A NOTE ON TOPOLOGICAL PONTRJAGIN CLASSES AND THE HIRZEBRUCH INDEX FORMULA

BY

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1. The topological Hirzebruch formula

Let $I(M)$ be the index of the closed, oriented, topological $4k$-manifold M. That is, $I(M)$ is the signature of the bilinear form on $H^{2k}(M)$ determined by the cup product pairing.[1] In [13], J. Schafer gives a sophisticated, but somewhat long and indirect proof of the following fact.

Theorem. *Let M and N be closed, oriented, topological $4k$-manifolds such that there exists an r-fold covering map $\pi : M \to N$ preserving orientation. Then*

$$I(M) = rI(N). \tag{1}$$

As Schafer remarks, an example of Wall [17, Cor. 5.4.1], shows that this result does not extend to the case in which M and N are finite Poincaré complexes. Thus, it cannot be proved, for example, by direct cohomology calculations.

Schafer's proof begins with the observation that the theorem is an easy consequence of the following two hypotheses:

(2) *Pontrjagin classes can be defined for topological microbundles.*
(3) *With respect to these classes the Hirzebruch Index Formula is valid.*

More exactly, suppose that to each topological microbundle ξ we can associate (rational!) classes $p_i(\xi) \in H^{4i}(B(\xi))$ that are *natural with respect to microbundle maps*. Set $p_i(M) = p_i(\tau M)$, where τM is the tangent microbundle of M, and let

$$L_i(M) = L_i(p_1(M), \cdots, p_i(M)),$$

where L_i is the Hirzebruch polynomial [4, p. 12]. Then, $L_i(M)$ is natural with respect to tangential maps—in particular, with respect to covering maps. If we now assume the Index Formula

$$\langle L_k(M), \mu_M\rangle = I(M), \tag{4}$$

where μ_M is the orientation class of M, then equation (1) follows from the simple equalities:

$$\langle L_k(M), \mu_M\rangle = \langle \pi^* L_k(N), \mu_M\rangle = \langle L_k(N), \pi_* \mu_M\rangle = r\langle L_k(N), \mu_N\rangle.$$

In this paper, we give two separate proofs of (2) and (3). The first proof (§1, below) is probably the shortest and easiest proof of these facts, and it

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[1] Cohomology will always have rational (= Q) coefficients.

avoids almost all of the machinery of [13]. In the second proof (§2), which is more elementary and, therefore, longer, we construct topological Pontrjagin classes so as to insure (3) by definition. The construction extends the original definition of Pontrjagin classes for *PL manifolds* due to Thom [15] and Milnor [9] and is probably much like the method that Milnor had in mind for defining Pontrjagin classes of *PL* microbundles (see [11, p. 6]). It follows easily that the Pontrjagin classes defined in §1 are the same as those defined in §2 (see the corollary and remark in §2.1).

All our results apply to the *PL* case as well. Our arguments in §1, below, depend, of course, on the recent important results of Kirby and Siebenmann [6], [7]. In §2, we use only their topological transversality theorem [7].

Proof of (2). It suffices to produce universal examples of Pontrjagin classes $p_i \in H^{4i}(BTOP)$ that map to their well-known counterparts $p_i \in H^{4i}(BO)$ under the homomorphism

$$H^*(BTOP) \to H^*(BO)$$

induced by the natural map $j : BO \to BTOP$. That this homomorphism is onto follows from Novikov's proof that (smooth) Pontrjagin classes are topological invariants [12]. This is enough for (2) and for our purposes, but we point out that we can do better. Namely, we appeal to the work of Kervaire-Milnor, Hirsch, and Cerf ([5], [2], and [1]) and the recent work of Kirby and Siebenmann [6], which imply that the fibre TOP/O of

$$j : BO \to BTOP$$

has only finite homotopy groups. It follows immediately that

$$j^* : H^*(BTOP) \to H^*(BO)$$

is an isomorphism, so that we can "extend" the $p_i \in H^{4i}(BO)$ to $H^{4i}(BTOP)$ in exactly one way.

Proof of (3). *Step* 1. Let Ω_O^* and Ω_{TOP}^* be the smooth and topological oriented cobordism rings, respectively, and let $\varphi : \Omega_O^* \to \Omega_{TOP}^*$ be the forgetful homomorphism. We observe that (3) follows easily from the assertion:

(5) coker $(\varphi : \Omega_O^{4k} \to \Omega_{TOP}^{4k})$ is finite, $k = 0, 1, 2, \cdots$.

