. An existence theorem

A *choice function* ϕ for R is a function from R into Ω such that (1) $\phi(H)$ is contained in H, for each H in R, and (2) if each of A and B is in R then there is a member M of D which partitions B such that if H is a member of M which contains a point of A, G is a member of R contained in H, and G contains a point of A, then $\phi(G)$ is in A only in case $\phi(H)$ is in A. A member A of R is said to be *properly situated* relative to a member B of R with respect to a collection of choice functions Φ on R provided either A and B are disjoint or for each ϕ in Φ and each member H of R which is contained in A and contains a point of B we have $\phi(H)$ is in B only in case $\phi(A)$ is in B.

THEOREM 3.1. *There is a choice function for R.*

Proof. There is a function ϕ from R into Ω such that, for each A in R and M in D which contains A, $\phi(A)$ is contained in A but no other member of M. Suppose that each of A and B is in R and M is a member of D which partitions B such that for each H in M either H is contained in A or $[H, A] = \emptyset$. Suppose that H is a member of M and G is a member of R which is contained in H and contains a point of A. If $\phi(G)$ is in A then G is contained in A and so H is contained in A. Hence $\phi(H)$ is in A. If $\phi(H)$ is in A then, similarly, $\phi(G)$ is in A. Therefore ϕ is a choice function for R.

THEOREM 3.2. *If ϕ is a choice function for R, (A, B) is in $R \times R$, and W is in BV then $\int_B 1_A\,[\phi]W$ exists.*

LEMMA. *For each positive number b there is a member M of D which partitions B such that*

$$\textstyle\sum_{H \text{ in } M, H\cap A\neq\emptyset}\sum_{G \text{ in } M', G\subseteq H, G\cap A=\emptyset} |W(G)| < b,$$

for each refinement M' of M in D.

Proof of the lemma. Suppose that the lemma is false. Then there is a positive number b and a sequence M with values in D such that $M(0)^* = B$ and, for each positive integer n, $M(n)$ refines $M(n-1)$ and

$$\textstyle\sum_{H \text{ in } M(n-1), H\cap A\neq\emptyset} \sum_{G \text{ in } M(n), G\subseteq H, G\cap A=\emptyset} |W(G)| \geq b.$$

If n is a positive integer then

$$\begin{aligned}
\int_B |W| &\geq \sum_{H \text{ in } M(0), H\cap A\neq\emptyset} \int_H |W| \\
&= \sum_{H \text{ in } M(0), H\cap A\neq\emptyset}\ \sum_{G \text{ in } M(1), G\subseteq H, G\cap A=\emptyset} \int_G |W| \\
&\quad + \sum_{H \text{ in } M(0), H\cap A\neq\emptyset}\ \sum_{G \text{ in } M(1), G\subseteq H, G\cap A\neq\emptyset} \int_G |W| \\
&\geq b + \sum_{G \text{ in } M(1), G\cap A\neq\emptyset} \int_G |W| \\
&\geq nb + \sum_{G \text{ in } M(n), G\cap A\neq\emptyset} \int_G |W|.
\end{aligned}$$

This contradicts the assumption that W is in BV.

Proof of Theorem 3.2. If A and B are disjoint then we are through. Suppose that A contains a point of B and b is a positive number. There is a member M of D which partitions B with the property that if H is in M then H is properly situated relative to A with respect to $\{\phi\}$. There is a member M' of D which refines M such that

$$\sum_{H \text{ in } M', H\cap A\neq\emptyset} \sum_{G \text{ in } M'', G\subseteq H, G\cap A=\emptyset} | W(G) | < b,$$

for each M'' in D which refines M'. Hence for each M'' in D which refines M' we have

$$\begin{aligned} &| \textstyle\sum_{M'} 1_A [\phi]W - \sum_{M''} 1_A [\phi]W | \\ &\qquad = | \textstyle\sum_{H \text{ in } M'} \sum_{G \text{ in } M'', G\subseteq H} \{1_A(\phi(H)) - 1_A(\phi(G))\} W(G) | \\ &\qquad \leq \textstyle\sum_{H \text{ in } M', H\cap A\neq\emptyset} \sum_{G \text{ in } M'', G\subseteq H, G\cap A=\emptyset} | W(G) | < b. \end{aligned}$$

Therefore $\int_B 1_A[\phi]W$ exists.

COROLLARY. *If ϕ is a choice function, A is in R, W is in BV, and f is in $B(\Omega, R)$ then $\int_A f[\phi]W$ exists.*

An integral K on $B(\omega, R) \times R$ is called a *refinement integral* provided there is a positive integer n and a sequence $\{\phi_p, W_p\}_1^n$, where, for $p = 1, 2, \cdots, n$, ϕ_p is a choice function for R and W_p is in BV, such that

$$K(f, A) = \sum_{p=1}^{n} \int_A f[\phi_p]W_p,$$

for each f in $B(\Omega, R)$ and A in R. Mac Nerney [1] has provided a partial answer to the question of what integrals are refinement integrals. In the next section we will extend Mac Nerney's representation theorem to give a better but still incomplete answer.

