ON MANIFOLDS WITH CONJUGATION

BY
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1. Introduction

The concept of a conjugation on an almost complex manifold M^{2n} was defined by Conner and Floyd [2, §24]. Specifically, an almost complex structure on M is a real linear bundle map $J : \tau_M \to \tau_M$ on the tangent bundle of M, covering the identity map of M, and such that $J^2 = -1$, and a conjugation on (M, J) is an involution $\sigma : M \to M$ whose differential $d\sigma : \tau_M \to \tau_M$ is a conjugate linear isomorphism (i.e. $d\sigma \cdot J = -J \cdot d\sigma$).

The object of this paper is to analyze the cobordism classification of such conjugations. One approach to this sort of problem occurs in Landweber [4], considering equivariant homotopy of the Thom spectrum MU. Lacking strong transverse regularity theorems, this homotopy question is not as closely related to the geometry of the problem as one would wish. The approach taken here is analogous to the geometric part of the Conner and Floyd study of involutions.

In order to perform a cobordism analysis, one must first enlarge the collection of objects under study to give a suitable boundary, since almost complex manifolds are always even dimensional. This is performed in Section 2 by defining the notion of a conjugation on a stably almost complex manifold (as studied by Milnor [7]). Another way to describe such a manifold structure is to be given a manifold M with involution σ and an isomorphism of the normal bundle of M with an Atiyah-real vector bundle over (M, σ) (see Atiyah [1]). If (M, σ, J) is a conjugation on an almost complex manifold, the inverse to the Atiyah-real bundle $(\tau_M, J, d\sigma)$ over (M, σ) provides a stably almost complex conjugation structure on M.

One may then form cobordism groups in the standard way, and the ring of cobordism classes of stably almost complex conjugations is denoted Ω_*^{AR}. By restricting to conjugations for which the underlying involution σ is fixed-point free, one may form the cobordism ring of free stably almost complex conjugations, denoted $\hat{\Omega}_*^{AR}$. In Section 3, the interrelationship of these rings is studied, making use of the relative cobordism group of conjugations on stably almost complex manifolds with free action on the boundary, denoted $\bar{\Omega}_*^{AR}$. One then has a rather obvious exact sequence

$$\begin{tikzcd} \hat{\Omega}_*^{AR} \arrow[r, "F"] & \Omega_*^{AR} \arrow[r, "i"] & \bar{\Omega}_*^{AR} \arrow[ll, bend left, "\partial"] \end{tikzcd}$$

similar to the sequences of Conner and Floyd [3] or [2, §28.1]. Further, using the fixed-point method, one may analyze $\bar{\Omega}_*^{AR}$ and reduce this to the calculation

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of ordinary (B, f) cobordism theories (as defined by Lashof [6]) corresponding to the fibrations $f : BO_k \times BO \to BO$, where f classifies $\gamma_k \oplus (\gamma \otimes C)$. These theories are basically uncomputable except for odd primary structure. The case $k = 0$ gives a Clifford algebra cobordism theory (briefly studied in [8]) related to the work of Wells (unpublished) on immersion cobordism, and at present this is a highly unknown quantity.

In Section 4, the analysis of the free stably almost complex conjugation ring $\hat{\Omega}_*^{AR}$ is carried out. The main step is a modified Smith-theory exact sequence

$$\begin{tikzcd} \hat{\Omega}_*^{AR} \arrow[r, "\Phi + 0"] & \Omega_*^U \oplus \Omega_*^U \arrow[r, "1 \pm c"] & \hat{\Omega}_*^{AR} \arrow[ll, bend left, "\Delta"] \end{tikzcd}$$

in which Δ is a Smith homomorphism, Φ takes the cobordism class of the underlying stably almost complex manifold, and $1 \pm c$ sends M into $M \cup \bar{M}$, where $\bar{M}$ is a complex conjugate of M. The crucial point of this analysis is that a given stably almost complex manifold has exactly two conjugates, and the two choices are used on the two copies of Ω_*^U. For an almost complex manifold there is one natural choice of conjugate, so the difficulties of the conjugate for the stable class of a bundle have not previously been noted. Using results (and examples) similar to those of Landweber [5] one may completely determine the image of Φ and thereby compute $\hat{\Omega}_*^{AR}$. In particular, $\hat{\Omega}_n^{AR}$ has rank the number of partitions of $n/2$, with all torsion of order 2 and with dimension as Z_2 vector space being computable inductively.

In Section 5, the computation of $\hat{\Omega}_*^{AR}$ is combined with the computation of the odd primary structure of $\tilde{\Omega}_*^{AR}$ to prove that all torsion of Ω_*^{AR} is two primary and to determine $\Omega_*^{AR} \otimes Z[\frac{1}{2}]$.

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2. Conjugations on stably almost complex manifolds

Let M^n be a compact differentiable manifold with boundary and $\mu : M^n \to M^n$ a differentiable involution. If $f : M^n \to R^{n+2r}$ is an imbedding with normal bundle ν, a complex conjugation on ν is a pair (J, μ^*), where J is a complex structure on ν; i.e. a real linear bundle map $J : \nu \to \nu$ covering the identity map on M with $J^2 = -1$; and $\mu^* : \nu \to \nu$ is an involution given by a real linear bundle map covering μ on M and satisfying $\mu^* J = -J\mu^*$. Two complex conjugations (J_0, μ_0^*) and (J_1, μ_1^*) are equivalent if there are complex conjugations (J_t, μ_t^*), $t \in [0, 1]$, so that the maps $\nu \times I \to \nu \times I$ defined by $(x, t) \to (J_t(x), t)$ and $(x, t) \to (\mu_t^*(x), t)$ are continuous.

Note. If $\nu(f) : M^n \to BO_{2r}$ is the map classifying the normal bundle of f, a complex conjugation is a deformation of $\nu(f)$ to an equivariant map of (M, μ) into BU_r with involution induced by complex conjugation. Two complex conjugations are equivalent if there is a homotopy through such deformations of the two given deformations.

Since any two imbeddings $f_1, f_2 : M \to R^{n+2r}$ for r sufficiently large are regularly homotopic and any two regular homotopies are themselves homotopic through regular homotopies keeping end points fixed, the normal bundles of any two such imbeddings are isomorphic and there is a well defined homotopy class of isomorphisms. If $\varphi : \nu_2 \to \nu_1$ is such an isomorphism, with (J, μ^*) a complex conjugation on ν_1, then $(\varphi^{-1}J\varphi, \varphi^{-1}\mu^*\varphi)$ is a complex conjugation on ν_2. This establishes a one-to-one correspondence between the equivalence classes of complex conjugations on the normal bundles of any two imbeddings of M provided the codimension is large.