For suppose that (5) is true. Then, any homorphism

$$\Omega_{TOP}^{4k} \to \mathbf{Q}, \quad k = 0, 1, 2, \cdots$$

is determined by its values on $\varphi(\Omega_O^{4k})$. Two such homomorphisms are given by the Index

$$\Omega_{TOP}^{4k} \xrightarrow{I} \mathbf{Z} \subset \mathbf{Q},$$

and the Hirzebruch polynomial

$$\Omega^{4k}_{TOP} \xrightarrow{L_k(p_1, \cdots, p_k)} \mathbf{Q},$$

which is now defined on Ω^{4k}_{TOP}, by (2). By the smooth Hirzebruch Index Formula, these homorphisms coincide on $\varphi(\Omega^{4k}_O)$. Thus, (3) follows.

Step 2. Let MSO and $MSTOP$ be the Thom spectra (cf. Williamson [18, p. 23]) associated to SO and $STOP$, the identity components of O and TOP (N.B., $\pi_0(O) \approx \pi_0(TOP) \approx \mathbf{Z}_2$), and let $i : MSO \to MSTOP$ be the natural map. We define the usual Thom homomorphisms θ_O and θ_{TOP} and observe that the diagram

$$\begin{array}{ccc} \Omega^*_O & \xrightarrow{\theta_O} & \pi_*(MSO) \\ \varphi \downarrow & & \downarrow i_* \\ \Omega^*_{TOP} & \xrightarrow{\theta_{TOP}} & \pi_*(MSTOP) \end{array}$$

commutes. The proof of commutativity in the PL case is harder (cf. [18, p. 26]) because the PL counterpart of φ must be defined via Whitehead C^1-triangulation theory.

Now we note that, by the usual argument, the isomorphism

$$H^*(BSO) \approx H^*(BO) \approx H^*(BTOP) \approx H^*(BSTOP)$$

implies that the homomorphism of homotopy groups

$$i_* : \pi_*(MSO) \to \pi_*(MSTOP)$$

is a $\mathcal{C}$-isomorphism, where $\mathcal{C}$ is the class of all finite groups. Moreover, that θ_O is an isomorphism is the classical result of Thom [16]. Thus, $\theta_{TOP} \circ \varphi = i_* \circ \theta_0$ is a $\mathcal{C}$-isomorphism.

We conclude Step 2 with a simple algebraic observation. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {}_{h}\searrow & \downarrow g \\ & & C \end{array}$$

be a commutative diagram of abelian groups. Then, there is induced an exact sequence

(6) $$\ker g \to \operatorname{coker} f \to \operatorname{coker} h.$$

If we set $f = \varphi$, $g = \theta_{TOP}$, and $h = \theta_{TOP} \circ \varphi$, then (6) becomes

(7) $$\ker \theta_{TOP} \to \operatorname{coker} \varphi \to \operatorname{coker} h$$

and, since h is a $\mathcal{C}$-isomorphism, it suffices to prove that $\ker \theta_{TOP}$ is finite.

Step 3. We assert that

$$\theta_{TOP} : \Omega^n_{TOP} \to \pi_n(MSTOP)$$

is an isomorphism for all n, except possibly $n = 4$, in which case it is a monomorphism.

For $n \geq 5$, this assertion follows in the same way as the corresponding assertion about θ_O, using the Kirby-Siebenmann transversality theorem [7] in place of that of Thom. Their theorem is more restrictive than Thom's, the restriction being that the submanifold produced must have dimension ≥ 5. Since potential cobordisms between 4-manifolds would have the right dimension, we can use Kirby-Siebenmann transversality in Thom's argument to conclude that θ_{TOP} is a monomorphism, for $n = 4$.

We leave the easy cases $0 \leq n \leq 3$ to the reader.

Therefore, we have shown that ker θ_{TOP} is zero—hence finite—in all dimensions, which completes the proof of (3).

Remarks. (a) For all n,

$$\ker (\varphi : \Omega^n_O \to \Omega^n_{TOP}) = 0.$$

The proof is just a repetition of Williamson's argument ([18, Theorem 5.1]) in the topological case: We note that Stiefel-Whitney classes can be defined for topological microbundles (using Thom's definition) just as in the linear bundle case. The corresponding Stiefel-Whitney numbers are then defined for all closed, oriented topological manifolds. But it is a classical result that these numbers, together with Pontrjagin numbers, detect *all* smooth cobordism classes. Since, by their definition, they must annihilate ker φ, it is zero.

