PIERCE'S REPRESENTATION AND SEPARABLE ALGEBRAS

BY
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Pierce [4] gives a representation of an arbitrary commutative ring R as the ring of global sections of a sheaf of connected rings over a compact, totally disconnected, Hausdorff space. Here we apply this representation to the study of central separable R-algebras. Pierce's sheaf has the interesting property that modules and algebras over the stalks may be extended to modules and algebras over R. We carry out these constructions in Section 1 below and use the results to compute the Brauer group of R in terms of the Brauer groups of the stalks. In Section 2 we establish that properties analogous to the Skolem-Noether Theorem, the existence of Galois splitting rings, and the generation of separable algebras by units holds for R if they are true at each stalk. Our results apply in particular to commutative Von Neumann (regular) rings, which are characterized [4, p. 41, 10.3] by the property that each stalk of the associated sheaf is a field.

We will assume all rings and algebras have identities and all modules are unitary. R always denotes the fixed commutative base ring and unsubscripted tensor means over R. The author wishes to thank Professor Daniel Zelinsky for his help and encouragement in the preparation of this paper.

1. We recall the description of the ringed space $(X(R), \mathcal{R})$ associated to R. $X(R)$ is the maximal ideal space of the Boolean algebra of all idempotents of R ($X(R)$ is topologized by taking the sets $U_e = \{x : 1 - e \in x\}$, for all idempotents e, as basic open sets) and $\mathcal{R}(U_e) = Re$. Note that $U_e \subset U_f$ if and only if $e \leq f$ (that is, $ef = e$). The stalk of $\mathcal{R}$ at x, which we denote R_x, is R/xR. R_x is flat, being the direct limit of projectives. If M is an R-module, let $M_x = M \otimes R_x$, and for each m in M, let m_x be the image of m in M_x. M_x is to be thought of as the stalk at x of a sheaf corresponding to M. The following lemma of Pierce makes this precise (4, p. 18]:

(1.1) Let M be an R-module, a, b elements of M. If $a_x = b_x$, there is an idempotent e with x in U_e such that $ea = eb$. If $a_y = b_y$ for all y in $X(R)$, then $a = b$.

Note that $ea = eb$ if and only if $a_y = b_y$ for all y in U_e.

(1.1) may be paraphrased as "if a finite system of equations among elements of M holds at x, it holds in a neighborhood of x".

If f is in $\mathrm{Hom}_R(M, N) = H$, let f_x denote the morphism $f \otimes R_x$ as well as the image of f in H_x. When M is finitely presented, this notation is unambiguous.

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(1.2) Let M be a finitely presented R-module. Then the canonical map

$$\mathrm{Hom}_R(M, N)_x \to \mathrm{Hom}_{R_x}(M_x, N_x)$$

is an isomorphism and hence the canonical map

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{R_x}(M_x, N_x)$$

is onto.

Proof. Since R_x is flat, the isomorphism is a special case of [2, p. 93, 2.8].

The second part of (1.2) says that homomorphisms at the stalks may be lifted. The first part, along with (1.1), says that if equations (diagrams) of maps between finitely presented modules hold (commute) at x, they do so in a neighborhood of x.

(1.3) Let M_0 be a finitely presented R_x-module. Then there is a finitely presented R-module M such that $M_x = M_0$.

Proof. Let $T_0 : (R_x)^n \to (R_x)^m$ be such that Coker $(T_0) = M_0$. Choose $T : R^n \to R^m$ such that $T_x = T_0$. Let $M =$ Coker (T); then $M_x = M_0$.

(1.4) Let S be an R-algebra, finitely generated and projective as an R-module. Let N_0 be a finitely generated projective S_x-module (x in $X(R)$). Then there is a finitely generated projective S-module N such that $N_x = N_0$.

