SUPREMUM NORM ESTIMATES FOR PARTIAL DERIVATIVES OF FUNCTIONS OF SEVERAL REAL VARIABLES

BY
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1. Introduction

Denote by $\| u \|_p$ the L_p-norm of the function u from R^n to the complex numbers C, let Q and P_j be polynomials in n variables with complex coefficients, and form the corresponding differential operators $Q(D)$ and $P_j(D)$, where $D_k = (1/i)\, \partial/\partial x_k$. Consider the problem of deciding whether there exists a constant C such that

$$(1.1) \qquad \| Q(D)u \|_p \leqq C \textstyle\sum_{j=1}^m \| P_j(D)u \|_p \quad \text{for all } u \in C_0^\infty(R^n).$$

Here $C_0^\infty(R^n)$ denotes the set of infinitely differentiable functions with compact support. It is easy to see that for an arbitrary p $(1 \leqq p \leqq \infty)$ the condition

$$(1.2) \qquad | Q(\xi) | \leqq C \textstyle\sum_{j=1}^m | P_j(\xi) | \qquad (\xi \in R^n)$$

is necessary for (1.1) to hold. To prove this one need only apply both members of (1.1) to the function $u(x) = g(\varepsilon x) \exp(i\langle x, \xi\rangle)$, where $g \in C_0^\infty(R^n)$, $g(x) = 1$ in a neighborhood of the origin, and ε is sufficiently small (we use the notation $\langle x, \xi\rangle = (x_1 \xi_1 + \cdots + x_n \xi_n)$. If $p = 2$ it is easily seen by means of Parseval's formula that (1.2) is also sufficient for (1.1) to hold. However, if $p \neq 2$, it is much harder to find necessary and sufficient conditions for (1.1) to hold, and so far only very few special cases have been treated.

In this paper we study the case where Q and P_j are all monomials. Then the inequality (1.1) can be written (for explanation of notation see Section 2)

$$(1.3) \qquad \| D^\beta u \|_p \leqq C \textstyle\sum_{\alpha \in A} \| D^\alpha u \|_p \qquad (u \in C_0^\infty(R^n)).$$

For the case $p = \infty$ we obtain a necessary and sufficient condition in geometric terms for the inequality (1.3) to hold (Theorem 1). This condition is close to β being an interior point of the convex hull of A. A sufficient condition close to ours was announced by Golovkin in [1] ($\beta \in \operatorname{int}(\operatorname{ch}(A))$ in the case where A is a simplex). In [2] Golovkin gives a proof of the same result in the case of two dimensions.

Recently it was proved by Il'in [4] that (1.3) holds for $1 < p < \infty$ (C depending on p) if and only if $\beta \in \operatorname{ch}(A)$ (note that $\beta \in \operatorname{ch}(A)$ is equivalent to (1.2)). Since Il'in's proof is quite complicated (Il'in treats also the case of L^p-norms over certain subdomains of R^n), we give a short proof of this result here. Our proof, like that of Il'in, is based on an L^p-multiplier theorem of Lizorkin [8].

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Finally we prove a result of de Leuuw and Mirkil [5] on the estimate (1.1) in the case $m = 2$, P_1 elliptic, $P_2 = 1$.

Our method of proving the inequalities in question is quite different from those of the cited authors. The essential idea in our method is the use of convolutions with a sequence of smooth functions whose sum is equal to the Dirac measure (for technical reasons we use a continuous parameter family instead of a sequence). This technique has been used by Peetre in works on Besov spaces [11], [12] and on partial regularity of vector valued distributions [13].

Some counter examples to L_p-estimates of the type (1.1) have been given by Littman, McCarthy, and Rivière [7].

2. The case $p = \infty$: Statement of result

Let N be the set of non-negative integers. If

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in N^n \quad \text{and} \quad \xi = (\xi_1, \cdots, \xi_n) \in R^n,$$

we write $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_k = (1/i)\, \partial/\partial x_k$. Let A be a finite subset of N^n. Considering A as a subset of R^n we can form the convex hull of A, denoted ch (A). Using the metric in R^n we can form the interior of ch (A), denoted int (ch (A)), or int_n (ch (A)). If B is a subset of a k-dimensional affine subspace L_k in R^n, we denote by int_k (B) the interior of B with respect to the induced metric in L_k. This concept does not depend on the particular choice of L_k, since if B is contained in two different k-dimensional affine subspaces L_k, then int_k (B) is empty.