If one has a complex conjugation (J, μ^*) on ν, then

$$f \times 0 : M \to R^{n+2r+2} = R^{n+2r} \times R^2$$

is an imbedding with normal bundle $\nu \oplus 2$ (the Whitney sum of ν with a trivial 2 plane bundle). One then defines the "suspension" of (J, μ^*) to be given by $(J \times i, \mu^* \times c)$ where

$$J \times i : E(\nu) \times C \to E(\nu) \times C : (x, \alpha) \to (Jx, i\alpha)$$

and

$$\mu^* \times c : E(\nu) \times C \to E(\nu) \times C : (x, \alpha) \to (\mu^* x, \bar{\alpha})$$

($\bar{\alpha}$ = complex conjugate of α), where the total space of $\nu \oplus 2$ is thought of as the product of that of ν with the complex numbers.

A stably almost complex conjugation structure on (M^n, μ) is then an equivalence class of triples (ν, J, μ^*) where ν is the normal bundle of an imbedding $f : M^n \to R^{n+2r}$ and (J, μ^*) is a complex conjugation on ν, with two such triples being equivalent if some sufficiently high suspensions of each define the same equivalence class under the correspondence established by means of the underlying normal bundles.

DEFINITION. A stably almost complex conjugation is a triple (M, μ, ξ) where M is a compact differentiable manifold with boundary, $\mu : M \to M$ is a differentiable involution, and ξ is a stably almost complex conjugation structure on (M, μ).

If (M^n, μ, ξ) is a stably almost complex conjugation, represented by (ν, J, μ^*), where ν is the normal bundle of an imbedding $f : M^n \to R^{n+2r}$ which imbeds M in half space $\{x \in R^{n+2r} \mid x_{n+2r} \geqq 0\}$ and ∂M in R^{n+2r-1} so that a tubular neighborhood imbeds orthogonally along $f(\partial M)$, then the normal bundle of M in R^{n+2r} restricts to the normal bundle of ∂M in R^{n+2r-1} and $(\nu|_{\partial M}, J, \mu^*)$ defines a stably almost complex conjugation structure, denoted $\partial\xi$, on $(\partial M, \mu|_{\partial M})$. The resulting triple $(\partial M, \mu|_{\partial M}, \partial\xi)$ is the boundary of (M, μ, ξ).

Note. The central point is that for the boundary one must *choose* an identification of $\nu|_{\partial M}$ with the normal bundle of ∂M, obtained by *choosing* a trivialization of the normal bundle of ∂M in M.

One then defines the cobordism group of stably almost complex conjugations as the group obtained from the semigroup (disjoint union inducing the operation $+$) of isomorphism classes of stably almost complex conjugations on closed manifolds by introducing the equivalence relation $\alpha \equiv \beta$ if there exist stably almost complex conjugations ξ and η with $\alpha + \partial\xi = \beta + \partial\eta$.

This cobordism group will be denoted Ω_*^{AR}, and is the direct sum of the subgroups Ω_n^{AR} formed of classes for which the underlying manifold has dimension n. The notation AR is meant to suggest Atiyah-real, for (ν, J, μ^*) is precisely a "real" vector bundle over (M, μ) in the sense of Atiyah.

In exactly the same way, one defines the group of cobordism classes of free stably almost complex conjugations, $\hat{\Omega}_*^{AR}$, using only those stably almost complex conjugations (M, μ, ξ) for which the involution μ has no fixed points.

Finally, one defines a relative cobordism group $\bar{\Omega}_*^{AR}$ of equivalence classes of stably almost complex conjugations (M, μ, ξ) for which $\mu|_{\partial M}$ has no fixed points. One says (M, μ, ξ) is cobordant to (M', μ', ξ') if there are stably almost complex conjugations (V, σ, η) and (W, ρ, λ) with σ having no fixed points such that

$$\partial(V, \sigma, \eta) = (\partial M', \mu'|_{\partial M'}, \partial\xi') \cup (\partial M, \mu|_{\partial M}, -\partial\xi)$$

and ∂W is the manifold formed from $M \cup V \cup M'$ by identifying boundaries, with ρ restricting to μ, σ, or μ' respectively, and λ restricting to ξ, η, or $-\xi'$ respectively.

Note. $(M, \mu, -\xi)$ is obtained by taking $M \times I \subset R^{n+2r} \times R$ with involution $\mu \times 1$, identifying the normal bundle of $M \times I$ with the pullback of ν to give an induced structure on $M \times I$ restricting to ξ on $M \times 0$, and letting $-\xi$ be the structure induced on $M \times 1$. This corresponds to reversing the trivialization of the normal bundle of M in $M \times I$, or the orientation of the normal bundle of M in R^{n+2r+1}.

Before beginning the analysis of these cobordism groups, one should note that a stably almost complex conjugation structure on (M^n, μ) may be defined using the tangent bundle rather than the normal bundle of M. Let τ denote the tangent bundle of M with involution $d\mu$, the differential of μ.

ASSERTION. *The stably almost complex conjugation structures on (M, μ) are in one-to-one correspondence with equivalence classes of triples (α, θ, j) where α is a trivial real vector bundle over M, $\theta : \alpha \to \alpha$ is an involution of α by real bundle maps covering μ, and $j : \tau \oplus \alpha \to \tau \oplus \alpha$ is a complex structure for which $d\mu \oplus \theta$ is a conjugation.*

Proof. Being given (α, θ, j), the inverse to $(\tau \oplus \alpha, j, d\mu \oplus \theta)$ as Atiyah-real bundle is an Atiyah-real structure on a bundle stably isomorphic to the normal bundle of M, and hence defines a stably almost complex conjugation structure on some normal bundle.

Conversely, suppose (ν, J, α^*) is a stably almost complex conjugation structure on the normal bundle of $f : M^n \to R^{n+2r}$, and choose an inverse for

(ν, J, μ^*) as Atiyah-real bundle, say (η, K, ρ) so that

$$(\nu \oplus \eta, J \oplus K, \mu^* \oplus \rho) \cong (M^n \times C^k, 1 \times i, \mu \times c).$$

Let (ξ, L, σ) be an inverse for the Atiyah-real bundle $(\tau \oplus \tau, I, (-d\mu) \oplus (d\mu))$ where $I(a, b) = (-b, a)$. Then $(\eta \oplus \xi \oplus \tau, \rho \oplus \sigma \oplus (-d\mu))$ is an involution on a trivial real vector bundle over M (covering μ) and the Whitney sum with the bundle with involution $(\tau, d\mu)$ admits a complex structure for which $[\rho \oplus \sigma \oplus (-d\mu)] \oplus d\mu$ is a conjugation. Specifically, $K \oplus L \oplus I$ is such a complex structure.**

In particular, if M is an almost complex manifold with conjugation, this gives a well defined stably almost complex conjugation structure by letting α be zero dimensional. For the more general case, one must consider pairs (α, θ). It should be noted that (α, θ) need not be simply the product $M \times V$ with V a representation space of Z_2.