(b) According to [6] and [7],

$$\begin{aligned} \pi_n(TOP/PL) &= 0, \quad n \neq 3, \\ &= \mathbf{Z}_2, \quad n = 3, \end{aligned}$$

whereas, according to [5], [2], and [1]

$$\pi_n(PL/O) = 0, \quad n \leq 6.$$

Let us consider the diagram in Step 2 when $n = 4$.

$$\begin{array}{ccccc} 0 \to & \pi_4(MSO) & \xrightarrow{i_*} & \pi_4(MSTOP) \\ & \theta_O \uparrow \cong & & \uparrow \theta_{TOP} \\ 0 \longrightarrow & \Omega^4_O & \xrightarrow[\varphi]{} & \Omega^4_{TOP} \\ & & & \uparrow \\ & & & 0 \end{array}$$

We know that $\Omega_O^4 \cong \pi_4(MSO) \cong \mathbf{Z}$. The above data imply that $\operatorname{coker} i_* \cong \mathbf{Z}_2$, and, thus, that $\operatorname{coker} \varphi = 0$ or $\mathbf{Z}_2$. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \Omega_O^4 & \xrightarrow{\ \varphi\ } & \Omega_{TOP}^4 \\ & {\scriptstyle I}\searrow \quad \swarrow{\scriptstyle I} & \\ & \mathbf{Z} & \end{array}$$

in which $I : \Omega_O^4 \to \mathbf{Z}$ is an isomorphism onto. Thus,

$$\Omega_{TOP}^4 \cong \Omega_O^4 \oplus \operatorname{coker} \varphi \cong \mathbf{Z} \oplus \operatorname{coker} \varphi.$$

From this it follows that if $\pi_4(MSTOP) \cong \mathbf{Z}$, then φ is an isomorphism but θ_{TOP} has cokernel $\mathbf{Z}_2$. On the other hand, if $\pi_4(MSTOP) = \mathbf{Z} \oplus \mathbf{Z}_2$, this gives no direct information about φ or θ_{TOP}.

2. Another definition of topological Pontrjagin classes

2.1. We want to define Pontrjagin classes for topological microbundles so that (3) holds. The point of this section is to show that it is sufficient to define them for manifolds so that (3) holds. Actually, it will be convenient to define Hirzebruch classes for manifolds, then bundles, and then solve for Pontrjagin classes in terms of these, using the Hirzebruch polynomials L_i. This is the point of view used by Thom [15] and Milnor [9] in constructing Pontrjagin classes for PL manifolds.

Accordingly, we assume that we are able to associate with every closed topological manifold M^n classes

$$l_i(M) \in H^{4i}(M^n), \quad 0 \leq 4i < n + 1,$$

such that $l_0(M) = 1$ and:

(A) The l_i are natural with respect to tangential maps (i.e., maps $f : M \to N$ such that $f^*\tau N$ is stably equivalent to τM).

(B) If M admits a smoothing, then the $l_i(M)$ are the usual Hirzebruch classes $L_i(M)$ associated with that smoothing.

(C) $l_i(M \times N) = \sum_{r+s=i} l_r(M) \times l_s(N)$.

(D) $\langle l_k(M^{4k}), \mu_M \rangle = I(M)$, when M^{4k} is oriented.[2]

PROPOSITION 1. *Let $\mathfrak{J}$ be the class of topological microbundles with base spaces that have the homotopy type of finite simplicial complexes. For every $\xi \in \mathfrak{J}$, there are defined unique classes $l_i(\xi) \in H^{4i}(B(\xi))$, $i = 0, 1, 2, \cdots$, such that $l_0(\xi) = 1$, and:*

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[2] Note that (A) and (D) are all that we need for the proof of the theorem, so that this section is unnecessary for an alternate proof of the theorem.

(E) *If* $f : B(\xi) \to B(\eta)$ *is covered by a stable bundle map* $\xi \to \eta$, *then*

$$f^* l_i(\eta) = l_i(\xi), \quad i = 0, 1, 2, \cdots.$$

(F) $l_i(\xi \oplus \eta) = \sum_{r+s=i} l_r(\xi) l_s(\eta)$.

(G) $l_i(\tau M) = l_i(M)$, *for all closed, topological manifolds* M.[3]

(H) *If* ξ *admits the structure of a vector bundle* α, *then*

$$l_i(\xi) = L_i(p_1(\alpha), \cdots, p_i(\alpha)).$$

Proof. Step 1. *Extend the definition of* $l_i(M)$ *to all compact manifolds, preserving properties* (A)–(C).

To do this, let $j_M M \to 2M$ be the inclusion of a manifold with non-empty boundary into its double and define $l_i(M) = j^* l_i(2M)$. Properties (A) and (B) are easily checked.

If $\partial M = \phi$, let us use the convention that $2M = M$ and $j_M = 1_M$. Then,

$$\begin{aligned} l_i(M \times N) &= (j_M \times 1_N)^* l_i(2M \times N) \\ &= (j_M \times 1_N)^* (1_{2M} \times j_N)^* l_i(2M \times 2N), \end{aligned}$$

by naturality. Since $2M$ and $2N$ are closed, we may apply (C) to $l_i(2M \times 2N)$, and, using the above equality, obtain (C) for $l_i(M \times N)$.