4. A representation theorem

A choice function ϕ_1 for R is said to *precede* a choice function ϕ_2 for R provided that if each of A and B is in R and A is properly situated relative to B with respect to $\{\phi_1, \phi_2\}$ then $1_B(\phi_1(A)) \leq 1_B(\phi_2(A))$. Suppose that Φ is a collection of choice functions for R. For each ϕ in Φ let f_ϕ denote the set of ordered pairs to which (x, k) belongs only in case x is an ordered pair (A, B) in $R \times R$ such that A is properly situated relative to B with respect to Φ and k is the least non-negative number m such that $1_B(\psi(A)) \leq m$, for each ψ in Φ different from ϕ which precedes ϕ. The collection Φ is said to be *complete* provided (1) if each of ϕ_1 and ϕ_2 is in Φ, ϕ_1 precedes ϕ_2, and ϕ_2 precedes ϕ_1 then $\phi_1 = \phi_2$; (2) if each of A, B, and C is in R, C is properly situated relative to each of A and B with respect to Φ, ϕ is a member of Φ, and

$$1_A(\phi(C)) - f_\phi(C, A) = 1 = 1_B(\phi(C)) - f_\phi(C, B)$$

then $[A, B] \neq \emptyset$ and the common part of A and C is the common part of B and

C; and (3) if each of A and B is in R, A is properly situated relative to B with respect to Φ, and A contains a point of B, then there is only one member ϕ of Φ such that $1_B(\phi(A)) - f_\phi(A, B) = 1$.

Furthermore, for each ϕ in Φ, let $I(\phi)$ denote the subset of Φ to which λ belongs only in case $\lambda \neq \phi$, ϕ precedes λ, and if λ' is in Φ and ϕ precedes λ' and λ' precedes λ then either $\lambda' = \lambda$ or $\lambda' = \phi$. Let $I^0(\phi)$ denote the set $\{\phi\}$ and if n is a positive integer let $I^{n+1}(\phi)$ denote the subset of Φ to which λ belongs only in case there is a member λ' of $I^n(\phi)$ such that $I(\lambda')$ contains λ. The collection Φ is said to be *coherent* provided if each of p and q is a non-negative number and F is a function from Φ to the plane then

$$\sum_{\lambda \text{ in } I^p(\phi)} \sum_{\mu \text{ in } I^q(\lambda)} F(\mu) = \binom{p+q}{q} \sum_{\nu \text{ in } I^{p+q}(\phi)} F(\nu).$$

THEOREM 4.1. *If K is an integral on $B(\Omega, R) \times R$, and Φ is a finite complete collection of choice functions for R which is coherent then there is a function W from Φ into BV such that*

$$K(f, A) = \sum_{\phi \text{ in } \Phi} \int_A f[\phi] W_\phi ,$$

for each (f, A) in $B(\Omega, R) \times R$.

Our proof of Theorem 4.1 follows in outline Mac Nerney's proof of Theorem 1 [1, p. 322] and requires the introduction as an intermediate step of a function V from Φ into BV from which W will be constructed. If each of M and M' is in D then M' is called a *proper refinement* of M with respect to Φ provided that M' is a refinement of M and if (A, B) is in $M' \times M$ then A is properly situated relative to B with respect to Φ. Let $V(\phi)$, for each ϕ in Φ, denote the set of ordered pairs to which (A, k) belongs only in case A is in R, k is a complex number, and for each positive number b there is a member M of D which contains a partition of A such that

$$|k - \textstyle\sum_{H \text{ in } M'} \sum_{G \text{ in } M'', G \subset A} K(\{1_H(\phi(G)) - f_\phi(G, H)\}1_H, G)| < b,$$

for each member M' of D which contains a refinement of M and each member M'' of D which is a proper refinement of M' with respect to Φ.

THEOREM 4.2. *V is a function from Φ into BV.*

Proof. Suppose that ϕ is a member of Φ and A is a member of R. If M is a member of D which contains a partition of A and M' is a proper refinement of M with respect to Φ then

$$\begin{aligned} &\textstyle\sum_{H \text{ in } M} \sum_{G \text{ in } M', G \subseteq A} |K(\{1_H(\phi(G)) - f_\phi(G, H)\}1_H, G)| \\ &\qquad \leq \textstyle\sum_{H \text{ in } M} \sum_{G \text{ in } M', G \subseteq A} \{1_H(\phi(G)) - f_\phi(G, H)\}\lambda(G) \leq \lambda(A). \end{aligned}$$

Suppose that each of M, M', and M'' is a member of D, M contains a partition of A, M' contains a refinement of M, and M'' contains a proper refinement of

M with respect to Φ and a proper refinement of M' with respect to Φ. For each F in M, let $N(F)$ denote the subset of M'' to which H belongs only in case H is contained in A and $1_F(\phi(H)) - f_\phi(H, F) = 1$ and $N'(F)$ the subset of M'' to which H belongs only in case H is contained in A and $1_G(\phi(H)) - f_\phi(H, G) = 1$, for some G in M' which is contained in F. $N(F)$ is contained in $N'(F)$, for each F in M.