The following result, which is our main theorem, asserts the equivalence of an estimate (E) in L_∞-norm and a certain geometric condition (G).

THEOREM 1. *Let $\beta \in N^n$ and let A be a finite subset of N^n. The following conditions are equivalent*:

(E) *There exists a constant C such that*

$$\| D^\beta u \|_\infty \leqq C \textstyle\sum_{\alpha \in A} \| D^\alpha u \|_\infty \qquad (u \in C_0^\infty (R^n)).$$

(G) *There exists an integer k, $0 \leqq k \leqq n$, and a k-dimensional affine subspace L_k in R^n, which is parallel to a k-dimensional coordinate plane in R^n, such that*

$$\beta \in \text{int}_k \ (\text{ch } (L_k \cap A)).$$

If $k = n$, the formula $\beta \in \text{int}_k$ (ch $(L_k \cap A)$) should of course be interpreted

$$\beta \in \text{int}_n \ (\text{ch } (A)) = \text{int } (\text{ch } (A)),$$

and if $k = 0$ it should be interpreted $\beta \in A$.

Example 1. As is well known, the inequality (we write $D = d/dx$)

$$\| D^j u \|_\infty \leqq C(\| D^k u \|_\infty + \| D^m u \|_\infty) \qquad (u \in C_0^\infty (R))$$

holds if and only if $k \leqq j \leqq m$.

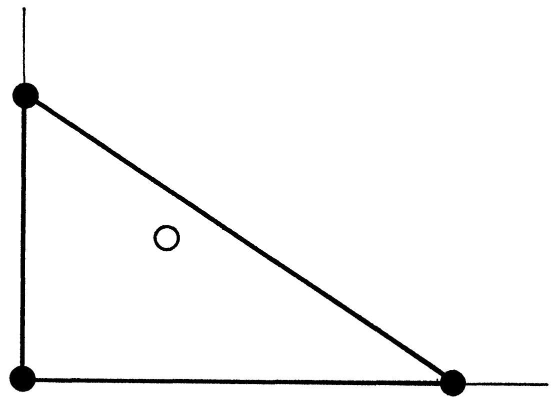

FIGURE 1

Example 2. The following estimate holds (see Figure 1):

$$\| D_1 D_2 u \|_\infty \leqq C(\| D_1^2 u \|_\infty + \| D_2^3 u \|_\infty + \| u \|_\infty) \qquad (u \in C_0^\infty(R^2)).$$

Example 3. There is no estimate

$$\| D_1 D_2 u \|_\infty \leqq C(\| D_1^2 u \|_\infty + \| D_2^2 u \|_\infty + \| D_1 u \|_\infty + \| D_2 u \|_\infty + \| u \|_\infty) \qquad (u \in C_0^\infty(R^2)),$$

since the point (1, 1) is not an interior point of ch (A), and is not an interior point of any horizontal or vertical line segment with end-points in A (see Figure 2).

The inequality (E) is closely related to the following statement.

(C) *If f is a continuous function defined in R^n such that $D^\alpha f$ is continuous for each $\alpha \in A$, then $D^\beta f$ is continuous.*

Here $D^\alpha f$ may be interpreted for instance in the sense of the theory of distributions. By means of the Closed Graph Theorem or by more elementary methods one easily proves that (C) implies

$$\text{(E}'\text{)} \quad \| D^\beta u \|_\infty \leqq C(\textstyle\sum_{\alpha \in A} \| D^\alpha u \|_\infty + \| u \|_\infty) \qquad (u \in C_0^\infty(R^n)).$$

Thus we conclude in particular (Example 3) that there exists a function f defined in R^2 such that f, $D_1 f$, $D_2 f$, $D_1^2 f$, and $D_2^2 f$ are continuous, but $D_1 D_2 f$ is discontinuous. A construction of such a function was given by Mitjagin [9]. On the other hand, if A has the property

$$\alpha \in A, \gamma \leqq \alpha \quad \textit{implies} \quad \gamma \in A,$$

then we also have the opposite implication (E′) = (E) $\Rightarrow$ (C). (Here $\gamma \leqq \alpha$ means that $\gamma_j \leqq \alpha_j$ for each j.) This statement is proved by standard arguments.