Step 2. *Extend the definition to microbundles over compact manifolds so that* (E)–(H) *are satisfied.*

Let ξ be such a microbundle. Once we find a compact manifold E and a map $f : B(\xi) \to E$ such that $f^* \tau E$ is stably equivalent to ξ, then we shall define $l_i(\xi) = f^* l_i(E)$. This definition will be independent of f and E, provided that we can choose some such f to be a homotopy equivalence.

Assuming this done, properties (E)–(G) follow easily. For example, property (F) is an immediate consequence of (C) and the equation $\tau(M \times N) = \tau M \times \tau N$.

To prove (H), let ξ be a microbundle that stably reduces to a vector bundle α, let $f : B(\xi) \to E$ be a homotopy equivalence with a compact manifold E, as above, such that $f^* \tau E$ is stably equivalent to ξ, and let β be a vector bundle over E with $f^* \beta$ stably equivalent to α. Then, β is a stable reduction of τE, and, by smoothing theory [10], $E \times \mathbf{R}^n$ is smoothable, n large, in such a way that the smooth tangent vector bundle $\tau_0(E \times \mathbf{R}^n)$ when restricted to $E \times 0$ is stably equivalent to β.

Let V be any smooth, co-dimension-0 compact submanifold of $2E \times \mathbf{R}^n$ containing $E \times 0 = E$. Then

$$\begin{aligned} l_i(E) &= l_i(V) \mid E, \quad \text{by (A)}, \\ &= L_i(p_1(V), \cdots, p_i(V)) \mid E, \quad \text{by (B)}, \\ &= L_i(p_1(\beta), \cdots, p_i(\beta)), \end{aligned}$$

[3] According to [6], each such M has the homotopy type of a finite simplicial complex.

so that $l_i(\xi) = f^* l_i(E) = L_i(p_1(\alpha), \cdots, p_i(\alpha))$, by the naturality of the p_j with respect to vector-bundle maps. This proves (H).

We now construct $f : B(\xi) \to E$. Let $-\tau$ be a stable inverse for $\tau B(\xi)$, and let $\eta = -\tau \oplus \xi$. A theorem of Mazur and Hirsch [3], together with Kister's theorem [8], implies that, by adding a trivial line bundle to η, if necessary, we may assume that there is a topological disc bundle over $B(\xi)$ with total space E whose interior is a bundle equivalent to η.[4] According to [10], $\tau E/B(\xi)$ is stably equivalent to

$$\tau B(\xi) \oplus \eta = \tau B(\xi) \oplus -\tau \oplus \xi,$$

which is stably equivalent to ξ. Let $f : B(\xi) \to E$ be the zero-section inclusion. This completes Step 2.

Step 3. *Extend the definition to all microbundles in* $\mathfrak{J}$.

Choose such a microbundle ξ, and let $f : B(\xi) \to K$ be a homotopy equivalence, with homotopy-inverse g, where K is a finite simplicial complex. We may assume that K is a compact manifold, for if not, triangulate the $\mathbf{R}^q$ in which K sits, so that K is a subcomplex and replace K by its second derived neighborhood. Let $\eta = g^*\xi$ and define $l_i(\xi) = f^* l_i(\eta)$.

Properties (E)–(H) and uniqueness are easily verified.

PROPOSITION 2. *Let BTOP be the classifying space for stable microbundles in* $\mathfrak{J}$. *There exist unique classes* $l_i \in H^{4i}(BTOP)$, $i = 0, 1, 2, \cdots$, *such that, for any* $\xi \in \mathfrak{J}$,

$$l_i(\xi) = f_\xi^*(l_i),$$

where $f_\xi : B(\xi) \to BTOP$ *is a classifying map for* ξ.

Moreover, if $j : BO \to BTOP$ *is the natural map, then*

$$j^*(l_i) = L_i(p_1, \cdots, p_i).$$

Proof. We recall that $BTOP$ is the direct limit of the sequence

$$BTOP(1) \xrightarrow{i_1} BTOP(2) \xrightarrow{i_2} \cdots \xrightarrow{i_{n-1}} BTOP(n) \xrightarrow{i_n} \cdots,$$

where $BTOP(n)$ is a countable CW complex classifying topological $\mathbf{R}^n$-bundles (or microbundles). To make sure that the result is suitably nice, we may replace i_1, i_2, i_3, etc., consecutively, by the inclusions of the corresponding mapping cylinder, and then take the union. For $n \gg m$, the inclusion $BTOP(n) \to BTOP$ determines an isomorphism

$$H^m(BTOP) \approx H^m(BTOP(n)).$$

Let u_n be the universal $\mathbf{R}^n$-bundle over $BTOP(n)$, and let $u_{n,k}$ be its restriction to the k-skeleton $BTOP(n, k)$ of $BTOP(n)$. For $0 \leq 4i < n + 1$, let

$$l_i(n, k) = l_i(u_{n,k}) \in H^{4i}(BTOP(n, k)).$$

[4] For these results to apply, we need $B(\xi)$ to be a polyhedron, or at least to have the homotopy type of one. As stated before, it does, by [6].