If F is in M then

$$\begin{aligned}\textstyle\sum_{G \text{ in } M', G \subseteq F} \sum_{H \text{ in } N(F)} K(\{1_G(\phi(H)) - f_\phi(H, G)\}1_G, H) \\ = \textstyle\sum_{H \text{ in } N(F)} \sum_{G \text{ in } M', G \subseteq F} K(\{1_G(\phi(H)) - f_\phi(H, G)\}1_F, H) \\ = \textstyle\sum_{H \text{ in } N(F)} K(1_F, H).\end{aligned}$$

Hence

$$\begin{aligned}&|\textstyle\sum_{F \text{ in } M} \{\sum_{H \text{ in } N(F)} K(1_F, H) - \sum_{G \text{ in } M', G \subseteq F} \sum_{H \text{ in } N'(F)} K(\{1_G(\phi(H)) \\ &\quad - f_\phi(H, G)\}1_G, H)\}| \\ &= |\textstyle\sum_{F \text{ in } M} \sum_{G \text{ in } M', G \subseteq F} \sum_{H \text{ in } N'(F) - N(F)} K(\{1_G(\phi(H)) - f_\phi(H, G)\}1_G, H)| \\ &\leq \textstyle\sum_{F \text{ in } M} \sum_{G \text{ in } M', G \subseteq F} \sum_{H \text{ in } N'(F) - N(F)} \{1_G(\phi(H) - f_\phi(H, G)\}\lambda(H) \\ &= \textstyle\sum_{G \text{ in } M'} \sum_{H \text{ in } M'', H \subseteq A} \{1_G(\phi(H)) - f_\phi(H, G)\}\lambda(H) \\ &\quad - \textstyle\sum_{F \text{ in } M} \sum_{H \text{ in } M'', H \subseteq A} \{1_F(\phi(H)) - f_\phi(H, F)\}\lambda(H).\end{aligned}$$

Therefore A is in the initial set of $V(\phi)$. It is easily seen that $V(\phi)$ is additive on R.

THEOREM 4.3. *If each of A and B is in R, A is properly situated relative to B with respect to Φ, ϕ is in Φ, and $1_B(\phi(A)) - f_\phi(A, B) = 1$, then*

$$K(1_B, A) = \int_A 1_B[\phi]V_\phi.$$

Proof. Suppose that b is a positive number. There is a member N of D which partitions A such that

$$\left| \int_A 1_B[\phi]V_\phi - \sum_{H \text{ in } N'} 1_B(\phi(H))V_\phi(H) \right| < b/3,$$

for each member $N \leq$ of D which refines N. There is a member M of D which refines N such that if M' is a member of D which refines M then

$$\textstyle\sum_{H \text{ in } M, H \cap B \neq \emptyset} \sum_{G \text{ in } M', G \subseteq H, G \cap B = \emptyset} \lambda(G) < b/3.$$

There is a member M' of D which contains a refinement of each of $\{B\}$ and M such that

$$\begin{aligned}&\textstyle\sum_{H \text{ in } M, H \cap B \neq \emptyset} | V_\phi(H) \\ &\quad - \textstyle\sum_{F \text{ in } M'} \sum_{G \text{ in } M'', G \subseteq H} K(\{1_F(\phi(G)) - f_\phi(G, F)\}1_F, G)| < b/3,\end{aligned}$$

for each member M'' of D which is a proper refinement of M' with respect to Φ. If M'' is a member of D which is a proper refinement of M' with respect to Φ then

$$
\begin{aligned}
&\left| K(1_B, A) - \int_A 1_B[\phi]V_\phi \right| \\
&\quad \le \left| \int_A 1_B[\phi]V_\phi - \sum_{H \text{ in } M} 1_B(\phi(H))V_\phi(H) \right| \\
&\qquad + \textstyle\sum_{H \text{ in } M, H\cap B\neq\emptyset} \mid V_\phi(H) - \sum_{F \text{ in } M'} \sum_{G \text{ in } M'', G\subseteq H} K(\{1_F(\phi(G)) \\
&\qquad\qquad - f_\phi(G, F)\}1_F, G) \mid \\
&\qquad + \textstyle\sum_{H \text{ in } M, H\cap B\neq\emptyset} \sum_{F \text{ in } M'} \sum_{G \text{ in } M'', G\subseteq H, G\cap B=\emptyset} \mid K(\{1_F(\phi(G)) \\
&\qquad\qquad - f_\phi(G, F)\}1_F, G) \mid \\
&\qquad + \mid K(1_B, A) - \textstyle\sum_{H \text{ in } M, H\cap B\neq\emptyset} \sum_{F \text{ in } M'} \sum_{G \text{ in } M'', G\subseteq H, G\cap B\neq\emptyset} K(\{1_F(\phi(G)) \\
&\qquad\qquad - f_\phi(G, F)\}1_F, G) \mid \\
&\quad < 2b/3 + \textstyle\sum_{H \text{ in } M, H\cap B\neq\emptyset} \sum_{G \text{ in } M'', G\subseteq H, G\cap B=\emptyset} \lambda(G) \\
&\qquad + \mid K(1_B, A) - \textstyle\sum_{G \text{ in } M'', G\subseteq A, G\cap B\neq\emptyset} \sum_{F \text{ in } M'} K(\{1_F((G)) \\
&\qquad\qquad - f_\phi(G, F)\}1_B, G) \mid < b.
\end{aligned}
$$

Therefore we have the theorem.