A very simple example of a function f of two variables such that f, $D_1 f$, $D_2 f$, $D_1^2 f$, and $D_2^2 f$ are continuous in a neighborhood of the origin, but $D_1 D_2 f$ does

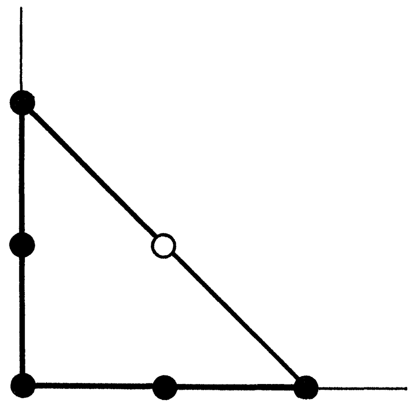

FIGURE 2

not exist at the origin, can be given as follows

$$f(x_1, x_2) = x_1 x_2 \log |\log (x_1^2 + x_2^2)| \quad \text{for } 0 < x_1^2 + x_2^2 < \tfrac{1}{2},$$
$$f(0, 0) = 0.$$

More generally, let $\beta \in N^n$, and define the function f by

$$f(x) = x^\beta \log \log (1/|x|) \quad \text{for } x \in R^n,\ 0 < |x| < \tfrac{1}{2},$$
$$f(0) = 0.$$

It is easy to prove that f has the following properties (write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha \in N^n$).

(i) $D^\alpha f$ exists and is continuous for each α such that $|\alpha| \leqq |\beta|$ and $\alpha \neq \beta$.
(ii) $D^\beta f(0)$ does not exist.

In these statements the mixed derivatives $D^\alpha f$ and $D^\beta f$ may be interpreted in an elementary sense being defined by repeated differentiation in an arbitrary order. Property (i) implies of course in particular that f belongs to C^k, the space of k times continuously differentiable functions, for $k = |\beta| - 1$.

Denote by M the set of complex-valued measures in R^n with finite total mass, and denote by $\hat{M}$ the set of Fourier transforms $\hat{\mu}$ of elements $\mu \in M$. Identify L_1 with a subset of M. The set of Fourier transforms of functions in L_1 is denoted $\hat{L}_1$. The total mass of $\mu \in M$ is denoted $\|\mu\|_M$, and if $h = \hat{\mu}$, $\mu \in M$, we write $\|h\|_{\hat{M}} = \|\mu\|_M$. Similarly if $h = \hat{f}, f \in L_1$, we write $\|h\|_{\hat{L}_1} = \|f\|_1$.

If $\sigma, \mu \in M$, we can form the convolution $\sigma * \mu$, which is also an element of M. We note also that if $\mu \in M$ and $u \in L_p$, we have the inequality

$$\|\mu * u\|_p \leqq \|\mu\|_M \cdot \|u\|_p \qquad (1 \leqq p \leqq \infty). \tag{2.1}$$

The proof of Theorem 1 is based on the following lemma.

LEMMA 1. *There exists a constant C such that*

$$\| Q(D)u \|_\infty \leqq C \textstyle\sum_{j=1}^m \| P_j(D)u \|_\infty \qquad (u \in C_0^\infty(R^n)) \tag{2.2}$$

if and only if there exist functions $h_j \in \hat{M}$ such that

$$Q(\xi) = \textstyle\sum_{j=1}^m h_j(\xi)P_j(\xi). \tag{2.3}$$

The sufficiency of the condition is an immediate consequence of (2.1), and the necessity is proved by a standard application of the Hahn-Banach Theorem (see Lemma 1 in [5]).

We do not know whether (2.3) is also implied by

$$\| Q(D)u \|_1 \leqq \textstyle\sum_{j=1}^m \| P_j(D)u \|_1 \qquad (u \in C_0^\infty(R^n)).$$

(The converse implication is obvious.) If this were proved, one would obtain the analogue of Theorem 1 for $p = 1$. This would generalize the result of Ornstein [10].

3. Sufficiency of the condition (G)

We shall need the following lemma.

LEMMA 2. *Assume that $g_1, \cdots, g_m$ are non-negative measurable functions defined in R^n and that*

$$\int_{R^n} \inf_{1 \leqq j \leqq m} g_j(t)\, dt < \infty .$$

Then there exist disjoint measurable sets G_j such that

$$\bigcup_{j=1}^m G_j = R^n \quad \text{and} \quad \int_{G_j} g_j(t)\, dt < \infty \qquad (j = 1, \ldots, m).$$

Proof. Set

$$H_j = \{t; t \in R^n, g_j(t) = \inf_i g_i(t)\}.$$

Then clearly

$$\bigcup_{j=1}^m H_j = R^n \quad \text{and} \quad \int_{H_j} g_j(t)\, dt < \infty ,$$

but the sets H_j are not necessarily disjoint. Define G_j inductively by

$$G_1 = H_1 \quad \text{and} \quad G_j = H_j \backslash (G_1 \cup \cdots \cup G_{j-1}) \qquad (j = 2, \cdots, m).$$

Then G_j, $j = 1, \cdots, m$, have all the required properties.