Note that for $k > m$, the inclusion $BTOP(n, k) \to BTOP(n)$ determines an isomorphism

$$H^m(BTOP(n)) \approx H^m(BTOP(n, k)).$$

Thus, $l_i(n, k)$ determines a unique class in $H^{4i}(BTOP)$, when $n \gg 4i$ and $k > 4i$. By naturality, it is independent of k and n. This is l_i.

The last statement follows easily from (H). This completes the proof of Proposition 2.

COROLLARY. *Define* $p_i^{TOP} \in H^{4i}(BTOP)$ *by the equalities (solved consecutively),*

$$L_i(p_i^{TOP}, \cdots, p_i^{TOP}) = l_i, \quad i = 0, 1, 2, \cdots.$$

Then, if $j : BO \to BTOP$ *is the natural map, we have*

$$j^* p_i^{TOP} = p_i.$$

Remarks. (a) Using the fact proved in §1 that j^* is an isomorphism, we see that the two definitions coincide.

(b) If we are willing to give up the uniqueness statements in the above results, then the proofs may be easily modified so as to avoid dependence on [6]. The most serious point is in the proof of Proposition 2, where we make tacit use of the finite generation of $H^i(BTOP)$.

2.2. *Hirzebruch classes for closed, topological manifolds.* The definition of topological Hirzebruch classes given below will differ only in certain technical respects from Milnor's definition of Hirzebruch classes for PL manifolds [9]. Moreover, the verification of properties (A)–(D) is essentially the same in both the topological and the PL cases. Therefore, we shall first verify (A)–(D) in the PL case and then indicate what modifications are needed for the topological case.

Note that we can restrict ourselves to orientable manifolds, because, for non-orientable M^n, we can take the orientation covering $\pi : \tilde{M}^n \to M^n$ and define

$$l_i(M^n) = (\pi^*)^{-1}(l_i(\tilde{M}^n)).$$

Let us now recall some key facts about Milnor's definition. Let M^n be a closed, PL, oriented, n-manifold, and let $f : M^n \to S^{n-4i}$ be a map. Then, there is associated with f an integer $I(f)$, depending only on the homotopy class of f. $I(f)$ can be defined as follows: Let $0 \in S^{n-4i}$ be a basepoint. Approximate f by a PL map also called f, and, by Williamson's transversality theorem [18], homotop f so that it is transversal to 0, still calling the resulting map f. Then $f^{-1}(0)$ is a closed $4i$-manifold, with orientation induced by those of M^n and S^{n-4i}, so that $I(f^{-1}(0))$ is defined. This if $I(f)$. (Milnor did not have Williamson's theorem available in [9] and used a slightly different method. We use this one because it generalizes to the topological case, whereas Milnor's does not.)

Milnor then shows, using a theorem of Serre, that for $8i + 2 \leq n$, there are classes $l_i(M^n) \in H^{4i}(M^n)$ *characterized* by the following fact: for every map $f : M^n \to S^{n-4i}$,

$$\langle l_i(M^n) \cup f^*(\nu_{S^{n-4i}}), \mu_{M^n} \rangle = I(f), \tag{8}$$

where $\nu_{S^{n-4i}}$ is the orientation cohomology class of S^{n-4i}.

For $8i + 2 > n$, let $q = 8i + 2 - n$ and define $l_i(M^n) \in H^{4i}(M^n)$ to be the restriction $l_i(M^n \times S^q) \mid M^n \times 0$.

Property (B) is verified in [9]. We leave the easy check that $l_0(M^n) = 1$ to the reader.

Verification of property (A) (*naturality*). It suffices to check this for $8i + 2 \leq n$. We shall use (8). Thus, suppose that $g : M^n \to N^q$ is a map with $g^*\tau N^q$ stably equivalent to τM^n, and let $f : M^n \to S^{n-4i}$ be any map. It suffices to show that

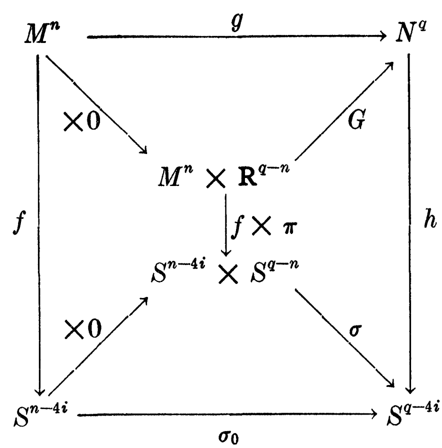
$$\langle g^* l_i(N^q) \cup f^*(\nu_{S^{n-4i}}), \mu_M \rangle = I(f). \tag{9}$$