3. The unrestricted-free relationship

Let $F : \hat{\Omega}_*^{AR} \to \Omega_*^{AR}$ denote the homomorphism obtained by considering a free stably almost complex conjugation as simply a stably almost complex conjugation by forgetting the freeness condition. Let $i : \Omega_*^{AR} \to \bar{\Omega}_*^{AR}$ be the homomorphism obtained by considering a closed manifold as a manifold with a free involution on its boundary, and let $\partial : \bar{\Omega}_*^{AR} \to \hat{\Omega}_*^{AR}$ be the homomorphism induced by sending (M, μ, ξ) to the class of $(\partial M, \mu|_{\partial M}, \partial\xi)$.

PROPOSITION 1. *The sequence*

$$\begin{tikzcd} \hat{\Omega}_*^{AR} \arrow[r, "F"] & \Omega_*^{AR} \arrow[r, "i"] & \bar{\Omega}_*^{AR} \arrow[ll, bend left, "\partial"] \end{tikzcd}$$

is exact.

Proof. (a) $\partial i = 0$, for $\partial i(M, \mu, \xi) = [(\partial M, \mu|_{\partial M}, \partial\xi)]$ and ∂M is empty.

(b) $F\partial = 0$, for $F\partial(M, \mu, \xi)$ is the class of $(\partial M, \mu|_{\partial M}, \partial\xi)$ which bounds (M, μ, ξ).

(c) $iF = 0$, for if (M, μ, ξ) is free, let $(W, \rho, \lambda) = (M \times I, \mu \times 1, \pi^*\xi)$ and (V, σ, η) the induced structure on $M \times 1$, giving a cobordism to zero of (M, μ, ξ).

(d) If $\partial[M, \mu, \xi] = 0$, then $(\partial M, \mu|_{\partial M}, \partial\xi) = \partial(X, \chi, \zeta)$ with χ free and let (T, τ, φ) be the stably almost complex conjugation structure on $M \cup X/\partial M \equiv \partial X$ with τ inducing μ and χ and φ inducing ξ and $-\rho$. Then

$$(W, \rho, \lambda) = (T \times I, \tau \times 1, \pi^*\varphi), \ (V, \sigma, \eta) = (X \times 0, \chi, -\zeta)$$

defines a cobordism of (M, μ, ξ) and $i(T, \tau, \varphi)$.

(e) If $F[M, \mu, \xi] = 0$, then (M, μ, ξ) bounds an unrestricted action (V, σ, η) and $\partial[V, \sigma, \eta] = [M, \mu, \xi]$.

(f) If $i[M, \mu, \xi] = 0$, with (W, ρ, λ) and (V, σ, η) as a cobordism to zero, then $\partial M = \partial V = \emptyset$ and (W, ρ, λ) is a cobordism of (M, μ, ξ) to $F(V, \sigma, \eta)$.**

In order to make this sequence of value for the study of Ω_*^{AR} one needs additional information about the other groups. Beginning with $\bar{\Omega}_*^{AR}$, one may apply a fixed point structure analysis.

If (M^n, μ, ξ) is a stably almost complex conjugation with ξ given by (ν, J, μ^*) where ν is the normal bundle of $f : M^n \to R^{n+2r}$, with μ free on ∂M, then the fixed point set F of μ is a closed submanifold imbedded in the interior of M. One may write $F = \bigcup_{k=0}^n F^k$, where F^k is the union of the k-dimensional components of F, and the normal bundle of F^k in R^{n+2r} is the Whitney sum of the normal bundle of F^k in M, an $n - k$ plane bundle denoted ν_{n-k}, and the restriction to F^k of the normal bundle of M. Over F^k, μ^* is a conjugation on $\nu\,|_{F^k}$ covering the identity map, so that $\nu\,|_{F^k}$ decomposes into the Whitney sum of $\xi_k = \xi_k^+$ and ξ_k^-, the $+1$ and -1 eigenvalue bundles of μ^*. The operation J is a real linear isomorphism interchanging these two bundles, so $\xi_k^- \cong \xi_k^+$ or $(\nu\,|_{F^k}, J, \mu^*)$ is isomorphic to $(\xi_k \otimes C, 1 \otimes i, 1 \otimes c)$.

Clearly, application of this same fixed point analysis to a cobordism gives a cobordism of the fixed point sets together with a decomposition of the normal bundle into a real bundle (of appropriate dimension $n - k$ over the $k + 1$ dimensional component) and the complexification of a real bundle.

One then introduces the (B, f) cobordism theories $\Omega_*(BO_s \times BO, f_s)$ defined by the map $f_s : BO_s \times BO \to BO$ classifying the Whitney sum of the canonical s plane bundle γ^s over BO_s and the complexification $\gamma \otimes C$ of the canonical bundle over BO. This is precisely the cobordism theory obtained by using manifolds together with a decomposition of their normal bundle as the Whitney sum of an s plane bundle and the complexification of a real bundle.

Thus, the fixed point analysis defines a homomorphism

$$f : \bar{\Omega}_n^{AR} \to \oplus_{k=0}^n \Omega_k(BO_{n-k} \times BO, f_{n-k}).$$

Being given a closed manifold F^k together with a decomposition of its normal bundle as $\nu_{n-k} \oplus (\xi_k \otimes C)$, one may consider the compact n-manifold with boundary given by $D(\nu_{n-k})$, the disc bundle of ν_{n-k}. Multiplication by -1 in the fibers of ν_{n-k} defines an involution on $D(\nu_{n-k})$, with fixed point set F^k (the zero section) and hence free on the boundary. The tangent bundle of $D(\nu_{n-k})$ is the Whitney sum of the pullback of the tangent bundle of F^k, $\pi^*\tau_{F^k}$, and the bundle tangent to the fibers, which is isomorphic to $\pi^*\nu_{n-k}$. Thus

$$\tau_{D(\nu_{n-k})} \oplus (\pi^*\xi_k \otimes C) \cong \pi^*(\tau_{F^k} \oplus \nu_{n-k} \oplus (\xi_k \otimes C))$$

is trivial and so the normal bundle of $D(\nu_{n-k})$ is stably isomorphic to $\pi^*\xi_k \otimes C$. Considering the total space of $\pi^*\xi_k \otimes C$ as a subspace of $D(\nu_{n-k}) \times E(\xi_k \otimes C)$, one has a complex structure given by $1 \times (1 \otimes i)$ and a conjugation given by $(-1) \times (1 \otimes c)$ covering (-1) on $D(\nu_{n-k})$. This defines a stably almost complex conjugation $(D(\nu_{n-k}), -1, \xi)$ and sending (F^k, ν_{n-k}, ξ_k) to this class defines a homomorphism

$$d : \oplus_{k=0}^{n} \Omega_k(BO_{n-k} \times BO, f_{n-k}) \to \bar{\Omega}_n^{AR}.$$