THEOREM 4.4. *If (A, B) is in $R \times R$, A is properly situated relative to B with respect to Φ, and Φ contains n elements then*

$$K(1_B, A) = \sum_{\phi \text{ in } \Phi} \int_A 1_B[\phi]\left\{V_\phi + \sum_{p=1}^{n} (-1)^p \sum_{\mu(p) \text{ in } I^p(\phi)} V(\mu_p)\right\}.$$

Proof. Suppose that λ is in Φ and $1_B(\lambda(A)) - f_\lambda(A, B) = 1$. Then

$$
\begin{aligned}
&\sum_{\phi \text{ in } \Phi} \int_A 1_B[\phi]\left\{V_\phi + \sum_{p=1}^{n} (-1)^p \sum_{\mu(p) \text{ in } I^p(\phi)} V(\mu_p)\right\} \\
&\qquad = \sum_{p=0}^{n} \sum_{\mu(p) \text{ in } I^p(\lambda)} \int_A 1_B[\mu(p)]\left\{\sum_{q=0}^{n-p} (-1)^q \sum_{\nu(q) \text{ in } I^q(\mu(p))} V(\nu(q))\right\} \\
&\qquad = \sum_{p+q=0}^{n} \sum_{\mu(p) \text{ in } I^p(\lambda)} \sum_{\nu(q) \text{ in } I^q(\mu(p))} (-1)^q \int 1_B[\lambda]V(\nu(q)) \\
&\qquad = \sum_{p+q=0}^{n} (-1)^q \binom{p+q}{q} \sum_{\mu \text{ in } I^{p+q}(\lambda)} \int_A 1_B[\lambda]V_\mu \\
&\qquad = \int_A 1_B[\lambda]V_\lambda = K(1_B, A).
\end{aligned}
$$

Proof of Theorem 4.1. Suppose that Φ contains n elements. Let W denote the function from Φ into BV defined by

$$W_\phi = V_\phi + \sum_{p=1}^{n} (-1)^p \sum_{\mu(p) \text{ in } I^p(\phi)} V(\mu_p).$$

If each of A and B is in R and M is a refinement of A in D such that each member of M is properly situated relative to B with respect to Φ then

$$\begin{aligned} K(1_B, A) &= \sum_{H \text{ in } M} K(1_B, H) \\ &= \sum_{H \text{ in } M} \sum_{\phi \text{ in } \Phi} \int_H 1_B[\phi] W_\phi \\ &= \sum_{\phi \text{ in } \Phi} \int_A 1_B[\phi] W_\phi . \end{aligned}$$

Hence we have the theorem.

5. Some examples

Suppose that R is a field and F is a continuous linear function from $B(\Omega, R)$ to the plane. Let K denote the function from $B(\omega, R) \times R$ to the plane defined by $K(f, A) = F(1_A f)$. K is an integral and any complete set of choice functions is degenerate. Hence

$$K(f, A) = \int_A f[\phi] K[1_\Omega, \ ]$$

for each choice function ϕ for R.

Suppose that n is a positive integer and Ω is the space of n-tuples of real numbers. A subset A of Ω is called a *rectangular interval* provided that there is an ordered pair (x, z) in $\Omega \times \Omega$ such that $x(p) < z(p) (p = 1, 2, \cdots, n)$ and a member w of Ω is in A only in case

$$x(p) \leq w(p) \leq z(p) (p = 1, 2, \cdots, n).$$

Briefly, $A = [x; z]$. Let R denote the set of all rectangular intervals contained in Ω.

THEOREM 5.1. *Suppose that each of* $[x; y]$ *and* $[w; z]$ *is in* R *and* $[x; y]$ *contains a point of* $[w; z]$. *For each integer* p *in* $[1, n]$, *let*

$$u(p) = \tfrac{1}{2}(x(p) + w(p) + |x(p) - w(p)|)$$

and

$$v(p) = \tfrac{1}{2}(y(p) + z(p) - |y(p) - z(p)|).$$

$[x; y]$ *is relatively prime to* $[w; z]$ *only in case* $u(p) = v(p)$ *for some integer* p *in* $[1, n]$.

Proof. One way is clear. Suppose that $[a; b]$ is a member of R contained in each of $[x; y]$ and $[w; z]$. Then $u(p) \leq a(p) < b(p) \leq v(p)$ for each integer p in $[1, n]$. Thus we have the theorem.