Suppose first that g is an imbedding with trivial normal microbundle. More precisely, let

$$M^n \times \mathbf{R}^{n-q} \xrightarrow{G} N^q$$

be an orientation-preserving PL imbedding onto a neighborhood of $g(M^n)$ such that $G \mid M^n \times 0 = g$. We define a map $h : N^q \to S^{q-4i}$ so that the following diagram commutes:

(10)



Here, $\pi : \mathbf{R}^{q-n} \to S^{q-n} = \mathbf{R}^{q-n} \cup \infty$ is an origin preserving degree-one map that sends the interior of the unit ball D^{q-n} piecewise-linearly onto $S^{q-n} - \infty$, and the rest of $\mathbf{R}^{q-n}$ to ∞. σ is the standard degree-one map sending 0×0 to 0 and $S^{n-4i} \times \infty \cup \infty \times S^{q-n}$ to ∞. The map h is uniquely defined by the diagram on image G. Define it elsewhere to be constantly ∞. It has two nice properties. First, by an arbitrarily small adjustment exterior to $G(M \times \frac{1}{2}D^{q-n})$, it can be made PL. Secondly, it is transversal to $0 \in S^{q-4i}$ if and only if f is transversal to $0 \in S^{n-4i}$ and then $f^{-1}(0)$ is sent homeomorphically to $h^{-1}(0)$ by g such that induced orientations are preseved. Thus, assuming that f is transversal to 0, let $V(f) = f^{-1}(0)$, $V(h) = h^{-1}(0)$, and let $j_1 : V(f) \to M^n$ and $j_2 : V(h) \to N^q$ be the inclusions. Then,

$$\begin{aligned} \langle g^* l_i(N^q) \cup f^*(\nu_{S^{n-4i}}), \mu_M\rangle &= \langle g^* l_i(N^q), j_{1*}(\mu_{V(f)})\rangle \\ &= \langle l_i(N^q), (gj_1)_*(\mu_{V(f)})\rangle \\ &= \langle l_i(N^q), j_{2*}(\mu_{V(h)})\rangle \\ &= I(h) \\ &= I(f), \end{aligned}$$

the first equality coming from Poincaré duality.

If $g : M^n \to N^q$ is not an imbedding with trivial normal microbundle, let

$$g' : M^n \to N^q \times S^p$$

be an imbedding approximating the composition

$$i_0\, g : M^n \to N^q \times S^p$$

where $i_0 : N^q \to N^q \times S^p$ sends x to $(x, 0)$ and p is large. By the results of [10] and [11], $g'(M^n)$ has a *trivial* normal microbundle in $N^q \times S^p$. This is where we use the condition that g (and hence g') is stably tangential. Therefore, applying the result already proved to g' and i_0, we have

$$(g')^* l_i(N^q \times S^p) = l_i(M^n) \qquad \text{and} \quad i_0^* l_i(N^q \times S^p) = l_i(N^q),$$

so that

$$g^*(l_i(N^q)) = g^* i_0^* l_i(N^q \times S^p) = (g')^* l_i(N^q \times S^p) = l_i(M^n),$$

as desired.

Verification of property (C) (*the product formula*). Given M^m and N^n, closed and oriented, we must show that

$$l_i(M^m \times N^n) = \sum\nolimits_{r+s=i} l_r(M^m) \times l_s(N^n).$$

We remark first that it is sufficient to verify this for $8i + 2 \leq \min(m, n)$, since, for larger i, we simply look at $(M^m \times S^p) \times (N^n \times S^q)$, p and q large. We impose this restriction on i, and we now look at the Künneth formula for stable cohomotopy tensored with $\mathbf{Q}$ (= cohomology):

$$(11) \qquad \pi_{\mathbf{Q}}^{m+n-4i}(M^m \times N^n) \cong \sum\nolimits_{r+s=m+n-4i} \pi_{\mathbf{Q}}^r(M^m) \otimes \pi_{\mathbf{Q}}^s(N^n).$$

Note that $\pi_{\mathbf{Q}}^r(M^m) = 0$, $r > m$, and $\pi_{\mathbf{Q}}^s(N^n) = 0$, $s > n$, so that we may suppose that $r \geq m - 4i$ and $s \geq n - 4i$. Together with the inequality $8i + 2 \leq \min(m, n)$, these imply

$$m \leq 2r - 2, \qquad n \leq 2s - 2, \qquad m + n \leq 2(m + n - 4i) - 2,$$

which allow us, by a theorem of Serre [14], to represent classes in

$$\pi_{\mathbf{Q}}^{m+n-4i}(M^m \times N^n), \qquad \pi_{\mathbf{Q}}^{r}(M^m) \quad \text{and} \quad \pi_{\mathbf{Q}}^{s}(N^n),$$

up to a rational multiple, by maps

$$M^m \times N^n \to S^{m+n-4i}, \qquad M^m \to S^r \quad \text{and} \quad N^n \to S^s,$$

respectively.

