HOMOLOGY THEORIES OF FUNCTORS[1]

BY
L. DEMERS

Functor here means a functor from the category of pointed functionally Hausdorff Kelley spaces to itself. D. B. Fuks has defined a duality theory on these functors, in order to give a firm foundation to Eckmann-Hilton duality [2], [3]. Here we will develop further aspects of this duality by defining homology and cohomology theories of functors and show that they are dual to each other in the sense that

$$\tilde{H}_n(DF; \mathbf{A}) \simeq \tilde{H}^{-n}(F; \mathbf{A})$$

where $\mathbf{A}$ is a spectrum of coefficients and DF is the dual of F. We will also define a slant and a cup product involving the composition of functors. Naturally, all these notions are nothing but the usual one when we restrict outselves to "spaces", i.e. functors of the form Σ_X, where X is a space.

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1. Duality of functors

We will deal with functors from the category of pointed functionally Hausdorff Kelley spaces to itself. As these terms require some explanation, we state the following definitions:

DEFINITION 1 [2, p. 8]. A Hausdorff topological space X is called a Kelley space if a subset $Y \subset X$ is closed if and only if its intersection with each compact subset of X is closed.

DEFINITION 2 [2, p. 8]. A space X is said to be functionally Hausdorff if for any two distinct points $x, y \in X$, there is a continuous map $f : X \to I = [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

Any Hausdorff space X can be made a Kelley space X^* by defining a new topology on it as follows: a closed set of X^* is any subset Y of X such that its intersection with each compact subset of X is closed.

For two spaces X and Y, let Y^X be the set of continuous maps from X to Y with the compact open topology; if X and Y are Kelley spaces, we define (X, Y) as $(Y^X)^*$.

Let us pass now to pointed Kelley spaces. In this category $\mathcal{K}$, (X, Y) will consist only of base point preserving maps. We can then define for each

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space X of $\mathcal{K}$, a functor $\Omega_X : \mathcal{K} \to \mathcal{K}$ as $\Omega_X(Y) = (X, Y)$. Moreover, we have now a smashed product $X \wedge Y = X \times Y/X \vee Y$, which gives us another functor Σ_X defined as $\Sigma_X(Y) = X \wedge Y$.

The important thing is that in this category Σ_X is left adjoint to Ω_X for any space X. The same thing is valid for the category $\mathcal{C}$ of pointed functionally Hausdorff Kelley spaces.

If $F : \mathcal{C} \to \mathcal{C}$ is a functor, we define the dual DF of F as follows: $DF(X) =$ set of natural transformations $F \to \Sigma_X$ with the following topology: A subbase for the topology of n.t. (F, Σ_X) consists of all inverse images of open sets of $(FY, X \wedge Y)$ under the maps

$$e_y : \text{n.t. } (F, \Sigma_X) \to (FY, X \wedge Y)$$

where Y runs through all the objects of $\mathcal{C}$ and e_y is the evaluation of a natural transformation at the space Y.

The fact that $DF(X)$ is indeed a set has been proved in [2] and more generally in [6]. In order to show this, a cogenerator is needed in the category, and that is why we take only functionally Hausdorff spaces. The unit interval I is then a cogenerator.

We will write $DF(X) = (F, \Sigma_X)$. The operator D is left adjoint to itself, in the sense that $(F, DG) \simeq (G, DF)$ naturally in F and G (the parentheses denote natural transformations).

A functor $F : \mathcal{C} \to \mathcal{C}$ will be called strong if the obvious map

$$(X, Y) \to (FX, FY)$$

is continuous. The category of functors from $\mathcal{C}$ to $\mathcal{C}$ will be denoted by $(\mathcal{C}, \mathcal{C})$ and that of strong functors by $(\mathcal{C}, \mathcal{C})_S$.

It will be noted that $D(\Sigma_X) \simeq \Omega_X$, so that we have a full and faithful embedding $X \to \Sigma_X$ of $\mathcal{C}$ into $(\mathcal{C}, \mathcal{C})$ (and even $(\mathcal{C}, \mathcal{C})_S$ since Σ_X is strong).

2. Spectra

Let $\mathbf{A} = \{A_n, \alpha_n : \Sigma A_n \to A_{n+1}\}$ be a spectrum. Then given a functor F and a natural transformation $\varphi : \Sigma \circ F \to F \circ \Sigma$, we can define a spectrum $(F, \varphi)(\mathbf{A})$ as follows: $(F, \varphi)(\mathbf{A})_n = F(A_n)$ and the maps

$$\Sigma \circ F(A_n) \to F(A_{n+1})$$

are the compositions

$$\Sigma \circ F(A_n) \xrightarrow{\varphi(A_n)} F(\Sigma A_n) \xrightarrow{F(\alpha_n)} F(A_{n+1}).$$

Two examples will be particularly important.

Example 1. *The natural transformation* $\varphi : \Sigma \circ D(F) \to D(F) \circ \Sigma$. For an arbitrary functor F, we define $\varphi_X : \Sigma \circ D(F)(X) \to D(F)(\Sigma X)$ as follows: let $T : F \to \Sigma_X$ be an element of $DF(X)$ and $t \in S^1$. Then $\varphi_X(t, T) : F \to \Sigma_{\Sigma X}$ is given by the formula

$$(\varphi_X(t, T))_Y(y) = (t, T_Y(y)) \in \Sigma(X \wedge Y)$$

where Y is an arbitrary space and y an arbitrary point of $F(Y)$.

Thus for any functor F and spectrum $\mathbf{A}$, there is a well-defined spectrum $(D, \varphi)(\mathbf{A})$ which will be simply denoted by $DF(\mathbf{A})$.

Example 2. *The case of reflexive functors.* The self adjointness properties of the operator D provides us with a natural transformation $\Phi : F \to D^2F$ of each F. Explicitly this is defined as follows: Let X be a space and $x \in FX$. Then

$$D^2FX = D(DF)(X) = \text{space of natural transformations } DF \to \Sigma_X .$$

Thus $\Phi(x)$ must be such a natural transformation. Given a space Y and an element $T \in DF(Y)$ (i.e. $T : F \to \Sigma_X$ is a natural transformation), we define $\Phi(x)_Y(T) = T_X(x) \in X \wedge Y$.