Suppose that $[x; y]$ is in R, $w(p) = \frac{1}{2}(x(p) + y(p)$ for $p = 1, 2, \cdots, n$, M is a member of D which contains $[x; y]$ and $[u; v]$ is a member of M which contains w. For each integer p in $[1, n]$, let

$$\bar{u}(p) = \tfrac{1}{2}(x(p) + u(p) + |x(p) - u(p)|)$$

and

$$\bar{v}(p) = \tfrac{1}{2}(y(p) + v(p) - |y(p) - v(p)|).$$

There is an integer p in $[1, n]$ such that $\bar{u}(p) = \bar{v}(p)$. But then

$$x(p) < \tfrac{1}{2}(x(p) + z(p)) = \bar{u}(p) = u(p) = \bar{v}(p) = u(p) = z(p)$$

and this is a contradiction. Hence no member of M other than $[x; y]$ contains w.

THEOREM 5.2. *Each member of R is R-measurable.*

Proof. Suppose that each of $[x; y]$ and $[w; z]$ is in R and $[x; y]$ contains a point of $[w; z]$. Let $\{N_p\}_1^n$ denote the sequence of sets defined as follows: N_p is the set to which u belongs only in case either $u = x(p)$ or $u = y(p)$ or $u = w(p)$ and $x(p) < w(p) < y(p)$ or $u = z(p)$ and $x(p) < z(p) < y(p)$. Let M denote the collection of subsets of R to which $[u; v]$ belongs only in case, for each integer p in $[1, n]$, $u(p)$ and $v(p)$ are in N_p and there is no member of N_p between $u(p)$ and $v(p)$. M partitions $[x; y]$. Suppose that each of $[u; v]$ and $[\bar{u}; \bar{v}]$ is in M and $[u; v]$ is not relatively prime to $[\bar{u}; \bar{v}]$ with respect to R. Then for each integer p in $[1, n]$

$$\tfrac{1}{2}(u(p) + \bar{u}(p) + |u(p) - \bar{u}(p)|) < \tfrac{1}{2}(v(p) + \bar{v}(p) - |v(p) - \bar{v}(p)|),$$

and so $u(p) = \bar{u}(p)$ and $v(p) = \bar{v}(p)$. Thus M is in D. Similarly, suppose that $[u; v]$ is in M and $[u; v]$ is not relatively prime to $[w; z]$ with respect to R. Then $[u; v]$ is contained in $[w; z]$. Therefore each member of R is R-measurable.

Clearly each pair of elements in R is contained in a third member of R. Theorem 2.1 shows that if (A, B) is in $R \times R$ then there is a member M of D which contains a refinement of each of A and B.

Let $\mathcal{S}$ denote the class of ordered pairs to which (S, T) belongs only in case each of S and T is a subset of the first n positive integers and S contains no member of T. For each member (S, T) of $\mathcal{S}$ let $P_{S,T}$ denote the class of functions from R into Ω to which ϕ belongs only in case, for each $[x; y]$ in R and integer p in $[1, n]$, $\phi([x; y])_p = x(p)$ if p is in S, $\phi([x; y])_p = y(p)$ if p is in T, and $x(p) < \phi([x; y])_p < y(p)$ otherwise.

THEOREM 5.3. *If (S, T) is in $\mathcal{S}$ and ϕ is in $P_{S,T}$ then ϕ is a choice function for R.*

Proof. Suppose that each of $[x; y]$ and $[w; z]$ is in R and $[x; y]$ contains a point of $[w; z]$. Let $\{N_p\}_1^n$ and M be as in the proof of Theorem 5.2. Suppose that $[u; v]$ is a member of M which contains a point of $[w; z]$, $[\bar{u}; \bar{v}]$ is a member of R contained in $[u; v]$ and $[\bar{u}; \bar{v}]$ contains a point of $[w; z]$. If $\phi([\bar{u}; \bar{v}])$ is in $[w; z]$ and p is in S then

$$\phi([u; v])_p = u(p) \leq \bar{u}(p) \leq v(p).$$

If p is in T then

$$\phi([u; v])_p = v(p) \geq \bar{v}(p) \geq u(p).$$

Since, for each integer p in the union of S and T, $w(p) \leq \bar{u}(p)$ and $\bar{v}(p) \leq z(p)$ we have $u(p) = \bar{u}(p)$ and $v(p) = \bar{v}(p)$. If p is an integer in $[1, n]$ and p is in neither S nor T then

$$u(p) < \phi([u; v])_p < v(p).$$

Again $u(p) \leq \bar{u}(p) < \phi([\bar{u}; \bar{v}])_p \leq \bar{v}(p) = v(p)$. Hence

$$w(p) < \phi([u; v])_p < z(p).$$

Therefore $\phi([u; v])$ is in $[w; z]$.