Now, choose any $f : M^m \times N^n \to S^{m+n-4i}$. The above remarks, together with (11) imply that, for some $\lambda \in \mathbf{Q}$, $\lambda \neq 0$, the class of f in $\pi_{\mathbf{Q}}^{m+n-4i}(M^m \times N^n)$ is obtained from maps $f_r : M^m \to S^r$ and $g_s : N^n \to S^s$ in the following way: Let

$$f_r \otimes g_s : M^m \times N^n \to S^{m+n-4i}$$

be the composition

$$M^m \times N^n \xrightarrow{f_r \times g_s} S^r \times S^s \xrightarrow{\quad\sigma\quad} S^{r+s}, \tag{12}$$

where σ is the collapsing map onto $S^r \times S^s / (S^r \times \infty) \cup (\infty \times S^s)$; then

$$\lambda[f] = \sum\nolimits_{r+s=m+n-4i} [f_r \otimes g_s].$$

Since the association $f \to I(f)$ determines a *homomorphism* $\pi_{\mathbf{Q}}^{m+n-4i} \to \mathbf{Q}$, we have

$$\lambda I(f) = \sum I(f_r \otimes g_s).$$

But, when both f_r and g_s are transversal to 0, so is $f_r \otimes g_s$; moreover, we always have $(f_r \otimes g_s)^{-1}(0) = f_r^{-1}(0) \times g_s^{-1}(0)$. Therefore, by the multiplicativity of I,

$$\lambda I(f) = \sum I(f_r) I(g_s). \tag{13}$$

Of course, note that the summand $I(f_r)I(g_s) = 0$ unless r is of the form $m - 4j$ *and* n is of the form $n - 4k$, $j + k = i$. We now compute

$$\lambda f^*(\nu_{S^{m+n-4i}}) = \sum\nolimits_{j+k=i} f_{m-4j}^*(\nu_{S^{m-4j}}) \times g_{n-4k}^*(\nu_{S^{n-4k}}),$$

so that

$$\lambda\langle (\sum\nolimits_{j+k=i} l_j(M) \times l_k(N)) \cup f^*(\nu_{S^{m+n-4i}}), \mu_{M \times N}\rangle$$
$$= \sum\nolimits_{j+k=i} I(f_{m-4j}) I(g_{n-4k}) = \lambda I(f).$$

Thus, by the fact that (8) characterizes l_i, we have

$$l_i(M \times N) = \sum\nolimits_{j+k=i} l_j(M) \times l_j(N).$$

This completes the verification.

Modifications for the topological case. The association $f \to I(f)$ in the topological case is defined just as in the PL case, for maps $f : M^n \to S^{n-4i}$, $i \neq 1$, using Kirby-Siebenmann transversality [7] in place of Williamson's when $i > 1$ and Hopf's theorem to get tranversality when $i = 0$. This procedure allows us to define $l_0, l_2, l_3, \cdots$ as before, to show that $l_0 = 1$, and to prove properties (A) (naturality), (B), and (D), for them as in the PL case. However, Kirby-Siebenmann transversality does not apply to maps $M^n \to S^{n-4}$ and so we cannot get l_1 this way. In this case, we use the following trick: given $f : M^n \to S^{n-4}$, $n \geq 10$, we let $\pi : M^n \times \mathbf{C}P^4 \to M^n$ be the first coordinate projection and define

$$I(f) = I(f\pi) = I((f\pi)^{-1}(0)).$$

Then $l_1(M^n) \in H^4(M^n)$ is defined to be the unique class satisfying

$$\langle l_1(M^n) \cup f^*(\nu_{S^{n-4}}), \mu_M\rangle = I(f).$$

Notice that if f is transversal to 0, then so is $f\pi$, and $(f\pi)^{-1}(0) = f^{-1}(0) \times \mathbf{C}P^4$, so that, because $I(\mathbf{C}P^4) = 1$,

$$\langle l_1(M^n) \cup f^*(\nu_{S^{n-4}}), \mu_M\rangle = I(f^{-1}(0) \times \mathbf{C}P^4) = I(f^{-1}(0)).$$

Thus, this definition of $I(f)$ and l_1 is a generalization of the PL and smooth cases. This proves (B) for l_1.