A functor F is called reflexive if $\Phi : F \to D^2F$ is an equivalence of functors.

Given a reflexive functor F we define $\Psi : \Sigma \circ F \to F \circ \Sigma$ as the composition

$$\Sigma \circ F \xrightarrow{\Sigma * \Phi} \Sigma \circ D^2F \xrightarrow{\varphi} D^2F \circ \Sigma \xrightarrow{\Phi^{-1} * \Sigma} F$$

where φ is the natural transformation of Example 1.

Thus for each reflexive functor F and spectrum $\mathbf{A}$, we obtain a spectrum $(F, \Psi)(\mathbf{A})$ which will be denoted by $F(\mathbf{A})$.

3. Cohomology theories

Cohomology theories are easier to deal with than homology theories. Moreover, their "domain of definition" can be given as the category $(\mathcal{C}, \mathcal{C})_s$ which is not so simple for homology, as we shall see later.

Definition 3.1. The *n-th* reduced cohomology group of a strong functor F with coefficients in a spectrum A is the group

$$\tilde{H}^n(F; \mathbf{A}) = \pi_{-n}(D\mathrm{F}(\mathbf{A})) = \lim \pi_{q-n}(DF(A_q))$$

Note that

$$\pi_{q-n}(DF(A_q)) = [S^{q-n}, DF(A_q)] = [S^{q-n}, (F, \Sigma_{A_q})] = [\Sigma^{q-n}F, \Sigma_{A_q}]$$

Thus if we make the sequence Σ_{A_q} a "spectrum of functors" via natural transformations

$$\Sigma \circ \Sigma_{A_q} \simeq \Sigma_{\Sigma A_q} \xrightarrow{\Sigma(\alpha_q)} \Sigma_{A_{q+1}}$$

we see that the above definition of cohomology groups is precisely the analogue of G. W. Whitehead's definition of the cohomology of a space with coefficients in a spectrum. It is then easy to show that we have even defined a cohomology theory in the following sense (see [8, p. 252)).

(1) We have a sequence of contravariant functors $\tilde{H}^n(\quad; \mathbf{A}) \to$ abelian groups.

(2) If $f_0, f_1 : F \to G$ are homotopic natural transformations (see [12],

[4]) the induced maps

$$f_0^* : \tilde{H}^n(G; \mathbf{A}) \to \tilde{H}^n(F; \mathbf{A}) \quad \text{and} \quad f_1^* : H^n(C; \mathbf{A}) \to \tilde{H}^n(F\mathbf{A})$$

are the same.

(3) For each n, there is a natural transformation

$$\sigma^n : \tilde{H}^{n+1}(\ \ ; A) \circ \Sigma \to \tilde{H}^n(\ \ ; A)$$

such that for all functors F, $\sigma^n(F)$ is an isomorphism.

(4) If $f : F \to G$ is a natural transformation and if C_f is the mapping cone of f (see [2]), and $i : G \to C_f$ is the inclusion, then the sequence

$$\tilde{H}^n(C_f; \mathbf{A}) \xrightarrow{i^*} \tilde{H}^n(G; \mathbf{A}) \xrightarrow{f^*} \tilde{H}^n(F; \mathbf{A})$$

is exact for all n.

The proof goes as in the case of spaces (see [8]). For the exactness in (4), note that we have a cofibration sequence of functors $F \to M_f \to C_f$, where M_f is the mapping cylinder of f. Then $(F, \Sigma_{A_q}) \leftarrow (M_f, \Sigma_{A_q}) \leftarrow (C_f, \Sigma_{A_q})$ is a fibration and hence induces a homotopy exact sequence.

4. Homology theories

(a) *The category* $(\mathcal{C}, \mathcal{C})_{s,\varphi}$. We have seen that if F is a strong functor and $\varphi : \Sigma \circ F \to F \circ \Sigma$ is a natural transformation, then for any spectrum A, we can define a spectrum $(F, \varphi)(\mathbf{A})$.

We will then define $(\mathcal{C}, \mathcal{C})_{s,\varphi}$ as follows: An object of this category is a pair (F, φ) where F is a strong functor and $\varphi : \Sigma \circ F \to F \circ \Sigma$ is a natural transformation. A morphism $f : (F, \varphi) \to (G, \psi)$ between two object of $(\mathcal{C}, \mathcal{C})_{s\varphi}$ is a natural transformation $f : F \to G$ such that the diagram

$$\begin{array}{ccc} \Sigma \circ F & \xrightarrow{\Sigma * f} & \Sigma \circ G \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ F \circ \Sigma & \xrightarrow{f * \Sigma} & G \circ \Sigma \end{array}$$

is commutative

(b) *Mapping cones in* $(\mathcal{C}, \mathcal{C})_{s,\varphi}$. If $f : (F, \varphi) \to (G, \psi)$ is a morphism of $(\mathcal{C}, \mathcal{C})_{s\varphi}$, let C_f be the unreduced mapping cone of f, i.e. $C_f(X)$ = mapping cone of $f_x : F(X) \to G(X)$.

Then C_f can be made an object of $(\mathcal{C}, \mathcal{C})_{s,\varphi}$ as follows.

We have a commutative diagram

$$(*) \qquad \begin{array}{ccc} \Sigma \circ F & \xrightarrow{\Sigma * f} & \Sigma \circ G \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ F \circ \Sigma & \xrightarrow{f * \Sigma} & G \circ \Sigma; \end{array}$$

by taking the adjoint of φ and ψ, we obtain

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\tilde{\varphi}"'] & G \arrow[d, "\tilde{\psi}"] \\ \Omega \circ F \circ \Sigma \arrow[r, "\Omega * f * \Sigma"] & \Omega \circ G \circ \Sigma. \end{tikzcd}$$

Now by definition of the mapping cone of a transformation it is clear that $C_{f*} = C_f \circ \Sigma$. If $i : G \to C_f$ is the inclusion in the base of the cone, we have that

$$(\Omega * i * \Sigma) \circ \tilde{\varphi} \circ f$$

is homotopic to zero.