Proof. N_0 is finitely presented as an R_x-module. Thus by (1.3) there is a finitely presented R-module M such that $M_x = N_0$ (as R_x-modules). To say that M has an S-module structure is to give a map $S \otimes M \to M$ satisfying an associative law, i.e. making a certain diagram (whose vertices are finitely presented R-modules) commute. Since $S \times M$ is finitely presented, we can choose an R-module map k lifting the S_x-module structure of M_0. Then the diagram for the associativity of k commutes at x, hence in a neighborhood of x; that is, there is an idempotent e with x in U_e such that ke makes Me and Se-module. Replace M by $S(1 - e) + Me$. Then M is an S-module and $M_x = N_0$ as S_x-modules. Since M is finitely generated over S, for some n there is an S-module epimorphism $g : S^n \to M$. Let h_0 be the S_x-module right inverse to g_x (which exists since M_x is S_x-projective). Let $h : M \to S^n$ be an R-module homomorphism lifting h. To say that h is an S-module homomorphism is again an assertion that a diagram, with finitely presented vertices, commutes. This diagram commutes at x, hence in a neighborhood of x. Moreover, since h is a right inverse to g at x, there is also a neighborhood of x on which h is a right inverse of g. Let U_f contain x and be contained in the intersection of these two neighborhoods. Then hf is an Sf-module inverse to gf, so Mf is Sf-projective. Then $N = S(1 - f) + Mf$ is a finitely generated projective S-module such that $N_x = N_0$.

Before extending these results to algebras, we need a preliminary lemma.

For a ring B and a subring A let

$$Z(B, A) = \{b \in B : ba = ab \text{ for all } a \text{ in } A\}.$$

$Z(B, A)$ is called the commutant of A in B.

(1.5) Let B be an R-algebra, A a separable R-subalgebra. Then
(a) $Z(B, A)$ is a direct summand of B,
(b) $Z(B, A)_x = Z(B_x, A_x)$.

Proof. Let $m : A^e \to A$ be the multiplication map and r its A^e-module right inverse. Recall that

$$h : \mathrm{Hom}_{A^e}(A, B) \to Z(B, A) \quad \text{and} \quad k : \mathrm{Hom}_{A^e}(A^e, B) \to B,$$

where both maps are evaluation at 1, are R-module isomorphisms. Then $k \circ \mathrm{Hom}_{A^e}(m, B) \circ h^{-1}$ is the inclusion of $Z(B, A)$ in B and $h \circ \mathrm{Hom}_{A^e}(r, B) \circ k^{-1}$ is a left inverse to the inclusion. This establishes (a); (b) is a special case of [2, p. 93, 2.8].

(1.6) Let A_0 be an R_x-algebra, finitely presented as an R_x-module. Then there is a finitely presented R-algebra A such that $A_x = A_0$. If A_0 is separable, A may be taken to be separable. If A_0 is central separable, A may be taken to be central separable.

Proof. By (1.3) we can find a finitely presented R-module B such that $B_x = A_0$. To say that B is an R-algebra is to give maps $B \otimes B \to B$ and $R \to B$ (multiplication and identity) which make certain diagrams commute. Since R and $B \otimes B$ are finitely presented we can choose maps lifting the multiplication and identify of A_0. Then the necessary diagrams commute at x and hence in a neighborhood. Thus there is an idempotent e with x in U_e such that $A = R(1 - e) + Be$ is an R-algebra. Clearly $A_x = A_0$ (as algebras). A is separable if the multiplication map $m : A^e \to A$ has a right A^e-module inverse. Exactly as in the proof of (1.4), if such an inverse exists at x it exists on a neighborhood U_f of x. Thus Af is Rf-separable; replacing A by $R(1 - f) + Af$, we have that A is separable. The center of A is $Z(A, A)$, which is finitely generated over R, and $Z(A, A)_x = Z(A_x, A_x)$, which is the center of A_x, both remarks following from (1.5). Thus if a finite set of generators of $Z(A, A)$ lies in R at x, it does so on a neighborhood and there is an e with x in U_e such that Re is the center of Ae. Replacing A by $R(1 - e) + Ae$, we have that A is central separable.

(1.7) Let S be a commutative R-algebra, finitely generated and projective as an R-module. Let A and B be S-algebras, finitely generated and projective as S-modules. Suppose A_x is isomorphic to B_x (x in $X(R)$) as S_x-algebras. Then there is an idempotent e with x in U_e such that Ae is isomorphic to Be as Se-algebras.