Clearly, $f \circ d = 1$, while $d \circ f[M, \mu, \xi]$ is represented by $(D(\nu), \mu, \xi)$ which may be identified as a tubular neighborhood of the fixed point set of μ in M. A cobordism of this to (M, μ, ξ) is given by

$$(W, \rho, \lambda) = (M \times I, \mu \times 1, \pi^*\xi)$$

and (V, σ, η) is given by restriction to

$$V = \partial M \times I \cup [M - \text{interior } (D(\nu))] \times 1.$$

Thus $d \circ f = 1$, proving:

PROPOSITION 2. *The fixed point homomorphism*

$$f : \bar{\Omega}_n^{AR} \to \oplus_{k=0}^{n} \Omega_k(BO_{n-k} \times BO, f_{n-k})$$

is an isomorphism.

This result is useful since one knows how to compute (B, f) cobordism theories by homotopy methods. Applying the Pontrjagin-Thom construction, one has

$$\Omega_p(BO_s \times BO, f_s) = \lim_{t\to\infty} \pi_{p+s+2t}(T[\gamma^s \oplus (\gamma^t \otimes C)], \infty)$$

where $T[\gamma^s \oplus (\gamma^t \otimes C)]$ is the Thom space of the bundle $\gamma^s \oplus (\gamma^t \otimes C)$ over $BO_s \times BO_t$.

From a practical point of view this isn't very helpful, since for example the 2-primary structure of $\Omega_*(BO_0 \times BO, f_0)$ is known to be difficult to compute. The odd primary structure is relatively easy and will be computed in Section 5.

Note. This calculational complication carries over to Ω_*^{AR} since

$$\Omega_*(BO_0 \times BO, f_0)$$

is identifiable with the direct summand of Ω_*^{AR} given by stably almost complex conjugations (M, μ, ξ) for which μ keeps M pointwise fixed. For example, $\Omega_1(BO_0 \times BO, f_0) = Z_4$ (Wells, unpublished) and hence Ω_*^{AR} has elements of order 4.

4. Cobordism of free conjugations

In order to analyze the groups $\hat{\Omega}_*^{AR}$, one may make use of the Smith homomorphism $\Delta : \hat{\Omega}_n^{AR} \to \hat{\Omega}_{n-1}^{AR}$. Being given (M, μ, ξ) with ξ represented by (ν, J, μ^*), and μ a free involution on M, one may find a submanifold N contained in M, invariant under μ, such that M is the union of a manifold with boundary V and its image under μ, μV, joined along their common boundary N. One may construct N as in [2, (26.1)] by finding an equivariant map $(M, \mu) \to (S^r, a)$, a the antipodal involution, $r > \dim M$, which is transverse regular on S^{r-1}. The normal bundle of N in M is then trivial, so that one may identify the normal bundle of N with the restriction to N of ν. Thus, by restriction to N, (ν, J, μ^*) defines a stably almost complex conjugation $(N, \mu|_N, \xi|_N)$. Clearly the same construction may be applied to a cobordism, so that assigning to (M, μ, ξ) the class of $(N, \mu|_N, \xi|_N)$ defibes a homomorphism $\Delta : \hat{\Omega}_*^{AR} \to \hat{\Omega}_*^{AR}$ of degree -1.

Clearly, if $(N, \mu|_N, \xi|_N)$ is obtained from (M, μ, ξ) in this way, then N bounds as a stably almost complex manifold, for in the above notation $N = \partial V$. Conversely, if (M, μ, ξ) is a free stably almost complex conjugation and $M = \partial W$ as a stably almost complex manifold with complex normal bundle ν_W restricting to ν_M, with (J, μ^*) on ν_M defining ξ, then one may form a closed stably almost complex manifold T from $W \times \{1, -1\}$ by identifying $M = \partial W \times 1$ with $\partial W \times (-1)$ using μ, with the normal bundle formed from ν_W over $W \times 1$ and the conjugate $\bar{\nu}_W$ over $W \times (-1)$ identified over the intersection by μ^*. The involutions on T and ν_T, ρ and ρ^*, given by the interchange of $W \times 1$ with $W \times (-1)$ and of ν_W with $\bar{\nu}_W$ then define a free stably almost complex conjugation (T, ρ, ξ'). Clearly $\Delta(T, \rho, \xi') = (M, \mu, \xi)$ since T is the union of $W \times 1$ and its image under ρ joined along their common boundary M. Thus, one has:

LEMMA 1. *The sequence*

$$\hat{\Omega}_*^{AR} \xrightarrow{\Delta} \hat{\Omega}_*^{AR} \xrightarrow{\Phi} \Omega_*^U$$

is exact, where Φ takes the cobordism class of the underlying stably almost complex manifold.

If on the other hand $\Delta[M, \mu, \xi] = 0$, then with the notation used in defining Δ, one has $(N, \mu|_N, \xi|_N) = \partial(W, \rho, \zeta)$ for some free stably almost complex conjugation (W, ρ, ζ) and one may form a manifold T with boundary by joining $M \times [0, 1]$ and $W \times [-1, 1]$ by identifying $\partial W \times [-1, 1]$ with a tubular neighborhood $N \times [-1, 1] \times 1$ of $N \times 1$ in $M \times 1$. This identification may be made so that the complex normal bundles are compatible, with $\mu \times 1$ agreeing with $\rho \times (-1)$ and $\mu^* \times 1$ agreeing with $\rho^* \times (-1)$ along the intersections. The resulting free stably almost complex conjugation on T has boundary in two parts, one being obtained from $M \times 0$ and being (M, μ, ξ), while the other portion consists of two disjoint manifolds interchanged under the involution (one piece P being formed from V minus a neighborhood of N and $W \times 1$ by joining along the common boundary). Thus, the remaining portion of the boundary is $P \times \{1, -1\}$ with involution $1 \times (-1)$, with normal bundle given by ν_P over $P \times 1$ and $\bar{\nu}_P$ over $P \times (-1)$, with conjugation given by interchanging the two summands.

If one begins with any closed stably almost complex manifold Q^n with $f : Q^n \to R^{n+2r}$ an imbedding, with normal bundle ν and complex structure J on ν, one may then form a free stably almost complex conjugation on two copies of Q, $Q \times \{1, -1\}$, with interchange involution, by taking the normal bundle ν over $Q \times 1$, its conjugate $\bar{\nu}$ over $Q \times (-1)$ and interchange involution on the normal bundle. This construction defines a class in $\hat{\Omega}_*^{AR}$, but depends on the choice of the normal bundle (ν, J) and not just upon its class as a stable bundle.