Hence there is a natural transformation $\tilde{\chi} : C_f \to \Omega C_f \circ \Sigma$ such that $(\Omega * i * \Sigma) \circ \tilde{\varphi} = \tilde{\chi} \circ i$. Taking the adjoint of $\tilde{\chi}$, we obtain a map

$$\chi : \Sigma \circ C_f \to C_f \circ \Sigma$$

such that $(i * \Sigma) \circ \psi = \chi \circ i$ and this implies that $\Sigma \circ C_f$ is naturally equivalent to $C_{\Sigma * f}$. It remains thus to construct a map $\chi : C_{\Sigma * f} \to C_f \circ \Sigma = C_{f * \Sigma}$. But this map is easily given by the commutative diagram $(*)$

(c) *Homotopy in* $(\mathcal{C}, \mathcal{C})_s$. Let I' be the disjoint union of I and a point $*$ serving as the base point. Then $\Sigma_{I'}(X) = I' \wedge X = I \times Y/I \times \{x_0\}$ where x_0 is the base-point of X. There are then two natural transformations $\varepsilon_0, \varepsilon_1$: identity $\to \Sigma_{I'}$ defined by sending x to $(0, x)$ and $(1, x)$ respectively.

If $f, g : F \to G$ are two natural transformations, a homotopy between them is a map $h : \Sigma_{I'} \circ F \to G$ such that $h \circ \varepsilon_0 * F = f$ and $h \circ \varepsilon_1 * F = g$. Since $\Sigma_{I'}$ commutes with Σ, it is clear that if $f, g : (F, \varphi) \to (G, \psi)$ are two maps of $(\mathcal{C}, \mathcal{C})_{s,\varphi}$ which are homotopic as maps of $(\mathcal{C}, \mathcal{C})_s$, then $\Sigma_I \circ F$ can be made an object of $(\mathcal{C}, \mathcal{C})_{s,\varphi}$ and the homotopy be can be made a map of $(\mathcal{C}, \mathcal{C})_{s,\varphi}$.

(d) *Homology theories in* $(\mathcal{C}, \mathcal{C})_{s,\varphi}$.

DEFINITION 4.1. If (F, φ) is an object of $(\mathcal{C}, \mathcal{C})_{s,\varphi}$ and $\mathbf{A}$ is a spectrum, the *n-th* homology group of (F, φ) with coefficients in $\mathbf{A}$ is defined as

$$\tilde{H}_n(F, \varphi; A) = \pi_n((F, \varphi)(\mathbf{A})) = \lim_q \pi_{n+q}(F(A_q)).$$

It is clear that if $f_0, f_1 : (F, \varphi) \to (G, \psi)$ are homotopic, then the maps

$$f_{0*}, f_{1*} : \tilde{H}_n(F, \varphi; \mathbf{A}) \to \tilde{H}_n(C, \psi; \mathbf{A})$$

coincide for all n. Moreover, there are natural transformations

$$\sigma_n : \tilde{H}_n(\quad ; \mathbf{A}) \to \tilde{H}_{n+1}(\Sigma(\quad); \mathbf{A})$$

inducing isomorphisms for all (F, φ).

Thus we will have obtained a bona fide homology theory once we have proved the exactness of the sequences

$$\tilde{H}_n(F, \varphi; \mathbf{A}) \xrightarrow{f_*} \tilde{H}_n(G, \varphi; \mathbf{A}) \xrightarrow{i_*} \tilde{H}_n(C_f, \chi; \mathbf{A}).$$

This will occupy the rest of the section.

If

$$\mathbf{A} = \{A_n, \alpha_n : \Sigma A_n \to A_{n+1}\} \text{ and } \mathbf{B} = \{B_n, \beta_n : \Sigma B_n \to B_{n+1}\}$$

are spectra, a map $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is a sequence of maps $f_n : A_n \to B_n$ such that

$$\beta_n \circ \Sigma f_n = f_{n+1} \circ \alpha_n$$

for all n. We can define the mapping cone $\mathbf{C} = \{C_n, \gamma_n : \Sigma C_n \to C_{n+1}\}$ of such a map: $C_n = C_{fn}$ and γ_n is given by the fact that $\Sigma C_{fn} = C_{\Sigma fn}$ and that $f_{n+1} \circ \alpha_n = \beta_n \circ \Sigma f_n$.

What we will show is that for all maps $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ and all n, we have an exact sequence

$$\pi_n(\mathbf{A}) \to \pi_n(\mathbf{B}) \to \pi_n(\mathbf{C}).$$

DEFINITION 4.2 (see [8, p. 242)). A spectrum $\mathbf{A}$ is said to be convergent if and only if there is an integer N such that A_{N+i} is i-connected for all $i \geq 0$.

LEMMA 4.1. *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, *and let N be an integer. Then there exist spectra* $\mathbf{A}'$, $\mathbf{B}'$ and maps $\mathbf{f}' : \mathbf{A}' \to \mathbf{B}'$, $\boldsymbol{\varepsilon} : \mathbf{A}' \to \mathbf{A}$ and $\boldsymbol{\eta} : \mathbf{B}' \to \mathbf{B}$ such that:

(1)
$$\begin{array}{ccc} A'_n & \xrightarrow{f'_n} & B'_n \\ {\scriptstyle \varepsilon_n}\downarrow & & \downarrow{\scriptstyle \eta_n} \\ A_n & \xrightarrow{f_n} & B_n \end{array}$$

is commutative for all n.

(2) *$A'_i = A_i$ and ε_i is the identity for all $i \leq N$. $B'_i = B_i$ and η_i is the identity for all $i \leq N$.*

(3) *A'_{N+i} and B'_{N+i} are $(i-1)$-connected for all $i \geq 0$.*

(4) *$\varepsilon_{i*} : \pi_j(A'_i) \to \pi_j(A_i)$ and $\eta_{i*} : \pi_j(B'_i) \to \pi_j(B_i)$ are isomorphisms for all $i \geq N + 1$ and $j \geq i - N$.*

Proof. The proof is adapted from a particular case in [8, Lemma 4.1, p, 242]. Assume that A_i and B_i are 0-connected for $i \geq N + 1$. (If not, we will do the following construction only on the path-components of their base point.) First construct A_i^* and B_i^* as spaces containing A_i and B_i respectively and such that:

(1) There exist maps $f_i^* : A_i^* \to B_i^*$ making commutative diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ \cap & & \cap \\ A_i^* & \xrightarrow{f_i^*} & B_i^*. \end{array}$$

(2) The inclusion maps induced isomorphisms

$$\pi_j(A_i) \to \pi_j(A_i^*) \quad \text{and} \quad \pi_j(B_i) \to \pi_j(B_i^*) \quad \text{for } j \leq i - N.$$

(3) $\pi_j(A_i^*) = \pi_j(B_i^*) = 0$ for $j \geq i - N + 1$.