Proof. A and B are finitely presented over R. Let $h_0 : A_x \to B_x$ be an

S_x-algebra isomorphism. Choose an R-module map $h : A \to B$ such that $h_x = h_0$. The statement that h is an S-algebra map is an assertion that certain diagrams (with finitely presented R-modules as vertices) commute. Since these diagrams commute at x they do so on a neighborhood, hence there is an e with x in U_e such that $he : Ae \to Be$ is an Se-algebra homomorphism. Since Be is finitely generated over Re so is the cokernel M of he; since $M_x = 0$, there is an $f \leq e$ (with x in U_f) such that $Mf = 0$ and hence hf is onto. Since Bf is projective the kernel N of hf is a direct sumand of Af and hence finitely generated. Since $N_x = 0$, there is a $g \leq f$ (with x in U_g) such that $Ng = 0$ and so hg is an Sg-algebra isomorphism.

We now consider functors.

(1.8) Let X be a topological space. A finite cover of X by pairwise disjoint open sets is called a *partition* of X. A presheaf F of Abelian groups on X is called *additive* if for each partition $\mathcal{P} = \{U_1, \cdots, U_n\}$ the induced map

$$F(X) \to F(U_1) \times \cdots \times F(U_n)$$

is an isomorphism.

Sheaves as well as presheaves are to be functors.

(1.9) Let $X = X(R)$ and let F be an additive presheaf on X. Let $\#F$ be the associated sheaf. Then $\#F(X) = F(X)$.

Proof. For every open cover $\mathfrak{U} = \{U_i\}$ of X, let $F(\mathfrak{U})$ be the difference kernel of the two standard projections of $\prod F(U_i)$ to $\prod F(U_i \cap U_j)$. The family of open covers of X is directed by refinement and by definition $\#F(X) = \text{dir lim } F(\mathfrak{U})$. Since X is compact, totally disconnected and Hausdorff, every open cover has a refinement which is a partition. Thus the direct limit may be taken over the cofinal subset of partitions, and for a partition $\mathcal{P} = \{U_1, \cdots, U_n\}$, $F(\mathcal{P})$ is $F(U_1) \times \cdots \times F(U_n)$ which is, by assumption, $F(X)$.

Now if F is any additive functor from commutative rings to Abelian groups (additive in the sense of preserving products) then $F \circ \mathcal{R}$ is an additive presheaf on $X(R)$. Let H be the associated sheaf. For every idempotent e with x in U_e there is a map $F(Re) \to F(R_x)$ and hence an induced map $H_x \to F(R_x)$, where $H_x = \text{dir lim } H(U_e)$, the limit being over U_e's with x in U_e.

THEOREM 1.10. *There is a sheaf on $X(R)$ whose global sections are the Brauer group of R and whose stalk at x is the Brauer group of R_x.*

Proof. We have to show that when $F = \text{Br}$ (Brauer group) the homomorphism $h : H_x \to F(R_x)$ defined above is an isomorphism. Let parentheses denote Brauer class and let A_0 be a central separable R_x-algebra. By (1.6) there is a central separable R-algebra A such that $A_x = A_0$. Then the image of A in H_x is sent by h to (A_0), and h is onto. If A is a central separable Re-algebra, with x in U_e, such that $(A_x) = 0$, A_x is isomorphic to

$\mathrm{Hom}_{R_x}(N_0, N_0)$ for some finitely generated projective R_x-module N_0. By (1.4) we can choose a finitely generated projective R-module N such that $N_x = N_0$. By (1.7) there is an $f \leq e$ (with x in U_f) such that Af is isomorphic to $\mathrm{Hom}_{Rf}(Nf, Nf)$. Then since (Af) is already trivial in Br (Rf), the image of A in H_x is zero. Hence h is one-one.

COROLLARY 1.11. *Let A be a central separable R-algebra. If A_x is split (that is, $(A_x) = 0$) for all x in $X(R)$ then A is split.*

As an example, let k be a commutative ring with no nontrivial idempotents and let X be a compact Hausdorff space. Let $S = C(X, k)$, the ring of continuous k-valued functions on X (k carries the discrete topology). Then $X(S)$ is an identification space of X, with components identified to points. S_x is k for all x and the map of S to S_x is evaluation on the component x, continuous functions being constant on components.

COROLLARY 1.12. *Let k be the integers, a finite field, or any other ring with zero Brauer group. Then* $\mathrm{Br}(C(X, k)) = 0$ *for all compact Hausdorff X. In particular, the Brauer group of a Boolean ring is zero.*

It is possible, using similar techniques, to prove the analogue of (1.10) for Pic (Picard group).