In proving that (G) implies (E) we may assume that (G) is satisfied with $k = n$, i.e. that $\beta \in \operatorname{int}(\operatorname{ch}(A))$. To see this assume that $\beta \in \operatorname{int}_k(\operatorname{ch}(L_k \cap A))$ and that $k \leqq n - 1$. If $k = 0$ this means that $\beta \in A$, so that (E) is trivially satisfied. If $1 \leqq k \leqq n - 1$, we may assume that L_k is of the form

$$L_k = K + \gamma,$$

where K is the k-dimensional coordinate plane

$$\{\alpha; \alpha_{k+1} = \cdots = \alpha_n = 0\} \quad \text{and} \quad \gamma = (0, \cdots, 0, \gamma_{k+1}, \cdots, \gamma_n).$$

Writing $\alpha = (\alpha', \alpha'')$, $\alpha' \in N^k$, $\alpha'' \in N^{n-k}$ and

$$A' = \{\alpha'; (\alpha', \gamma'') \in A\} \subset N^k,$$

we observe that $\beta = (\beta', \gamma'') \in \operatorname{int}_k (\operatorname{ch} (L_k \cap A))$ means the same as

$$\beta' \in \operatorname{int}_k(\operatorname{ch} (A')) = \operatorname{int} (\operatorname{ch} (A')).$$

Hence by the special case of the theorem we have, writing $D = (D', D'')$,

$$\| D'^{\beta'} v \|_\infty \leqq C \textstyle\sum_{\alpha' \in A'} \| D'^{\alpha'} v \|_\infty \qquad (v \in C_0^\infty (R^k)).$$

Taking $v = D^\gamma u$, $u \in C_0^\infty (R^n)$ and keeping $x_{k+1}, \cdots, x_n$ fixed we obtain (E).

Take $\varphi \in C_0^\infty (R)$ such that $\varphi(y) = \varphi(-y)$, $\varphi(y) = 0$ in a neighborhood of 0, and

$$\int_{-\infty}^{\infty} \varphi(e^{-y})\, dy = 1 . \tag{3.1}$$

For $t = (t_1, \cdots, t_n) \in R^n$ and $\xi \in R^n$, set

$$\Phi_t(\xi) = \textstyle\prod_{i=1}^n \varphi(\xi_i e^{-t_i}).$$

Then in view of (3.1),

$$\int \Phi_t(\xi)\, dt = 1$$

if $\xi_i \neq 0$ for each i. For fixed t_i the function $\varphi(\xi_i e^{-t_i})$ is equal to zero in a neighborhood of $\xi_i = 0$. Hence for any $\alpha \in N^n$, $\beta \in N^n$ and $t \in R^n$, the function $\xi^{\beta-\alpha}\Phi_t(\xi)$ belongs to $C_0^\infty (R^n)$, and hence

$$\xi^{\beta-\alpha}\Phi_t(\xi) \in \hat{L}_1 .$$

We now study the t-dependence of the norm $\| \quad \|_{\hat{L}_1}$ of this function. Since for an arbitrary $\psi(\xi_i) \in \hat{L}_1$ we have $\|\psi(\xi_i e^{-t_i})\|_{\hat{L}_1} = \| \psi(\xi_i) \|_{\hat{L}_1}$, we obtain

$$\begin{aligned} \| \xi^{\beta-\alpha}\Phi_t(\xi) \|_{\hat{L}_1(R^n)} &= \textstyle\prod_{i=1}^n \| \xi_i^{\beta_i-\alpha_i}\varphi(\xi_i e^{-t_i}) \|_{\hat{L}_1(R)} \\ &= C \textstyle\prod_{i=1}^n e^{t_i(\beta_i-\alpha_i)} \\ &= Ce^{\langle t,\beta-\alpha\rangle} . \end{aligned} \tag{3.2}$$

Next we prove that if $\beta \in \operatorname{int} (\operatorname{ch} (A))$, then

$$\int_{R^n} \inf_{\alpha\in A} e^{\langle t,\beta-\alpha\rangle}\, dt < \infty . \tag{3.3}$$