These conditions can be realized simultaneously as follows. First kill $\pi_{i-N+1}(A_i)$ (resp. $\pi_{i-N+1}(B_i)$) by attaching cells to A_i (resp. B_i) via *all maps* $S^{i-N+1} \to A_i$ (resp. B_i). Let $A_i(i - N + 1)$ and $B_i(i - N + 1)$ be the spaces so obtained. From the function

$$(S^{i-N+1}, f_i) : (S^{i-N+1}, A_i) \to (S^{i-N+1}, B_i),$$

we obtain a map $A_i(i - N + 1) \to B_i(i - N + 1)$ making the following diagram commutative:

$$\begin{tikzcd} A_i \arrow[r, phantom, "\subset"] \arrow[d, "f_i"] & A_i(i - N + 1) \arrow[d] \\ B_i \arrow[r, phantom, "\subset"] & B_i(i - N + 1). \end{tikzcd}$$

We then repeat this process to kill

$$\pi_{i-N+1}(A_i(i - N + 1)) \quad \text{and} \quad \pi_{i-N+1}(B_i(i - N + 1)).$$

We obtain a commutative ladder

$$\begin{tikzcd} A_i \arrow[r, phantom, "\subset"] \arrow[d] & A_i(i - N + 1) \arrow[r, phantom, "\subset"] \arrow[d] & \cdots \arrow[d] \\ B_i \arrow[r, phantom, "\subset"] & B_i(i - N + 1) \arrow[r, phantom, "\subset"] & \cdots. \end{tikzcd}$$

Call the direct limits A_i^* and B_i^* respectively and let $f_i^* : A_i^* \to B_i^*$ be the map induced by the above diagram.

Now let A_i' (resp. B_i') be the spaces of paths in A_i^* (resp. B_i^*) which start at the base point and end in A_i (resp. B_i). Since

$$\begin{tikzcd} A_i \arrow[r, phantom, "\subset"] \arrow[d, "f_i"] & A_i^* \arrow[d, "f_i^*"] \\ B_i \arrow[r, phantom, "\subset"] & B_i^* \end{tikzcd}$$

is commutative, we clearly obtain a map $f_i' : A_i' \to B_i'$. We then define

$$\varepsilon_i : A_i' \to A_i \quad \text{and} \quad \eta_i : B_i' \to B_i$$

as the end point maps.

Clearly

$$\begin{tikzcd} A' \arrow[r, "\varepsilon_i"] \arrow[d, "f_i'"'] & A_i \arrow[d, "f_i"] \\ B_i' \arrow[r, "\eta_i"] & B_i \end{tikzcd}$$

is commutative for all i.

A'_i is in fact the fibre of the inclusion $A_i \subset A^*_i$ transformed into a fibration. Thus we have an exact sequence

$$\pi_j(A_i) \to \pi_{j+1}(A^*_i) \to \pi_j(A'_i) \to \pi_j(A_i) \to \pi_j(A^*_i).$$

Because of [2], $\pi_j(A'_i) = 0$ for $j \leq i - N - 1$ and because of (3), $\pi_j(A'_i) \simeq \pi_j(A_i)$ for $j \leq i - N$

Thus in particular $\pi_j(A'_{N+i}) = 0$ for $j \leq N + i - N - 2 = i - 2$.

The same is obviously true with B instead of A.

The spaces A'_i and B'_i thus satisfy conditions (1)–(4) of the lemma. It remains only to define maps $\alpha'_i : \Sigma A'_i \to A'_{i+1}$ and $\beta_i : B'_i \to B'_{i+1}$ making the following diagrams commutative:

$$\begin{array}{ccc} \Sigma A'_i & \xrightarrow{\alpha'_{i+1}} & A'_{i+1} \\ {\scriptstyle \Sigma\varepsilon_i}\downarrow & & \downarrow{\scriptstyle \varepsilon_{i+1}} \\ \Sigma A_i & \xrightarrow{\alpha_i} & A_{i+1} \end{array} \qquad \begin{array}{ccc} \Sigma B'_i & \xrightarrow{\beta'_i} & B'_{i+1} \\ {\scriptstyle \Sigma\eta}\downarrow & & \downarrow{\scriptstyle \eta_{i+1}} \\ B_i & \xrightarrow{\beta_i} & B_{i+1} \end{array} \qquad \begin{array}{ccc} \Sigma A'_i & \xrightarrow{\alpha'_i} & A'_{i+1} \\ {\scriptstyle \Sigma f'_i}\downarrow & & \downarrow{\scriptstyle f'_{i+1}} \\ \Sigma B'_i & \xrightarrow{\beta'_i} & B'_{i+1} \end{array}.$$

Define first canonical maps

$$\Sigma A^*_i \xrightarrow{\alpha^*_i} A^*_i$$

as follows. A^*_i is the direct limit of a sequence $A_i \subset A_i(i - N + 1) \subset \cdots$.

Since Σ commutes with direct limits, ΣA^*_1 is the direct limit of the sequence $\Sigma A_i \subset A_i(i - N + 1) \subset \cdots$.

We will then define α^*_i step by step.

We have

$$\begin{array}{ll} \Sigma A_i & \subset \Sigma A_i(i - N + 1) \\ {\scriptstyle \alpha_i}\downarrow & \\ A_{i+1} & \subset A_{i+1}(i - N + 2). \end{array}$$

To extend α_i to a map $\Sigma A_i(i - N + 1) \to A_{i+1}(i - N + 2)$, let $f : S^{i-N+1} \to A_i$ be a map. Then

$$\Sigma f : S^{i-N+2} \to \Sigma A_i \quad \text{and} \quad \Sigma A_i \cup_{\Sigma f} e^{i-N+3} = \Sigma(A_i \cup_f e^{i-N+2}).$$

Then we simply extend to $\Sigma(A_i \cup_f e^{i-N+2})$ by coning. We do the same thing for all maps $S^{i-N+1} \to A_i$ and obtain

$$\alpha_i(i - N + 1) : \Sigma A_i(i - N + 1) \to A_{i+1}(i - N + 2).$$

We can then repeat the process to obtain finally a map $\alpha^*_i : \Sigma A^*_i \to A^*_{i+1}$ of the direct limits.