To prove (A) (naturality) for l_1, we augment diagram (10) as follows:

$$\begin{array}{ccc} M^n \times \mathbf{C}P^4 & \xrightarrow{g \times 1} & N^q \times \mathbf{C}P^4 \\ \pi\downarrow & & \downarrow\pi \\ M^n & \xrightarrow{g} & N^q \\ f\downarrow & & \downarrow h \\ S^{n-4} & \xrightarrow[\sigma_0]{} & S^{q-4}. \end{array} \tag{14}$$

Here, g is supposed to be any stably tangential map, but, as in the argument for naturality in the PL case, we may assume that g is an imbedding with trivial normal microbundle. The map h is defined as in the PL case, except that here we do not have to worry about making it PL. Note that when $f\pi$ is homotoped to be transversal to $0 \in S^{n-4}$, this produces a homotopy between $h\pi$ and a map transversal to $0 \in S^{q-4}$. Also note that when $f\pi$ is transversal to 0, so is $h\pi$, and $g \times 1$ sends $(f\pi)^{-1}(0) = V(f)$ homeomorphically onto $(h\pi)^{-1}(0) = V(h)$. Let

$$j_1 : V(f) \to M^n \times \mathbf{C}P^4 \quad \text{and} \quad j_2 : V(h) \to N^q \times \mathbf{C}P^4$$

be the inclusions, and let $\beta \in H^8(\mathbf{C}P^4)$ be the canonical generator. Then,

$$\begin{aligned}\langle g^* l_1(N) \cup f^* \nu_{S^{n-4}}, \mu_M\rangle &= \langle (g \times 1)^*(l_1(N) \times \beta) \cup (f\pi)^* \nu_{S^{n-4}}, \mu_{M \times \mathbf{C}P^4}\rangle \\ &= \langle (g \times 1)^*(l_1(N) \times \beta), j_{1*}(\mu_{V(f)})\rangle \\ &= \langle l_1(N) \times \beta, j_{2*}(\mu_{V(h)})\rangle \\ &= \langle l_1(N) \times \beta) \cup (h\pi)^* \nu_{S^{q-4}}, \mu_{N \times \mathbf{C}P^4}\rangle \\ &= \langle l_1(N) \cup h^* \nu_{S^{q-4}}, \mu_N\rangle \\ &= I(V(h)), \quad \text{by definition of } l_1, \\ &= I(V(f)), \quad \text{since } V(h) \approx V(f), \\ &= I(f).\end{aligned}$$

Since this last equality, if valid for all $f : M^n \to S^{n-4}$, determines $l_1(M)$, we must have

$$g^* l_1(N) = l_1(M).$$

This completes the proof of naturality for l_1.

To prove (C) (the product formula), we proceed exactly as in the PL case until we reach the point (cf. (13)) where we show that

$$(15) \qquad I(f_r \otimes g_s) = I(f_r) I(g_s).$$

Of course, when f_r and g_s can be made transversal to 0, we obtain (15) as before. If one or the other cannot be made transversal to 0 (that is, if $r = m - 4$ or $s = n - 4$), then we must modify the argument to obtain (15). The conclusion of the argument then is a copy of the PL case.

To prove (15), first observe that if $f : M^m \to S^{m-4i}$ is any map, and if

$$\pi_M : M^m \times \mathbf{C}P^4 \to M^m$$

is the projection, then

$$(16) \qquad I(f) = I(f\pi_M)$$

and, similarly, if $g : N^n \to S^{n-4j}$ is another map,

$$(17) \qquad I(f \otimes g) = I((f \otimes g)(\pi_M \times \pi_N)).$$

When f and $f \otimes g$ can be made transversal to 0, these equalities follow from the relations

$$(f\pi_M)^{-1}(0) = f^{-1}(0) \times \mathbf{C}P^4,$$

$$(f \otimes g)(\pi_M \times \pi_N)^{-1}(0) = t((f \otimes g)^{-1}(0) \times \mathbf{C}P^4 \times \mathbf{C}P^4),$$

where $t : M^m \times N^n \times \mathbf{C}P^4 \times \mathbf{C}P^4 \to M \times \mathbf{C}P^4 \times N \times \mathbf{C}P^4$ is the orientation-preserving (!) homeomorphism that switches factors. Otherwise, the relations are true by definition.

Since, by definition (12),

$$(f \otimes g)(\pi_M \times \pi_N) = (f\pi_M) \otimes (g\pi_N),$$

and since both $f\pi_M$ and $g\pi_N$ can be made transversal to 0, we have, using (16) and (17),

$$I(f \otimes g) = I((f \otimes g)(\pi_M \times \pi_N)) = I((f\pi_M) \otimes (g\pi_N))$$
$$= I(f\pi_M)I(g\pi_N) = I(f)I(g),$$

which proves (15) and completes the proof of the product formula.

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