Suppose that $\phi([u; v])$ is in $[w; z]$. Let a be a point of $[\bar{u}; \bar{v}]$ in $[w; z]$. If p is in S than $w(p) \leq \bar{u}(p) \leq u(p) \leq a(p) \leq z(p)$. If p is in T then $w(p) \leq a(p) \leq \bar{v}(p) \leq v(p) \leq z(p)$. If p is an integer in $[1, n]$ and p is in neither S nor T then $u(p) < \phi([u; v])_p < v(p)$. Hence

$$w(p) \leq u(p) \leq \bar{u}(p) \leq \phi([\bar{u}; \bar{v}])_p < \bar{v}(p) \leq v(p) \leq z(p).$$

Therefore $\phi([\bar{u}; \bar{v}])$ is in $[w; z]$ and so ϕ is a choice function for R.

THEOREM 5.4. *Let Φ be a collection of choice functions for R with the property that, for each (S, T) in $\mathcal{S}$, Φ contains exactly one member of $P_{S,T}$. Φ is a finite complete collection of choice functions for R which is coherent.*

LEMMA 5.1. *Suppose that each of $[x; y]$ and $[w; z]$ is in R. If there is a member (S, T) of $\mathcal{S}$ such that a member u of $[x; y]$ is in $[w; z]$ only in case $u(p) = x(p)$ for each p in S and $u(p) = y(p)$ for each p in T then $[x; y]$ is properly situated relative to $[w; z]$ with respect to Φ.*

Proof. Suppose that $[u; v]$ is a member of R contained in $[x; y]$ which contains a member of $[w; z]$, (S', T') is a member of $\mathcal{S}$, and ϕ is the member of Φ in $P_{S',T'}$. If $\phi([u; v])$ is in $[w; z]$ and p is in S then

$$x(p) \leq u(p) \leq \phi([u; v])_p = x(p).$$

Hence S is contained in S'. Similarly, T is contained in T'. Therefore $\phi([x; y])$ is in $[w; z]$. Suppose that $\phi([x; y])$ is contained in $[w; z]$. Let a be a member of $[u; v]$ in $[w; z]$. If p is in S then $x(p) \leq u(p) \leq a(p) = x(p)$ and if p is in T then $y(p) = a(p) \leq v(p) \leq y(p)$. Hence $\phi([u; v]$ is in $[w; z]$. Therefore $[x; y]$ is properly situated relative to $[w; z]$ with respect to Φ.

LEMMA 5.2. *If each of* (S, T) *and* (S', T') *is contained in* $\mathcal{S}$, ϕ_1 *is the member of* Φ *in* $P_{S,T}$, *and* ϕ_2 *is the member of* Φ *in* $P_{S',T'}$, *then these are equivalent*:

(1) ϕ_1 *precedes* ϕ_2,
(2) S *is contained in* S' *and* T *is contained in* T'

Proof. Suppose that (1) holds and $[x; y]$ is a member of R. Let (w, z) be an ordered pair in $\Omega \times \Omega$ such that $[w; z]$ is in R, if p is in S then $z(p) = x(p)$, if p is in T then $w(p) = y(p)$, and $w(p) \leq x(p) < y(p) \leq z(p)$ otherwise. $[x; y]$ is properly situated relative to $[w; z]$ with respect to Φ. Since

$$1_{[w;z]}(\phi_1([x; y])) \leq 1_{[w;z]}(\phi_2([x; y])),$$

S is contained in S', and T is contained in T'.

Suppose that (2) holds, each of $[x; y]$ and $[w; z]$ is in R, $[x; y]$ is properly situated relative to $[w; z]$ with respect to $\{\phi_1, \phi_2\}$ and

$$1_{[w;z]}(\phi_1([x; y])) > 1_{[w;z]}(\phi_2([x; y])).$$

There is an integer p in $[1, n]$ such that either

$$\phi_2([x; y])_p < w(p) \quad \text{or} \quad \phi_2([x; y])_p > z(p).$$

Suppose the former. For each integer q in $[1, n]$, let $v(q) = w(q)$ if $q = p$ and $v(q) = y(q)$ otherwise. $[x; v]$ is a member of R contained in $[x; y]$ and $[x, v]$ contains a member of $[w; z]$. Hence $\phi_1([x; v])$ is in $[w; z]$ and so p is in T. But p is not in T'. We have a similar situation if $\phi_2([x; y])_p > z(p)$. Therefore (2) implies (1).

The proof of the second part of Lemma 2 also shows that if

$$1_{[w;z]}(\phi_1([x; y])) = 1$$

and p is an integer in $[1, n]$ which is in neither S nor T then

$$w(p) \leq x(p) < y(p) \leq z(p).$$

LEMMA 5.3. *Suppose that each of* $[x; y]$ *and* $[w; z]$ *is in* R, $[x; y]$ *is properly situated relative to* $[w; z]$ *with respect to* Φ, (S, T) *is in* $\mathcal{S}$, ϕ *is the member of* Φ *in* $P_{S,T}$, *and*

$$1_{[w;z]}(\phi([x; y])) - f_\phi([x; y], [w; z]) = 1,$$

then for each member u *of* $[x; y]$ *these are equivalent*:

(1) u *is in* $[w; z]$,
(2) $u(p) = x(p)$ *for each* p *in* S *and* $u(p) = y(p)$ *for each* p *in* T.