The same thing can be done for **B** instead of **A** to obtain $\beta^*_i : \Sigma B^*_i \to B^*_{i+1}$. Finally, it is clear that the following diagrams are commutative.

$$\begin{array}{ccccccc}
\Sigma A_i \subset \Sigma A_i^* & & \Sigma B_i \subset \Sigma B_i^* & & \Sigma A_i^* & \xrightarrow{\Sigma f_i^*} & \Sigma B_i^* \\
\alpha_i \downarrow \quad \downarrow \alpha_i^* & & \beta_i \downarrow \quad \downarrow \beta_i^* & & \alpha_i^* \downarrow & & \downarrow \beta_i^* . \\
A_{i+1} \subset A_{i+1}^* & & B_{i+1} \subset B_{i+1}^* & & A_{i+1} & \xrightarrow{f_{i+1}^*} & B_{i+1}^*
\end{array}$$

Taking the adjoint of α_i and α_i^* we obtain a commutative diagram

$$\begin{array}{ccc}
A_i & \subset & A_i^* \\
\tilde{\alpha}_i \downarrow & & \downarrow \tilde{\alpha}_i^* \\
\Omega A_{i+1} & \longrightarrow & \Omega A_{i+1}^*
\end{array}$$

Let $A_i' =$ space of paths in A_i^* starting at the base point and ending in A_i, $A_{i+1}' =$ space of paths in A_{i+1}^* starting at the base point and ending in A_{i+1}.

Define $\tilde{\alpha}_i' : A_i' \to \Omega A_{i+1}'$ as follows. Let λ be a path in A_i^* starting at $*$ and ending in A_i. Then $\alpha_i'(\lambda)(t) = \alpha_i^*(\lambda(\quad))(t)$. In other words,

$$(\tilde{\alpha}_i'(\lambda)(t))(s) = \alpha_i^*(\lambda(s))(t).$$

It is easy to verify that $\tilde{\alpha}_i'$ is really a map $A_i' \to \Omega A_{i+1}'$. Taking the adjoint of $\tilde{\alpha}_i$ we obtain $\alpha_i' : \Sigma A_i' \to A_{i+1}'$ which has all the properties we want.

This concludes the proof of Lemma 4.1.

PROPOSITION 4.2. *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *be a map of spectra and let* $\mathbf{C}$ *be the mapping cone of* $\mathbf{f}$. *Then for all* n, *we have an exact sequence*

$$\pi_n(\mathbf{A}) \to \pi_n(\mathbf{B}) \to \pi_n(\mathbf{C}).$$

Proof. In this proof, we will assume that $\mathbf{B}$ has been replaced by the mapping cylinder of $\mathbf{f}$ and $\mathbf{f}$ by the inclusion of $\mathbf{A}$ as the top of the cylinder.

Suppose first that $\mathbf{A}$ and $\mathbf{B}$ are convergent, and choose N large enough so that A_{N+i} and B_{N+i} are both i-connected for $i > 0$, and assume that $n + N \geq 2$ (n is here a fixed integer). Then the pair (B_{N+i}, A_{N+i}) is also i-connected, by the relative Hurewicz isomorphism Theorem. Consider the diagram

$$(*) \qquad \begin{array}{ccccc}
\pi_{n+k}(A_k) & \xrightarrow{f_*} & \pi_{n+k}(B_k) & \xrightarrow{j_*} & \pi_{n+k}(B_k, A_k) \\
& & & {\scriptstyle p_*}\searrow \quad \swarrow{\scriptstyle p_*'} & \\
& & & \pi_{n+k}(C_k) &
\end{array} .$$

From the Blakers-Marsey theorem (see [1]), it follows that p_*' is an isomorphism for $n + k = j \leq 2i$, where $k = N + i$. Suppose that $k \geq n + 2N$ (i.e. $i \geq n + N$). Then A_k is $(k - N)$-connected and $(k - N) \geq n + N \geq 2$. Hence $n + k \leq 2\ (k - N) = 2i$, and p_*' is an isomorphism for $i = k - N$, $j = n + k$. Thus in the diagram $(*)$, $\ker p_* = \operatorname{im} f_*$ for k large enough. Since a direct limit of exact sequences is exact, the sequence

$$\pi_n(\mathbf{A}) \longrightarrow \pi_n(\mathbf{B}) \longrightarrow \pi_n(\mathbf{C})$$

is exact provided both **A** and **B** are convergent.

Now suppose that they are not convergent, and let $\alpha \in \ker \mathbf{p}_*$, where $\mathbf{p}_* : \pi_n(\mathbf{B}) \longrightarrow \pi_n(\mathbf{C})$. Choose a representative $\alpha' \in \pi_{n+k}(B_k)$ of α. Increasing k if necessary, we may assume that α' is in tee kernel of

$$p_* : \pi_{n+k}(B_k) \longrightarrow \pi_{n+k}(C_k).$$

By the preceding lemma, there are convergent spectra $\mathbf{A}'$, $\mathbf{B}'$ and maps

$$\boldsymbol{\varepsilon} : A' \longrightarrow \mathbf{A}, \quad \mathbf{n} : \mathbf{B} \longrightarrow \mathbf{B}, \quad \mathbf{f}' : \mathbf{A}' \longrightarrow \mathbf{B}'$$

such that $\mathbf{n} \circ \mathbf{f}' = \mathbf{f} \circ \boldsymbol{\varepsilon}$. Moreover, $A_i = A_i$, $B_i' = B_i$ and ε_i and η_i are identity maps for $i \leq k$. Let $\mathbf{C}'$ = mapping cone of $\mathbf{f}'$. Then we have a commutative diagram

$$\begin{tikzcd}
A_i' \arrow[r, "f_i'"] \arrow[d, "\varepsilon_i"] & B_i' \arrow[r, "p_i''"] \arrow[d, "\eta_i"] & C_i' \arrow[d, "\gamma_i"] \\
A_i \arrow[r, "f_i"] & B_i \arrow[r, "p_i"] & C_i
\end{tikzcd}.$$

Since $A_i' = A_i$ and $B_i' = B_i$ for $i \leq k$, we have in particular that ε_k, η_k and γ_k are identity maps.