Specifically, if $g : Q^n \times \{1, -1\} \to R^{n+2r}$ is an imbedding with normal bundle $\nu \cup \bar{\nu}$, stabilization gives normal bundle $(\nu \cup \bar{\nu}) \times C$ with involution (interchange $\times$ conjugation), while first stabilizing ν gives $\nu \times C \cup \bar{\nu} \times \bar{C}$ with interchange involution. That these are inequivalent is most easily seen by noting that the orientation of the fibers (as induced by the complex structure) is preserved by one involution and reversed by the other. The difficulty is that the space of complex structures on R^2 has two components, one containing i, the other containing $-i$.

If one stabilizes twice the problem disappears, for on $C \oplus C$ the complex structures $(x, y) \to (ix, iy)$ and $(x, y) \to (-ix, -iy)$ may be joined by a path of complex structures. Specifically, letting H denote the quaternions, $ie^{\pi jt}$, $0 \leq t \leq 1$, is a path of complex structures joining i and $-i$, so that $C \oplus C \cong \bar{C} \oplus \bar{C}$.

Thus one obtains precisely two free stably almost complex conjugations depending on the complex dimension of the representative normal bundle mod 2. Clearly, the same process may be used for a stably almost complex manifold with boundary, and thus one has defined two homomorphisms from Ω_*^U to $\hat{\Omega}_*^{AR}$. Rather than distinguish these more precisely, one may simply add them together to define a homomorphism

$$1 \pm c : \Omega_*^U \oplus \Omega_*^U \to \hat{\Omega}_*^{AR}.$$

This obviously has image consisting of all classes of disjoint unions of two manifolds which are interchanged by the involution, and hence:

LEMMA 2. *The sequence*

$$\Omega_*^U \oplus \Omega_*^U \xrightarrow{1 \pm c} \hat{\Omega}_*^{AR} \xrightarrow{\Delta} \hat{\Omega}_*^{AR}$$

is exact.

In order to understand the homomorphism $1 \pm c$ more fully, let M be a stably almost complex manifold of dimension $2n$, $f : M^{2n} \to R^{2n+2r}$ an imbedding with complex normal bundle ν, and form $M \times \{1, -1\}$ with normal bundle $\nu \cup \bar{\nu}$. This defines the stably almost complex manifold structure (for the class of r mod 2) underlying the $1 \pm c$ operation. One then has:

ASSERTION. *The class of $M^{2n} \times \{1, -1\}$ with normal bundle $\nu \cup \bar{\nu}$ in R^{2n+2r} as an element of Ω_{2n}^U is $\{1 + (-1)^{n+r}\}[M]$.*

Proof. Let $[M, \nu] \in H_{2n}(M; Z)$ denote the orientation class induced by the complex structure ν on the normal bundle of M. The complex structure $\bar{\nu}$ then induces the orientation $[M, \bar{\nu}] = (-1)^r[M, \nu]$. Since the Chern class $c(\bar{\nu})$ is given by

$$c(\bar{\nu}) = 1 - c_1(\nu) + c_2(\nu) - c_3(\nu) + \cdots$$

one has for each ω a partition of n, that $c_\omega(\bar{\nu}) = (-1)^n c_\omega(\nu)$ in $H^{2n}(M; Z)$.

Thus

$$c_\omega(\bar{\nu})[M, \bar{\nu}] = (-1)^{n+r} c_\omega(\nu)[M, \nu]$$

and the class of $(M, \nu) \cup (M, \bar{\nu})$ in Ω_*^U is $\{1 + (-1)^{n+r}\}[M]$.**

Note. If dim M is odd, everything must bound, so no analysis is required.

Now let M be a stably almost complex manifold such that $M \cup \bar{M}$ ($\bar{M}$ being one conjugate structure) bounds in $\hat{\Omega}_*^{AR}$ (notice that this completely determines r) and let $\partial(V, \mu, \xi) = M \cup \bar{M}$. For dim $M = 0$, one may suppose by cobordism that M consists of k points with the same orientation. Then V is a union of intervals joining one point in M and one in $\bar{M}$ and a collection of circles. Since an interval has no free involution, k must be even (with intervals being interchanged). Since two points with the same orientation admits the interchange conjugation, $[M]$ lies in the image of Φ. If dim $M > 0$, then by a cobordism one may assume M is connected (using complex surgery). Let $N \subset V$ be a submanifold of codimension one such that V is the union of W and μW along their common boundary N. Since μ interchanges the boundary components M and $\bar{M}$ of V, N may be chosen to be a closed manifold contained in the interior of V. One may then label as W that manifold whose boundary is $M \cup N$, which defines a stably almost complex cobordism of M and N. N is, however, stable under μ and ξ restricts to a free stably almost complex conjugation structure on $(N, \mu|_N)$. Thus $[M] \in \Omega_*^U$ belongs to the image of $\Phi : \hat{\Omega}_*^{AR} \to \Omega_*^U$.

If (M, μ, ξ) is a free stably almost complex conjugation, one may form the manifold $M \times [-1, 1]$ with involution $\mu \times (-1)$ having normal bundle $\nu \times [-1, 1] = \pi^*(\nu)$, complex structure $J \times 1$ and conjugation $\mu^* \times (-1)$ covering $\mu \times (-1)$, where ξ is given by (ν, J, μ^*). The boundary of the resulting manifold is a free stably almost complex conjugation interchanging two stably almost complex manifolds, one of which is M with normal structure given by ν, and hence the other is the conjugate of M cobordant to $-M$.

Combining this with Lemmas 1 and 2 gives:

PROPOSITION 3. *The sequence*

$$\begin{tikzcd} \hat{\Omega}_*^{AR} \arrow[r, "\Phi + 0"] & \Omega_*^U \oplus \Omega_*^U \arrow[r, "1 \pm c"] & \hat{\Omega}_*^{AR} \arrow[ll, bend left, "\Delta"] \end{tikzcd}$$

is exact.