2. This section discusses certain properties which R has if each R_x has them.

A commutative R-algebra S is said to be weakly Galois [5, 3.1] provided S is separable over R, finitely generated projective and faithful as an R-module, and that the S-module $\mathrm{Hom}_R(S, S)$ is generated by R-algebra automorphisms of S.

THEOREM 2.1. *Every central separable R-algebra has a weakly Galois splitting ring if the analogous property holds for each stalk.*

Proof. Let A be a central separable R-algebra, x a point of $X(R)$ and S_0 a weakly Galois R_x-algebra splitting A_x. By (1.6) there is a separable R-algebra S, finitely presented as an R-module, such that $S_x = S_0$. Since S_0 is projective and commutative, arguments similar to those advanced in section 1 show that S may be taken to be commutative and projective. The annihilator B of S is a direct summand of R since S is finitely generated projective and thus finitely generated. Since $B_x = 0$ there is an e with x in U_e such that $Be = 0$ and replacing S by $Se + R(1 - e)$ we have that S is also faithful and still $S_x = S_0$. By [5, 3.5 and 3.6], choose a finite group G_0 of R_x-algebra automorphisms of S_0 such that $S[G_0] \to \mathrm{Hom}_{R_x}(S_0, S_0)$ is an isomorphism ($S[G_0]$ denotes the trivial crossed product). As in [5, Section 3] there is an idempotent e (with x in U_e) and a finite group G of Re-algebra automorphisms of Se such that $G_x = G_0$ and $Se[G] \to \mathrm{Hom}_{R_x}(Se, Se)$ is an isomorphism. Se is a weakly Galois Re-algebra. Then S_0 and S were chosen such that $(A \otimes S)_x = A_x \otimes_{R_x} S_x$ is isomorphic to $\mathrm{Hom}_{S_x}(N_0, N_0)$ for some finitely generated projective S_x-module N_0. Choose, by (1.4), a finitely generated projective S-module N such that $N_x = N_0$. Since $A \otimes S$ and $\mathrm{Hom}_S(N, N)$ are S_x-isomorphic at x, by (1.7) there is a neighborhood U_e (with x in U_e) on which they are isomorphic. For each x we have such a U_e, and this cover of $X(R)$ has a refinement which is a partition. Thus there are pairwise orthogonal idempotents e_i, $i = 1, \cdots, n$, summing to unity, and for each i a weakly Galois Re_i-algebra S_i and a finitely generated projective S_i-module N_i such that $Ae_i \otimes_{Re_i} Se_i$ is isomorphic to $\mathrm{Hom}_{Se_i}(N_i, N_i)$. (The e_i's correspond to the neighborhoods forming the partition; see [4, p. 12].) Now let $S = \prod S_i$ and $N = \prod N_i$. Then S is weakly Galois [5, section 3], N is a finitely generated projective S-module and $A \otimes S = \mathrm{Hom}_S(N, N)$.

Corollary 2.2. *Let R be a commutative Von Neumann ring. Then* $\mathrm{Br}(R) =$ dir lim Br (S/R) *where S ranges over weakly Galois R-algebras.*

Proof. Since a finite Galois field extension of a field is clearly a weakly Galois algebra, the result follows from the theorem and the existence of Galois splitting fields of central simple algebras [1, p. 78, 8.3E].

Call an R-algebra A *locally connected* if for each x in $X(R)$ A_x has no nontrivial central idempotents. Since $X(R_x)$ is a single point (4, p. 15, 4.2] if A is a locally connected R-algebra then A_x is trivially a locally connected R_x-algebra. Note that if R is a field, a separable R-algebra is simple if and only if it is locally connected.