Let $\alpha'' \in \pi_{n+k}(B_k')$ be $\eta_{k*}^{-1}(\alpha')$. Then $p_{k*}\alpha' = 0$ implies that $p_{k*}''\alpha'' = 0$ so that α'' represents an element $\alpha^* \in \pi_n(\mathbf{B}')$ such that $\mathbf{n}_*(\alpha^*) = \alpha$ and $\mathbf{p}_*''(\alpha^*) = 0$.

Since $\mathbf{A}'$ and $\mathbf{B}'$ are both convergent, the sequences for this pair is exact so that there exists $\beta \in \pi_n(\mathbf{A}')$ such that $\mathbf{f}_*'\beta = \alpha^*$. Then $\mathbf{n}_*'\mathbf{f}_*'(\beta) = \alpha = f_*\boldsymbol{\varepsilon}_*(\beta)$ and α is in the image of $\mathbf{f}_*$, Q.E.D.

This concludes the proof that the functors $\tilde{H}_n(\;;\mathbf{A})$ define a homology theory.

5. Duality between homology and cohomology

Let F be a functor and $\varphi : \Sigma \circ DF \longrightarrow DF \circ \Sigma$ the natural transformation defined in §2, Example 1. Then

$$\tilde{H}_n(DF, \varphi; \mathbf{A}) = \pi_n(DF, \varphi)(\mathbf{A})) = \pi_n(DF(\mathbf{A})) = \tilde{H}^{-n}(F; \mathbf{A}).$$

Since the transformation $\varphi : \Sigma \circ DF \longrightarrow DF \circ \Sigma$ is the standard one associated with a functor of the form DF, we can state in short $\tilde{H}_n(DF; \mathbf{A}) = \tilde{H}^{-n}(F; \mathbf{A})$.

If F is a reflexive functor, we also have

$$\tilde{H}_n(F; \mathbf{A}) = \pi_n(F(\mathbf{A})) \cong \pi_n(D(DF)(\mathbf{A})) = \tilde{H}^{-n}(DF; \mathbf{A})$$

6. Relation with homology and cohomology theories of spaces

Suppose that $F = \Sigma_X$ for some space $X \in \mathcal{C}$. Then

$$\begin{aligned}\tilde{H}^n(F; \mathbf{A}) = \pi_{-n}(DF(\mathbf{A})) = \pi_{-n}(\Omega_X(\mathbf{A})) &= \lim_q \pi_{q-n}((X, A_q)) \\ &= \lim_q [S^{q-n}X, A_q] = \tilde{H}^n(X; \mathbf{A})\end{aligned}$$

in the sense of G. W. Whitehead (see [8]).

Similarly since F is a reflexive functor, we have

$$\tilde{H}_n(F;\mathbf{A}) = \pi_n(F(\mathbf{A})) = \pi_n(X \wedge \mathbf{A}) = \tilde{H}_n(X;\mathbf{A}).$$

Thus the homology and cohomology theories of functors are a good generalization of that of spaces.

7. Some examples of computations

(a) *Functors of the form* $\Sigma \circ F$ *and* $\Omega \circ F$. We know that

$$\tilde{H}_{n+1}(\Sigma \circ F;\ \mathbf{A}) \simeq \tilde{H}_n(F;\ \mathbf{A}) \quad \text{and} \quad \tilde{H}^{n+1}(\Sigma \circ F;\ \mathbf{A}) \simeq \tilde{H}^n(F;\ \mathbf{A}).$$

But if F is reflexive, $D(\Omega \circ F) \simeq \Sigma \circ DF$ (see [4]).
Thus

$$\tilde{H}_n(\Omega \circ F;\ \mathbf{A}) \simeq \tilde{H}^{-n}(\Sigma \circ DF;\ \mathbf{A}) \simeq \tilde{H}^{-n-1}(DF;\ \mathbf{A}) \simeq \tilde{H}_{n+1}(F;\ \mathbf{A})$$

and

$$\tilde{H}^n(\Omega \circ F;\ \mathbf{A}) \simeq \tilde{H}_{-n}(\Sigma \circ DF;\ \mathbf{A}) \simeq \tilde{H}_{-n-1}(DF;\ \mathbf{A}) \simeq \tilde{H}^{n+1}(F;\ \mathbf{A})$$

(b) *The functors* $J(X) = X * X$ = *reduced join of* X *with* X *and* $K(X) = DJ(X)$ = *space of paths in* $X \vee X$ *starting in the left summand and ending in the right summand.* Suppose that $\mathbf{A}$ is a spectrum of Eilenberg Mac-Lane spaces $K(A, n)$. (We will write A instead of $\mathbf{A}$ for the coefficients in this case.) Then

$$\tilde{H}_n(J;A) = \lim_k \pi_{n+k}(J(A_k)) = \lim_k \pi_{n+k}(K(A,k) * K(A,k)).$$

Now the space $K(A, k) * K(A, k)$ is $2k$-connected and if k increases $2k > n + k$. Thus $\tilde{H}_n(J;A) = 0$ for all n, and by duality $\tilde{H}^n(K;A) = 0$ for all n.

To compute $\tilde{H}_n(K;\ A)$, note that we have a functorial fibration

$$\Omega(X \vee X) \rightarrow K(X) \rightarrow X \times X \qquad \text{(see [5, p. 122))}.$$

We will denote the functors $X \rightarrow X \vee X$ and $X \rightarrow X \times X$ by W and P respectively. Then we have a fibration

$$\Omega \circ W \rightarrow K \rightarrow P$$

which induces an exact sequence

$$\rightarrow \pi_n(\Omega \circ W(A)) \rightarrow \pi_n(K(A)) \rightarrow \pi_n(P(A)) \rightarrow \pi_{n-1}(\Omega \circ W(A) \rightarrow$$

and this is nothing but the sequence

$$(*) \qquad \rightarrow \tilde{H}_n(\Omega \circ W;\ \mathrm{A}) \rightarrow \tilde{H}_n(K : A) \rightarrow \tilde{H}_n(P; A) \rightarrow \cdots .$$

Now if $I^{\bullet}$ is a space with only two points, W is the functor $\Sigma_{(I^{\bullet} \vee I^{\bullet})}$ and $P = DW = \Omega_{(I^{\bullet} \vee I^{\bullet})}$.