In order to make use of this sequence, it is necessary to determine the image of $\Phi : \hat{\Omega}_*^{AR} \to \Omega_*^U$. The result is:

LEMMA 3. *The image of $\Phi : \hat{\Omega}_*^{AR} \to \Omega_*^U$ is precisely the kernel of the homomorphism $\Psi : \Omega_*^U \to \mathfrak{N}_*$ sending each stably almost complex manifold to its unoriented cobordism class.*

Proof. To see that im $\Phi \subset$ ker Ψ one notes that for (M, μ, ξ) a free stably almost complex conjugation, μ acts freely on M so M bounds as manifold (M

bounds the disc bundle of the double cover $M \to M/\mu$). To see that $\ker \Psi \subset \operatorname{im} \Phi$, one first proves that every class in Ω_*^U contains a manifold with conjugation; i.e. the homomorphism $\Phi' : \Omega_*^{AR} \to \Omega_*^U$ is epic. For this, one notes that it is trivial in dimension zero (since a point with either orientation has the trivial conjugation) while in positive dimensions every class in Ω_*^U is represented as an integral polynomial in the classes of complex projective spaces and hypersurfaces $H(m, n) \subset CP(m) \times CP(n)$ defined by equations with real coefficients. (This is a result of Milnor, but the only exposition is in Thom [9].) Since $CP(n)$ and $H(m, n)$ have conjugations given by conjugation of the complex number spaces underlying their definition, while $\operatorname{im} \Phi'$ is clearly closed under sums, products, and additive inverses (using appropriately chosen conjugates for the normal bundle) this shows that Φ' is epic. Then $\operatorname{im} \Phi$ is an ideal, for if $\alpha \in \operatorname{im} \Phi$ is represented by M in $\hat{\Omega}_*^{AR}$ and $\beta \in \Omega_*^U$ is represented by N in Ω_*^{AR}, then $\alpha \cdot \beta$ is represented by the product conjugation $M \times N$, which is free.

As already noted $\operatorname{im} \Phi \supset 2\Omega_*^U$ by taking $M \cup \bar{M}$ for an appropriate conjugate and hence it suffices to prove that $\operatorname{im} \Phi$ contains generators for the ideal given by the kernel of $\Psi' : \Omega_*^U/2\Omega_*^U \to \mathfrak{N}_*$. Representative generators for this ideal are well known and are given by the nonsingular complex algebraic varieties of dimension $2^{s+2} - 2$, $s \geqq 0$, $H(2^s, 2^s) \subset CP(2^s) \times CP(2^s)$ defined by the equations $\sum_{i=0}^{2^s} z_i w_i = 0$, where

$$CP(2^s) \times CP(2^s) = \{([z_0, \cdots, z_{2^s}], [w_0, \cdots, w_{2^s}])\}$$

in standard homogeneous coordinates. A fixed point free conjugation for the usual complex manifold structure on $H(2^s, 2^s)$ is given by

$$T([z_0, \cdots, z_{2^s}], [w_0, \cdots, w_{2^s}]) = ([\bar{w}_0, \cdots, \bar{w}_{2^s}], [\bar{z}_0, \cdots, \bar{z}_{2^s}]).$$

[*Note.* $T([z], [w]) = ([z], [w])$ implies $w = \alpha\bar{z}$, $\alpha \neq 0$, so $0 = \sum z_i w_i = \alpha \sum z_i \bar{z}_i$, but this gives $z_i = 0$ for all i which is impossible.]**

One may apply this result, together with Proposition 3, and $\Omega_{\text{odd}}^U = 0$ to obtain exact sequences

$$(1) \quad 0 \to \Omega_{2k}^U \oplus \operatorname{im} (\Psi_{2k} : \Omega_{2k}^U \to \mathfrak{N}_{2k}) \xrightarrow{1 \pm c} \hat{\Omega}_{2k}^{AR} \xrightarrow{\Delta} \hat{\Omega}_{2k-1}^{AR} \to 0$$

and

$$(2) \qquad 0 \to \hat{\Omega}_{2k+1}^{AR} \xrightarrow{\Delta} \hat{\Omega}_{2k}^{AR} \xrightarrow{\Phi} \ker (\Psi_{2k} : \Omega_{2k}^U \to \mathfrak{N}_{2k}) \to 0.$$

Using these sequences, one may describe $\hat{\Omega}_*^{AR}$ as a group by the inductive result:

PROPOSITION 4. (a) *The group $\hat{\Omega}_{2k-1}^{AR}$ is a vector space over Z_2 of dimension a_{2k-1}; and*

(b) *The group $\hat{\Omega}_{2k}^{AR}$ is the direct sum of a free abelian group of rank the number of partitions of k and a vector space over Z_2 of rank a_{2k}, where*

$$a_{2k+1} = a_{2k} = a_{2k-1} + \dim (\operatorname{im} \Psi'_{2k}) - \dim (\ker \Psi'_{2k})$$

with $\Psi'_{2k} : \Omega_{2k}^U/2\Omega_{2k}^U \to \mathfrak{N}_{2k}$, and $a_{-1} = 0$.

Note. $\Omega_*^U/2\Omega_*^U$ is a Z_2 polynomial ring on classes z_{2k}, $k > 0$, of dimension $2k$ and $\mathfrak{N}_*$ is a Z_2 polynomial ring on classes x_k, $k > 0$ and not of the form $2^s - 1$, of dimension k, with Ψ'_* the homomorphism of algebras defined by $\Psi'_*(z_{2k}) = x_k^2$ for $k \neq 2^s - 1$ and $\Psi'_*(z_{2k}) = 0$ for $k = 2^s - 1$. Thus a_q is inductively computable by the given formula.

Proof. Suppose inductively that $\hat{\Omega}_{2k-1}^{AR}$ is a Z_2 vector space of dimension a_{2k-1} (which is clearly true for $k = 0$ with $a_{-1} = 0$). Then the exact sequence (1) becomes

$$0 \rightarrow Z^{|\pi(k)|} \oplus Z_2^{b_{2k}} \rightarrow \hat{\Omega}_{2k}^{AR} \rightarrow Z_2^{a_{2k-1}} \rightarrow 0$$

where $\Omega_{2k}^U = Z^{|\pi(k)|}$ is a free abelian group of rank $|\pi(k)|$, the number of partitions of k, and $\operatorname{im}\Psi_{2k} = \operatorname{im}\Psi'_{2k} = Z_2^{b_{2k}}$. Define a homomorphism

$$\theta : \hat{\Omega}_{2k}^{AR} \rightarrow \mathfrak{N}_{2k}$$

by sending (M, μ, ξ) into the unoriented bordism class of M/μ. Then

$$\theta\cdot(1 \pm c) : \Omega_{2k}^U \oplus \Omega_{2k}^U \rightarrow \mathfrak{N}_{2k}$$

sends $([M], [N])$ into $\Psi_{2k}[M] + \Psi_{2k}[N]$ and thus the composite

$$\operatorname{im}\Psi_{2k} \xrightarrow{1 \pm c} \hat{\Omega}_{2k}^{AR} \xrightarrow{\theta} \mathfrak{N}_{2k}$$

is an isomorphism onto the direct summand $\operatorname{im}\Psi_{2k} \subset \mathfrak{N}_{2k}$. Thus the Z_2 vector space $(1 \pm c)(\operatorname{im}\Psi_{2k})$ is a direct summand of $\hat{\Omega}_{2k}^{AR}$. If T is a complementary summand, one then has an exact sequence

$$0 \rightarrow Z^{|\pi(k)|} \xrightarrow{\alpha} T \xrightarrow{\beta} Z_2^{a_{2k-1}} \rightarrow 0.$$

The homomorphism $\Phi : \hat{\Omega}_{2k}^{AR} \rightarrow \Omega_{2k}^U = Z^{|\pi(k)|}$ clearly sends the torsion group $(1 \pm c)(\operatorname{im}\Psi_{2k})$ to zero and induces an epimorphism $T \rightarrow \ker\Psi_{2k}$, with $\alpha(Z^{|\pi(k)|})$ mapping precisely onto $2\Omega_{2k}^U$. This induces an exact sequence

$$0 \rightarrow \text{Torsion }(T) \rightarrow Z_2^{a_{2k-1}} \rightarrow \ker\Psi'_{2k} \rightarrow 0$$

so that T is the direct sum of a Z_2 vector space of dimension $a_{2k-1} - \dim(\ker\Psi'_{2k})$ and a free abelian group of rank $|\pi(k)|$. This completes part b) and the calculation of a_{2k}.