Theorem 2.3. *Every R-algebra isomorphism between two locally connected separable subalgebras of a central separable R-algebra is the restriction of an inner automorphism of the algebra, provided the analogous property holds at each stalk.*

Proof. Let B be a central separable R-algebra, A' and A'' locally connected separable subalgebras of B and h an R-algebra isomorphism of A' to A''. By assumption, for each x we have that h_x is the restriction to A'_x of an inner automorphism of B_x by a unit u. Let a and b be in B such that $a_x = u$ and $b_x = u^{-1}$. Then $a_x b_x = 1_x$, so there is an e with x in U_e such that $(ae)(be) = e$. Thus ae is a unit in Be; let g denote the inner automorphism of Be associated to it. Since g_x restricted to A'_x is h_x, there is a $U_f \subset U_e$ such that hf is also given by inner automorphism by af. $X(R)$ is covered by such U_f's; this cover has a refinement which is a partition. Thus there are pairwise orthogonal idempotents e_i, $i = 1, \cdots, n$, summing to unity, and for each i elements a_i, b_i of Be_i such that $(a_i e_i)(b_i e_i) = e_i$ and $he_i(y) = (a_i e_i)y(b_i e_i)$ for all y in A'. Letting $a = \sum a_i e_i$ and $b = \sum b_i e_i$ we have $ab = 1$ and $h(y) = ayb$ for all y in A'.

We call the property of R with which the theorem is concerned SN (for Skolem-Noether).

We remak that if instead of SN we consider the weaker property that every automorphism of every central separable R-algebra is inner, then a result similar to (2.3) also holds.

COROLLARY 2.4. *Let R be a commutative Von Neumann ring. Then R satisfies SN.*

Proof. When R is a field, the Skolem-Noether Theorem [1, p. 66, 7.2C] implies that R satisfies SN, and hence the result follows from (2.3).

(2.3) indicates that any Galois theory of central separable algebras over a ring with the property SN should be inner. In this context it is useful to know if separable algebras are generated by their units.

THEOREM 2.5. *Let A be a separable R-algebra, finitely generated as an R-module, and suppose that for each x in $X(R)$ A_x is generated as an R_x-algebra by a finite set of units. Then A is generated as an R-algebra by a finite set of units.*

Proof. By assumption, for each x, A_x is generated over R_x by units u_i, $i = 1, \cdots, k$. Choose w_i's in A such that $(w_i)_x = u_i$. Exactly as in the proof of (2.3), for each i there is an idempotent h_i with x in U_{h_i} such that $w_i h_i$ is a unit of Ah_i. Let f be the product of the h_i's. Then each $w_i f$ is a unit of Af; let B be the R-subalgebra of Af generated by them. Since $(Af/B)_x = A_x/B_x = 0$ and Af/B is a finitely generated R-module, there is an idempotent $e \leq f$, with x in U_e, such that $Ae = Be$. Such U_e's cover $X(R)$. As usual, we refine the cover by a partition; let e_i, $i = 1, \cdots, n$, be the corresponding pairwise orthogonal idempotents summing to one. Then $A = Ae_1 + \cdots + Ae_n$, where each Ae_i is generated, as an Re_i-algebra, by a finite set of units of Ae_i. We now note that if w is a unit of Ae_i then $1 - e_i + w$ is a unit of A, and hence find that A is generated as an R-algebra by a finite set of units.

If R is a field with more than two elements (so that 1 is the sum of two units) then every finite-dimensional separable (indeed semi-simple) R-algebra is generated over R by a finite set of units, as is well known.

COROLLARY 2.6. *Let R be a commutative Von Neumann ring in which no idempotent equals its own negative. Then every separable R-algebra, finitely generated as an R-module, is generated by a finite set of units.*

Proof. The condition on idempotents guarantees that for each x in $X(R)$ R_x has characteristic unequal to two, hence surely more than two elements. The result now follows from (2.5) and the remark above.

With (2.4), (2.6) and Kanzaki's generalization of the Double Commutant Theorem, we can, exactly as in the classical case of fields, give a Galois theory for central separable algebras over a Von Neumann ring. For completeness, we give the full proof of this:

(2.7) Let R be a commutative Von Neumann ring in which no idempotent is its own negative and let A be a central separable R-algebra. Then every separable subalgebra of A is the fixed ring of some finite set of inner automorphisms of A.

Proof. Let B be a separable subalgebra of A. By [3, p. 105, 2], $Z(A, B)$ is separable over R and $B = Z(A, Z(A, B))$. By (1.5), $Z(A, B)$ is a direct summand of A and hence finitely generated as an R module. By (2.6), $Z(A, B)$ is generated by a finite set F of units. Then B, being the commutant of $Z(A, B)$, is the subring of A left elementwise fixed by inner automorphism by elements of F.

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