Proof. Suppose that (1) holds, p is a member of S, and $u(p) > x(p)$. Let S' denote $S - \{p\}$ and ϕ' the member of $P_{S',T}$ in Φ. For each integer q in $[1, n]$, let $v(q) = u(q)$ if $q = p$ and $v(q) = y(q)$ otherwise. Then $[x; v]$ is a member of R contained in $[x; y]$ and $[x; v]$ contains a member of $[w; z]$. Furthermore, $\phi'([x; v])$ is in $[w; z]$ and so $\phi'([x; y])$ is in $[w; z]$. A similar situation holds if p is a member of T and $u(p) < y(p)$. Hence

$$1_{[w;z]}(\phi([x; y])) - f_\phi([x; y], [w; z]) = 0.$$

This is a contradiction and so (1) implies (2).

Suppose that (2) holds and u is not in $[w; z]$. There is an integer p in $[1, n]$ such that either $u(p) < w(p)$ or $u(p) > z(p)$. Suppose the former. For each integer q in $[1, n]$, let $v(q) = w(q)$ if $q = p$ and $v(p) = y(q)$ otherwise. Then $[x; v]$ is a member of R contained in $[x; y]$ and $[x; v]$ contains a member of $[w; z]$. Hence $\phi([x; v])$ is in $[w; z]$. But this is impossible. A similar situation holds if $u(p) > z(p)$. Hence u is in $[w; z]$ or (2) implies (1).

Lemma 5.4. *If each of* $[x; y]$ *and* $[w; z]$ *is in* R, $[x; y]$ *is properly situated relative to* $[w; z]$ *with respect to* Φ, *and* $[x; y]$ *contains a point of* $[w; z]$, *then there is a member* ϕ *of* Φ *such that* $\phi([x; y])$ *is in* $[w; z]$.

Proof. Let a be a member of $[x; y]$ in $[w; z]$. For each integer p in $[1, n]$, let $u(p) = x(p)$ if $a(p) = z(p)$ and

$$u(p) = \tfrac{1}{2}(x(p) + w(p) + |x(p) - w(p)|)$$

otherwise and $v(p) = y(p)$ if $a(p) = w(p)$ and

$$v(p) = \tfrac{1}{2}(y(p) + z(p) - |y(p) - w(p)|)$$

otherwise. $[u; v]$ is in R and is contained in $[x; y]$. Furthermore, $[u; v]$ contains a member of $[w; z]$. Let S be the set of integers in $[1, n]$ to which p belongs only in case $a(p) = z(p)$. Let T be the set of integers in $[1, n]$ to which p belongs only in case $a(p) = w(p)$. Let ϕ be the member of $P_{S,T}$ in Φ. Then $\phi([u; v])$ is in $[w; z]$ and so $\phi([x; y])$ is in $[w; z]$.

Proof of Theorem 5.4. Clearly Φ is finite. Suppose that each of (S, T) and (S', T') is in $\mathcal{S}$, ϕ_1 is the member of Φ in $P_{S,T}$, ϕ_2 is the member of Φ in $P_{S',T'}$, ϕ_1 precedes ϕ_2, and ϕ_2 precedes ϕ_1. Then by Lemma 5.2 we have $S = S'$ and $T - T'$. Hence $\phi_1 = \phi_2$.

Suppose that each of $[x; y]$, $[w; z]$, and $[u; v]$ is in R, $[u; v]$ is properly situated relative to each of $[x; y]$ and $[w; z]$, (S, T) is a member of $\mathcal{S}$, ϕ is the member of $P_{S,T}$ in Φ, and

$$1_{[x;y]}(\phi([u; v])) - f_\phi([u; v], [x; y]) = 1 = 1_{[w;z]}(\phi([u; v])) - f_\phi([u; v], [w; z]).$$

Again by Lemma 5.2 a member a of $[u; v]$ is in $[x; y]$ only in case $a(p) = u(p)$ for each p in S and $a(p) = v(p)$ for each p in T. The same holds for $[w; z]$. Hence the common part of $[x; y]$ and $[u; v]$ is the common part of $[w; z]$ and $[u; v]$. For each integer p in $[1, n]$, let $b(p) = v(p)$ if p is in T and $b(p) = u(p)$ otherwise and

$$c(p) = \tfrac{1}{2}(y(p) + z(p) - |y(p) - z(p)|)$$

if p is in T and

$$c(p) = \tfrac{1}{2}(x(p) + w(p) + |x(p) - z(p)|)$$

otherwise. $[b; c]$ is a member of R.

Suppose that each of $[x; y]$ and $[w; z]$ is in R, $[x; y]$ is properly situated relative to $[w; z]$ with respect to Φ, and $[x; y]$ contains a point of $[w; z]$. By Lemma 5.4 and the finiteness of Φ there is a least member ϕ of Φ such that

$$1_{[w;z]}(\phi([x; y])) = 1.$$

But this means that $1_{[w;z]}(\phi([x; y])) - f_\phi([x; y], [w; z]) = 1$. Lemma 5.3 shows that there is no more than one such ϕ in Φ.