Applying exact sequence (2), the kernel of $\Phi : \hat{\Omega}_{2k}^{AR} \rightarrow \ker\Psi_{2k} = Z^{|\pi(k)|}$ is precisely the torsion subgroup, so $\hat{\Omega}_{2k+1}^{AR} \cong \text{Torsion }(\hat{\Omega}_{2k}^{AR})$ is a Z_2 vector space of rank $a_{2k} = a_{2k+1}$. This completes the induction step and thereby proves the proposition.**

Note. The product in $\hat{\Omega}_*^{AR}$ given by the product of manifolds makes $\hat{\Omega}_*^{AR}$ a ring, with $\Phi : \hat{\Omega}_*^{AR} \rightarrow \Omega_*^U$ a ring homomorphism. Tensoring with $Z[\frac{1}{2}]$, Φ becomes an isomorphism of rings

$$\Phi : \hat{\Omega}_*^{AR} \otimes Z[\tfrac{1}{2}] \cong \Omega_*^U \otimes Z[\tfrac{1}{2}] = Z[\tfrac{1}{2}][x_{2i}] \quad (\dim x_{2i} = 2i).$$

5. Odd primary structure of the unrestricted groups

Returning to the exact sequence of Proposition 1, one has the diagram

$$\begin{tikzcd} \hat{\Omega}_*^{AR} \arrow[r, "F"] \arrow[dr, "\Phi"'] & \Omega_*^{AR} \arrow[r, "i"] \arrow[d, "\Phi'"] & \bar{\Omega}_*^{AR} \arrow[ll, bend right, "\partial"'] \\ & \Omega_*^U & \end{tikzcd}$$

where Φ and Φ' take the cobordism class of the underlying manifold. Applying the analysis of $\hat{\Omega}_*^{AR}$, one has $\ker F \subset \ker \Phi$ which is the torsion subgroup of $\hat{\Omega}_*^{AR}$, consisting of elements of order 2. Since cokernel Φ is also a Z_2 vector space, Φ' defines a splitting of the exact sequence provided one ignores the prime 2. Thus one has:

LEMMA 4. *Modulo the Serre class of 2-primary groups, one has an isomorphism*

$$\Omega_*^{AR} \cong \hat{\Omega}_*^{AR} \oplus \bar{\Omega}_*^{AR}.$$

Note. Ω_*^{AR}, $\hat{\Omega}_*^{AR}$, and $\bar{\Omega}_*^{AR}$ are all rings with the product given by the product of manifolds, and F, i, and Φ' are ring homomorphisms. Thus Ω_*^{AR} contains the ideals $\operatorname{im} F = \ker i$ and $\ker \Phi'$ (isomorphic mod 2 primary groups with $\bar{\Omega}_*^{AR}$). Thus $\Omega_*^{AR} \otimes Z[\frac{1}{2}]$ is isomorphic to $\hat{\Omega}_*^{AR} \otimes Z[\frac{1}{2}] \oplus \bar{\Omega}_*^{AR} \otimes Z[\frac{1}{2}]$ as rings (both summands being ideals).

Thus, the odd primary structure of Ω_*^{AR} may be determined by analyzing $\bar{\Omega}_*^{AR}$, or applying Proposition 2, by analyzing

$$\Omega_*(BO_s \times BO, f_s) = \operatorname{dir lim} \pi_*(T(\gamma^s \oplus (\gamma^t \otimes C)), \infty).$$

LEMMA 5. *Let $\varphi : BO \to BSO$ be the map classifying $\gamma \otimes C$ with the orientation given by the complex structure. φ is a homotopy equivalence modulo the Serre class of 2-primary groups, and thus the induced map of Thom spectra*

$$T\varphi : \{T(\gamma^t \otimes C)\} \to \{TBSO_{2n}\}$$

is also a homotopy equivalence modulo the Serre class of 2-primary groups.

Proof. Both $H^*(BSO; Z[\frac{1}{2}])$ and $H^*(BO; Z[\frac{1}{2}])$ are the polynomial rings over $Z[\frac{1}{2}]$ on the Pontrjagin classes $\mathfrak{p}_i$ (dimension $4i$) of the universal bundles. By the Whitney sum formula,

$$\varphi^*(\mathfrak{p}) = \mathfrak{p}(\gamma \otimes C) = \mathfrak{p}(\gamma \oplus \gamma) = \mathfrak{p}(\gamma)\cdot\mathfrak{p}(\gamma)$$

so $\varphi^*(\mathfrak{p}_i) = 2\mathfrak{p}_i +$ decomposables. Thus

$$\varphi^* : H^*(BSO; Z[\tfrac{1}{2}]) \to H^*(BO; Z[\tfrac{1}{2}])$$

is an isomorphism. Both Thom spectra are orientable for integral cohomology, and by the Thom isomorphism

$$T\varphi^* : \tilde{H}^*(TBSO; Z[\tfrac{1}{2}]) \to \tilde{H}^*(T(\gamma \otimes C); Z[\tfrac{1}{2}])$$

is also an isomorphism.**

COROLLARY. *The spectral morphism defined by*

$$T(\gamma^s \oplus (\gamma^t \otimes C)) = T(\gamma^s) \wedge T(\gamma^t \otimes C) \xrightarrow{1 \wedge T\varphi} T(\gamma^s) \wedge TBSO_{2n}$$

induces an isomorphism modulo the Serre class of 2-primary groups

$$1 \wedge T\varphi_* : \Omega_*(BO_s \times BO, f_s) \to \tilde{\Omega}_*^{SO}(TBO_s)$$

into the reduced oriented bordism of the Thom space $TBO_s = T(\gamma^s)$.