Thus $\tilde{H}_n(\Omega W;A) \simeq \tilde{H}_{n+1}(W;A) \simeq \tilde{H}_{n+1}(I^{\bullet} \vee I^{\bullet};A) = 0$ unless $n + 1 = 0$ and $\tilde{H}_0(I^{\bullet} \vee I^{\bullet};A) = A \oplus A$.

Similarly, $\tilde{H}_n(P; A) \simeq \tilde{H}^{-n}(W; A) \simeq \tilde{H}^{-n}(I^\bullet \vee I^\bullet; A) = 0$ unless $n = 0$ amd $\tilde{H}^0(I^\bullet \vee I^\bullet; A) = A \oplus A$.

We are thus left with the exact sequence

$$(**) \qquad 0 \to \tilde{H}_0(K; A) \to A \oplus A \to A \oplus A \to \tilde{H}_{-1}(K; \mathrm{A}) \to 0.$$

Now by [5 p. 122], the inclusion $\Omega \circ W \to K$ has an inverse.

Thus we have, for all n, split short exact sequences

$$(***) \qquad 0 \to \tilde{H}_{n+1}(P; \mathrm{A}) \to \tilde{H}_n(\Omega W; A) \to \tilde{H}_n(K; A) \to 0$$

This and (**) imply that $\tilde{H}_n(K; A) = 0$ for all n. By duality, $\tilde{H}^n(J; A) = 0$ for all n.

8. The slant product

Given a pairing of spectra $\mathbf{f} : (\mathbf{A}, \mathbf{B}) \to \mathbf{C}$ (see definition below) there is defined, for all spaces X and Y a slant product

$$\tilde{H}^n(X \wedge Y; \mathbf{A}) \otimes \tilde{H}_q(Y; \mathbf{B}) \to \tilde{H}^{n-q}(X; \mathbf{C}).$$

We want to define the analogue for functors, with the condition that it agrees with the usual slant product when we consider functors of the form Σ_X and Σ_Y. Since $\Sigma_{X\wedge Y} = \Sigma_X \circ \Sigma_Y$, the generalized slant product will involve the composition of functors, and not their "smashed product".

We will assume then that we have three spectra:

$$\mathbf{A} = \{A_p, \alpha_p : \Sigma A_p \to A_{p+1}\}, \quad \mathbf{B} = \{B_q, \beta_q : \Sigma B_q \to B_{q+1}\},$$
$$\mathbf{C} = \{C_r, \gamma_r : \Sigma C_r \to C_{r+1}\}$$

and a pairing $\mathbf{f} : (\mathbf{A}, \mathbf{B}) \to \mathbf{C}$. This is defined (see [8, p. 254–255]) as a family of maps $f_{p,q} : A_p \wedge B_q \to C_{p+q}$ such that for each pair (p, q) we have a diagram

$$\begin{tikzcd}
(\Sigma A_p) \wedge B_q \arrow[rr, "\alpha_p \wedge 1"] & & A_{p+1} \wedge B_q \arrow[d, "f_{p+1,q}"] \\
\Sigma(A_p \wedge B_q) \arrow[u, "\lambda"] \arrow[r, "\Sigma f_{p,q}"] \arrow[d, "\mu"'] & \Sigma C_{p+q} \arrow[r, "\gamma_{p+q}"] & C_{p+q+1} \\
A_p \wedge (\Sigma B_q) \arrow[rr, "1 \wedge \beta_q"] & & A_p \wedge B_{q+1} \arrow[u, "f_{p,q+1}"']
\end{tikzcd}$$

with the following property. Let

$$f_{p+1,q} \circ (\alpha_p \wedge 1) \circ \lambda = \theta', \quad \gamma_{p+q} \circ \Sigma f_{p,q} = \theta, \quad f_{p,q+1} \circ (1 \wedge \beta_q) \circ \mu = \theta''.$$

Then in the group $[\Sigma(A_p \wedge B_q), C_{p+q+1}]$, $\theta' = \theta$ and $\theta = (-1)^p \theta''$.

From now on, we will assume that all functors are reflexive. Our aim is to define a pairing

$$(D(F \circ G)(\mathbf{A}),\ G(\mathbf{B})) \to DF(\mathbf{C})$$

but since F and G are reflexive and hence satisfy $D(F \circ G) = DF \circ DG$ (see [4]), this is equivalent to defining a pairing

$$\varphi : (DF \circ G(\mathbf{A}),\ DG(\mathbf{B})) \to DF(\mathbf{C}).$$

We will define φ as follows: Let $T : F \to \Sigma_{G(A_p)}$ be an element of $DF(G(A_p))$ and $T' : G \to \Sigma_{B_q}$ an element of $DG(B_q)$. Then $\varphi_{p,q}(T, T')$ is defined as the composition

$$F \xrightarrow{\quad T \quad} \Sigma_{G(A_p)} \xrightarrow{\quad \Sigma(T'_{A_p}) \quad} \Sigma_{A_p \wedge B_q} \xrightarrow{\quad \Sigma(f_{p,q}) \quad} \Sigma_{C_{p+q}}.$$

To prove that φ is a pairing, we will break it into the composition of two pairings easier to handle.