Suppose that (S, T) is in $\mathcal{S}$, ϕ is the member of $P_{S,T}$ in Φ; each of p and q is a non-negative integer, and F is a function from Φ to the number plane. If $I^{p+q}(\phi)$ is empty then

$$\textstyle\sum_{\lambda \text{ in } I^p(\phi)} \sum_{\mu \text{ in } I^q(\lambda)} F(\mu) = 0 = \binom{p+q}{q} \sum_{\nu \text{ in } I^{p+q}(\phi)} F(\nu).$$

Clearly the proposition holds if either p or q is 0. Suppose that $I^{p+q}(\phi)$ is not empty and $p \neq 0 \neq q$. Then there are at least $p + q$ integers in $[1, n]$ which are in neither S nor T. Suppose that (S', T') is in S, ν is the member of Φ in (S', T'), and ν is in $I^{p+q}(\phi)$. Let H denote the set of integers in the union of S' and T' which are not in the union of S and T. H contains exactly $p + q$ elements and there are $\binom{p+q}{p}$ subsets of H which contain p elements. Hence

$$\sum_{\lambda \text{ in } I^p(\phi)} \sum_{\mu \text{ in } I^q(\lambda)} F(\mu) = \binom{p+q}{q} \sum_{\nu \text{ in } I^{p+q}(\phi)} F(\nu)$$

Hence we have Theorem 5.4.

THEOREM 5.5. *$B(\Omega, R)$ is an algebra.*

Proof. A function f from Ω to the number-plane is said to be *quasicontinuous* provided if x is a point in Ω; $[w; z]$ is a member of R which contains x in its interior; for each integer p in $[1, n]$, N_p is the set of numbers to which u belongs only in case $u = w(p)$ or $u = x(p)$ or $u = z(p)$; M is the collection of subsets of R to which $[u; v]$ belongs only in case, for each integer p in $[1, n]$ each of $u(p)$ and $v(p)$ is in N_p and no member of N_p lies between $u(p)$ and $v(p)$; (S, T) is in $\mathcal{S}$; $[u; v]$ is in M; and z is a sequence with values in $[u; v]$ such that for each integer p in $[1, n]$ and positive integer q, $z_q(p) = x(p)$ if p is in either S or T and $z_q(p)$ is between $u(p)$ and $v(p)$ and $z(p)$ has limit $x(p)$ otherwise; then $f[z]$ has a limit. The set M is the partition of $[w; z]$ in D which contains both $[w; x]$ and $[x; z]$ with the fewest members. Let $\mathfrak{M}$ denote the space of functions from Ω to the number-plane which are quasi-continuous and have compact support.

Suppose that f is in $\mathfrak{M}$, $[x; y]$ is a member of R which contains the support of f, and b is a positive number. Let F denote the set of ordered pairs to which (a, A) belongs only in case a is in $[x; y]$; $A = [w; z]$ is a member of R which contains a in its interior; and if M is the partition of $[x; z]$ in D which contains $[x; a]$ and $[a; z]$ with the fewest members, (S, T) is in $\mathcal{S}$, $[u; v]$ is in M, each of r and s is in $[u; v]$, and, for each integer p in $[1, n]$, $r(p) = s(p) = a(p)$ if p is in either S or T and each of $r(p)$ and $s(p)$ is between $u(p)$ and $v(p)$ otherwise;

then $|f(s) - f(r)| < b$. There is a finite subset A of $[x; y]$ such that the interiors of the elements of the final set of the contraction of f to A covers $[x; y]$.

For each integer p in $[1, n]$, let N_p denote the set to which u belongs only in case there is an a in A such that either $u = a(p)$ or $u = w(p)$ or $u = z(p)$, where $F(a) = [w; z]$. Let M denote the subset of R to which $[u; v]$ belongs only in case, for each integer p in $[1, n]$, $u(p)$ and $v(p)$ are in N_p and no member of N_p lies between $u(p)$ and $v(p)$. Let M' denote the collection of subsets of Ω to which B belongs only in case there is a member $[u; v]$ of M and a member (S, T) of $\mathcal{S}$ such that a point a of Ω is in B only in case, for each integer f in $[1, n]$, $a(p) = u(p)$ if p is in S, $a(p) = v(p)$ is p is in T, and $u(p) < a(p) < v(p)$ otherwise. There is a function ψ from M' into Ω such that $\psi(B)$ is in B for each B in M'. Let g denote the function from Ω to the plane defined by

$$g = \sum_{B \text{ in } M'} f(\psi(B))1_B .$$

g is in $B(\Omega, R)$ and $|f - g| < b$. Since $\mathfrak{M}$ is an algebra the closure of $\mathfrak{M}$, which is $B(\Omega, R)$, in the space of functions from Ω to the plane which have bounded final sets with respect to $|\cdot|$ is an algebra.

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