Proof. The Thom space of an external Whitney sum is the smash product of the Thom spaces, so $1 \wedge T\varphi_*$ is just the induced homotopy homomorphism of the odd primary homotopy equivalence $1 \wedge T\varphi$.**

In order to analyze these bordism groups one considers the pair $(D(\gamma^s), S(\gamma^s))$ consisting of the disc and sphere bundle of γ^s, giving the cofibration sequence

$$S(\gamma^s) \xrightarrow{i} D(\gamma^s) \xrightarrow{p} (D(\gamma^s)/S(\gamma^s)) = T(\gamma^s).$$

Projection on the base space gives a homotopy equivalence of $D(\gamma^s)$ and BO_s, while taking the orthogonal complement of a unit vector in $D(\gamma^s)$ defines a fibration $S(\gamma^s) \to BO_{s-1}$ with fiber the infinite sphere, which is contractible. Thus one has the cofibration sequence

$$BO_{s-1} \xrightarrow{i} BO_s \xrightarrow{p} T(\gamma^s)$$

and i pulls γ^s back to $\gamma^{s-1} \oplus 1$. Then $H^*(BO_n; Z[\frac{1}{2}])$ is the polynomial ring over $Z[\frac{1}{2}]$ on the Pontrjagin classes $\mathfrak{p}_i$, $1 \leq i \leq [n/2]$, [ ] denoting integral part, of the universal bundle γ^n. In particular

$$i^* : H^*(BO_s; Z[\tfrac{1}{2}]) \to H^*(BO_{s-1}; Z[\tfrac{1}{2}])$$

is epic, and thus

$$\begin{aligned} \tilde{H}^*(T(\gamma^s); Z[\tfrac{1}{2}]) &= 0 && \text{if } s \text{ is odd} \\ &= Z[\tfrac{1}{2}][\mathfrak{p}_1, \cdots, \mathfrak{p}_{s/2}]\cdot \mathfrak{p}_{s/2} && \text{if } s \text{ is even,} \end{aligned}$$

the latter being the ideal consisting of multiples of $\mathfrak{p}_{s/2}$.

This implies that $\tilde{H}_*(T(\gamma^s); Z)$ has no odd torsion, so that by [2, (15.2)]:

$$\tilde{\Omega}_*^{SO}(TBO_s) \cong \tilde{H}_*(TBO_s; \Omega_*^{SO}).$$

Combining this with the corollary and the structure of Ω_*^{SO} gives:

LEMMA 6. *$\Omega_*(BO_s \times BO, f_s)$ is a 2-primary group if s is odd, while for s even, all torsion is two primary and $\Omega_*(BO_s \times BO, f_s)$ has the same rank as a* *free module on one generator of dimension* $2s$ *over the polynomial ring on generators* p_i, $1 \leq i \leq s/2$, *of dimension* $4i$, *and generators* x_j, $j \geq 1$, *of dimension* $4j$.

Combining Lemmas 4 and 6, one then has:

PROPOSITION 5. Ω_*^{AR} *has no odd primary torsion, and* $\Omega_*^{AR} \otimes Z[\frac{1}{2}]$ *is isomorphic to the direct sum of* $\Omega_*^U \otimes Z[\frac{1}{2}] (= Z[\frac{1}{2}][x_{2i}], i > 0)$ *and a polynomial ring over* $Z[\frac{1}{2}]$ *on generators* y_{2j}, $j > 1$ *(where* $\dim u_n = n$ *throughout), both summands being ideals.*

Proof. The ring structure in $\tilde{\Omega}_*^{AR}$ given by the product of manifolds coincides with the ring structure in $\oplus_{k=0}^{*} \Omega_k(BO_{*-k} \times BO, f_{*-k})$ given by the product of manifolds (with Whitney sum defining the normal structure). On the spectral level, this is the homotopy product induced by the maps

$$\begin{aligned} &T(\gamma^s \oplus (\gamma^t \otimes C)) \wedge T(\gamma^{s'} \oplus (\gamma^{t'} \otimes C)) \\ &\qquad\qquad = T(\gamma^s) \wedge T(\gamma^t \otimes C) \wedge T(\gamma^{s'}) \wedge T(\gamma^{t'} \otimes C) \\ &\qquad\qquad\downarrow \\ &T(\gamma^{s+s'} \oplus (\gamma^{t+t'} \otimes C)) = T(\gamma^{s+s'}) \wedge T(\gamma^{t+t'} \otimes C) \end{aligned}$$

and this being compatible with $T\varphi$, the homomorphism

$$\lambda : \tilde{\Omega}_*^{AR} \to \oplus_{s=0}^{*} \tilde{\Omega}_{*-s}^{SO}(TBO_s)$$

is a ring homomorphism, where the product is given by the maps

$$TBO_s \wedge TBO_{s'} \to TBO_{s+s'}.$$

In turn, one has the cofibrations

$$BO_{s-1} \xrightarrow{i} BO_s \xrightarrow{p} TBO_s$$

with the maps p defining a ring homomorphism

$$p_\# : \oplus_{s=0}^{*} \Omega_{*-s}^{SO}(BO_s) \to \oplus_{s=0}^{*} \tilde{\Omega}_{*-s}^{SO}(TBO_s)$$

where the product in the first term is induced by the Whitney sum maps

$$BO_s \times BO_{s'} \to BO_{s+s'}.$$

Tensoring with $Z[\frac{1}{2}]$, λ becomes an isomorphism and $p_\#$ becomes epic, with kernel given by the image of the homomorphism

$$i_\# : \oplus_{s=0}^{*} \Omega_{*-s}^{SO}(BO_s) \to \oplus_{s=0}^{*} \Omega_{*-s}^{SO}(BO_s)$$

induced by the maps $i : BO_{s-1} \to BO_s$ adding a trivial line bundle. It is, of course, clear that $\oplus_{s=0}^{*} \Omega_{*-s}^{SO}(BO_s) \otimes Z[\frac{1}{2}]$ is the polynomial ring over $\Omega_*^{SO} \otimes Z[\frac{1}{2}] = Z[\frac{1}{2}][z_{4j}]$ on the classes $u_1 = [pt, 1] \in \Omega_0^{SO}(BO_1)$ (the class of the trivial line bundle over a point) and $u_{4j+2} = [CP(2j), \xi] \in \Omega_{4j}^{SO}(BO_2)$, $j > 0$, given by the canonical complex line bundle over $CP(2j)$ thought of as a real

2-plane bundle. It is clear that $i_{\#}$ is precisely multiplication by u_1, and hence

$$\oplus_{s=0}^{*} \tilde{\Omega}_{*-s}^{SO}(TBO_s) \otimes Z[\tfrac{1}{2}] \cong Z[\tfrac{1}{2}][y_{2j} \mid j > 1]$$

where $y_{4j} = p_{\#}(z_{4j})$ and $y_{4j+2} = p_{\#}(u_{4j+2})$, $j > 0$.**

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