(a) Given a pairing $\mathbf{f} : (\mathbf{A}', \mathbf{B}') \to \mathbf{C}'$ and a functor F, define a pairing

$$\psi : (DF(\mathbf{A}'),\ \mathbf{B}') \to DF(\mathbf{C}')$$

as follows. Let $T : F \to \Sigma_{A_p'}$ be an element of $DF(A_p')$ and $b_q \in B_q'$. Let $\tilde{b}_q : I^\bullet \to B_q'$ be the map such that $\tilde{b}_q(1) = b_q$. We define then $\psi_{p,q}(T, b_q)$ as the composition

$$F = \Sigma_{I^\cdot} \circ F \xrightarrow{\quad \Sigma_{I^\cdot} * T \quad} \Sigma_{I^\cdot} \circ \Sigma_{A_p'} \xrightarrow{\quad \Sigma(\tilde{b}_q) * \Sigma_{A_p} \quad}$$

$$\Sigma_{B_q'} \circ \Sigma_{A_p'} \simeq \Sigma_{A_p' \wedge B_q'} \xrightarrow{\quad \Sigma(f_{p,q}) \quad} \Sigma_{C_{p+q}'}$$

Explicitly, let X be a space, $x \in FX$ and let $T_X(x) = (a_p,\ x') \in A_p' \wedge X$. Then $\psi_{p,q}(T, b_q)_X(x) = (f_{p,q}(a_p, b_q),\ x') \in C_{p+q}' \wedge X$.

(b) Given a pairing $\mathbf{f} : (\mathbf{A}, \mathbf{B}) \to \mathbf{C}$ define

$$\chi : (G(\mathbf{A}),\ DG(\mathbf{B})) \to \mathbf{C}$$

as follows. Let $a_p \in G(A_p)$, $T : G \to \Sigma_{B_q}''$.

Then

$$\chi_{p,q}(a_p, T) = f_{p,q}\, T_{A_p}(a_p).$$

Assume for the moment that ψ and χ are pairings. We will then prove that φ is a pairing.

In case (a), replace $\mathbf{A}'$ by $G(\mathbf{A})$, $\mathbf{B}'$ by $DG(\mathbf{B})$ and f by χ the latter being obtained from (b). We obtain then a pairing

$$\psi : (DF \circ G(\mathbf{A}),\ DG(\mathbf{B})) \to DF(\mathbf{C})$$

defined as follows. Let $T : F \to \Sigma_{G(A_p)}$, $T' : G \to \Sigma_{B_q}$, let X be a space, $x \in FX$ and $T_X(x) = (a, x') \in G(A_p) \wedge X$. Then

$$\psi_{p,q}(T, T')_X(x) = (f_{p,q} \circ T'_{A_p}(a),\ x')$$

On the other hand,

$$\varphi_{p,q}(T, T')_X = \Sigma(f_{p,q})_X \circ \Sigma(T'_{A_p})_X \circ T_X(x) = (f_{p,q} \circ T'_{A_p}(a), x')$$

Thus $\psi_{p,q} = \varphi_{p,q}$ so that $\varphi_{p,q}$ is a pairing if both ψ and χ are pairings.

The proof that ψ and χ are pairings is long but straightforward. In fact, only the reflexivity of G is needed.

Thus we obtain a slant product

$$\tilde{H}^n(F \circ G; \mathbf{A}) \otimes \tilde{H}_p(G; \mathbf{B}) \to \tilde{H}^{n-p}(F; \mathbf{C}).$$

It is easy to check that if $F = \Sigma_X$ and $G = \Sigma_Y$ this slant product coincides up to sign with the usual one.

9. The cross-product and the cup product

We can define a cross-product

$$\tilde{H}^p(F; \mathbf{A}) \otimes H^q(G; \mathbf{B}) \to \tilde{H}^{p+q}(F \circ G; \mathbf{C})$$

via a pairing

$$\psi : (DF(\mathbf{A}), DG(\mathbf{B})) \to D(F \circ G)(\mathbf{C})$$

given by the following formula. Let $T : F \to \Sigma_{A_p}$, $T' : G \to \Sigma_{B_q}$. Then

$$\psi_{p,q}(T, T') : F \circ G \to \Sigma_{C_{p+q}}$$

is the composition

$$F \circ G \xrightarrow{T * G} \Sigma_{A_p} \circ G \xrightarrow{\Sigma_{A_p} * T'} \Sigma_{A_p} \circ \Sigma_{B_q} \simeq \Sigma_{A_p \wedge B_q} \xrightarrow{\Sigma(f_{p,q})} \Sigma_{C_{p+q}}.$$

As for the slant product, this cross-product coincides up to sign with the usual one when $F = \Sigma_X$ and $G = \Sigma_Y$.

Moreover, if $\mathbf{B} = \mathbf{C} = \mathbf{A}$, i.e. if we have a pairing $\mathbf{f} : (\mathbf{A}, \mathbf{A}) \to \mathbf{A}$ and if we have a natural transformation $\sigma : F \to F \circ F$, we can define a cup product as the composition

$$\tilde{H}^p(F; \mathbf{A}) \otimes \tilde{H}^q(F; \mathbf{A}) \to \tilde{H}^{p+q}(F \circ F; \mathbf{A}) \xrightarrow{\sigma^*} \tilde{H}^{p+q}(F; \mathbf{A}).$$

Then we have the following result: If $\mathbf{A}$ is a spectrum of Eilenberg Mac Lane spaces $K(A, n)$ (or any spectrum which behaves like a ring with unit) where A is a ring with unit and if F is a cotriple, then the cup product makes $\tilde{H}^*(F; \mathbf{A})$ a graded ring with unit.

10. Relations with Spanier-Whitehead duality

An n-duality map between two connected polyhedra X and Y has been defined by Spanier as a continuous map $u : X \wedge Y \to S^n$ such that the slant product $u^*S_n/H_q(X) \to H^{n-q}(Y)$ is an isomorphism, S_n being a generator of $H^n(S^n)$ (see [7, p. 338]). Moreover, G. W. Whitehead has shown that if u is such a duality map, then for any spectrum A,

$$u^*s/H_q(X; \mathbf{A}) \to H^{n-q}(Y; \mathbf{A})$$

is an isomorphism, where s is a generator of $H^n(S^n; \mathbf{S})$ and $\mathbf{S}$ is the spectrum of spheres (see [8, p. 281, Corollary 8.2]).

Now the map $u : X \wedge U \to S^n$ induces a natural transformation

$$\Sigma(u) : \Sigma_X \circ \Sigma_Y \to \Sigma_{S^n} = \Sigma^n.$$

Call $\omega : \Sigma_Y \to \Omega_X \circ \Sigma^n$ the adjoint natural transformation.

Then we can show the following: u is an n-duality map if and only if ω induces an isomorphism in homology and cohomology for all spectra of coefficients.

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UNIVERSITY OF OTTAWA
OTTAWA, CANADA