In fact

$$\inf_{\alpha\in A} \exp \langle t, \beta - \alpha\rangle = \exp (-\sup_{\alpha\in A} \langle t, \alpha - \beta\rangle) = \exp (-H_{\tilde{A}}(t)),$$

where $H_{\tilde{A}}(t)$ is the supporting function for the convex set

$$\tilde{A} = \operatorname{ch} (A) - \beta = \operatorname{ch} (\{\alpha - \beta; \alpha \in A\}).$$

But the assumption $\beta \in \operatorname{int}(\operatorname{ch}(A))$ is equivalent to

$$0 \in \operatorname{int}(\operatorname{ch}(\tilde{A})),$$

and hence implies that

$$H_{\tilde{A}}(t) \geqq c|t|$$

for some $c > 0$. This proves (3.3).

Applying Lemma 2 with $\{g_j(t)\}_{j=1}^m = \{e^{\langle t, \beta-\alpha\rangle}\}_{\alpha \in A}$ we conclude that there exist disjoint sets $G_\alpha \subset R^n$ such that

$$\bigcup_{\alpha \in A} G_\alpha = R^n \quad \text{and} \quad \int_{G_\alpha} e^{\langle t, \beta-\alpha\rangle}\, dt < \infty$$

for each $\alpha \in A$. Define $h_\alpha(\xi)$ for $\xi_i \neq 0$ by

$$h_\alpha(\xi) = \int_{G_\alpha} \xi^{\beta-\alpha}\, \Phi_t(\xi)\, dt.$$

According to (3.2) we have

$$\| h_\alpha \|_{\hat{L}_1} \leqq C \int_{G_\alpha} e^{\langle t, \beta-\alpha\rangle}\, dt < \infty,$$

i.e. $h_\alpha \in \hat{L}_1 \subset \hat{M}$. Finally

$$\sum_{\alpha \in A} \xi^\alpha h_\alpha(\xi) = \sum_{\alpha \in A} \int_{G_\alpha} \xi^\beta \Phi_t(\xi)\, dt = \xi^\beta \int_{R^n} \Phi_t(\xi)\, dt = \xi^\beta.$$

Applying Lemma 1 we conclude that (E) holds.

4. Necessity of the condition (G)

In view of Lemma 1 it is enough to prove that the existence of $h_\alpha \in \hat{M}$ such that

$$\xi^\beta = \textstyle\sum_{\alpha \in A} \xi^\alpha h_\alpha(\xi) \tag{4.1}$$

implies (G). The idea of the proof is as follows. It is easy to see that β cannot be an exterior point of ch (A) if (E) holds. To see this one need only take

$$u_\lambda(x) = g(\lambda^{r_1} x_1, \cdots, \lambda^{r_n} x_n),$$

where $g \in C_0^\infty(R^n)$, $g \neq 0$, and the real numbers r_i are chosen such that

$$\langle \beta, r\rangle > \sup_{\alpha \in A} \langle \alpha, r\rangle,$$

and the real parameter $\lambda \to +\infty$. (The same argument applies if one considers, instead of (E), the corresponding inequality for an arbitrary fixed p, $1 \leqq p \leqq \infty$.) Hence it is enough to study the case where β lies on the boundary of ch (A). We will prove that in this case (4.1) implies roughly that ξ^β is linearly dependent on the ξ^α. As an illustration of this point we mention the following example. Assume that for some $h_j \in \hat{M}$

$$\xi_1 \xi_2 = \xi_1^2 \xi_2^2 h_1(\xi) + 1 \cdot h_2(\xi).$$

Replacing ξ_1 by $\lambda\xi_1$ and ξ_2 by $\lambda^{-1}\xi_2$, $\lambda > 0$, we obtain

$$\xi_1 \xi_2 = h_1(\lambda\xi_1, \lambda^{-1}\xi_2)\xi_1^2 \xi_2^2 + h_2(\lambda\xi_1, \lambda^{-1}\xi_2).$$

Since $h_j \in \hat{M}$, the functions $h_j(\lambda\xi_1, \lambda^{-1}\xi_2)$ converge in a certain weak topology (see below) to constant functions as $\lambda \to \infty$ (Lemma 3). This gives a contradiction.

We denote by C_0 the set of continuous functions $R^n \to C$ with compact support. The integral of $\varphi \in C_0$ with respect to the measure μ will be denoted $\langle\mu, \varphi\rangle$. We shall use the following notion of convergence in M. If $\mu_\lambda \in M$ for each $\lambda > 0$, we say that μ_λ tends to $\mu \in M$ as $\lambda \to \infty$ if

$$\lim_{\lambda\to\infty} \langle\mu_\lambda, \varphi\rangle = \langle\mu, \varphi\rangle \quad \text{for every } \varphi \in C_0 .$$

This will be written

$$\mu_\lambda \to \mu \quad \text{as } \lambda \to \infty.$$

We use the same notation for the corresponding convergence of the Fourier transforms, i.e.

$$\hat{\mu}_\lambda \to \hat{\mu} \quad \text{as } \lambda \to \infty.$$

This is actually an abuse of language, but there will be no confusion.

LEMMA 3. *Let $h \in \hat{M}(R^n)$, let $r_1, \cdots, r_q$ $(q \leqq n)$ be real numbers all different from zero, and define the measure μ_λ for $\lambda > 0$ by*

$$\hat{\mu}_\lambda(\xi) = h(\lambda^{r_1}\xi_1, \cdots, \lambda^{r_q}\xi_q, \xi_{q+1}, \cdots, \xi_n).$$

Then

$$\mu_\lambda \to \mu \quad \textit{as } \lambda \to \infty, \tag{4.2}$$

where μ is a measure with support in the hyperplane

$$\{x; x_1 = \cdots x_q = 0\},$$

or equivalently, $\hat{\mu}(\xi)$ depends only on $\xi_{q+1}, \cdots, \xi_n$.

Proof. If $\varphi \in C_0$ we have

$$\langle\mu_\lambda, \varphi\rangle = \langle\mu_1, \varphi_\lambda\rangle \quad \text{where } \varphi_\lambda(x) = \varphi(\lambda^{r_1}x_1, \cdots, \lambda^{r_q}x_q, x_{q+1}, \cdots, x_n).$$

Assume that $r_i > 0$ when $1 \leqq i \leqq k$ and that $r_i < 0$ when $k < i \leqq q$. Let χ be the characteristic function for the origin in R. Then it is easily seen that

$$\lim_{\lambda\to\infty} \varphi_\lambda(x) = \varphi(0, \cdots, 0, x_{q+1}, \cdots, x_n) \textstyle\prod_{i=1}^{k} \chi(x_i) \qquad (x \in R^n).$$

Denote the function in the right member by Ψ. Clearly Ψ is a μ-measurable function for any $\mu \in M$. Since the functions φ_λ are uniformly bounded, it follows from a general form of the Lebesgue Dominated Convergence Theorem that

$$\lim_{\lambda\to\infty} \langle\mu_1, \varphi_\lambda\rangle = \langle\mu_1, \Psi\rangle.$$

This proves (4.2) with μ defined by

$$\langle\mu, \varphi\rangle = \langle\mu_1, \Psi\rangle \quad \text{for } \varphi \in C_0 .$$

If φ is equal to zero in the plane

$$L = \{x; x_1 = \cdots x_q = 0\},$$

then Ψ is identically zero, hence the support of μ is contained in L.

To prove that (E) implies (G) we use induction with respect to the dimension n. If $n = 1$, the statement is easily seen to be true, for in this case ch (A) is an interval $[\alpha_0, \alpha_1]$, where α_0 and α_1 are non-negative integers. In fact, by a remark in the beginning of Section 4 it is sufficient to consider the case where β is a boundary point of ch (A), and this case is of course trivial when $n = 1$.

We now assume that we have proved that (E) implies (G) when the number of variables $x_1, \cdots, x_m$ is less than or equal to $n - 1$. We have to prove the same statement when $m = n$. Thus assume that (E) holds. By Lemma 1 this is equivalent to the existence of $h_\alpha \in \hat{M}$ such that

$$\xi^\beta = \textstyle\sum_{\alpha\in A} \xi^\alpha h_\alpha(\xi). \tag{4.3}$$

Again we may assume that β is a boundary point of ch (A), or equivalently, that there exists a non-zero vector $r = (r_1, \cdots, r_n)$, r_i real numbers, such that

$$\langle\beta, r\rangle = \sup_{\alpha\in A} \langle\alpha, r\rangle.$$

Now replace ξ_i by $\lambda^{r_i}\xi_i$, $\lambda > 0$, in (4.3). Let A_0 be the set of $\alpha \in A$ such that $\langle\alpha, r\rangle = \langle\beta, r\rangle$. Setting

$$h_{\alpha,\lambda}(\xi_1, \cdots, \xi_n) = h_\alpha(\lambda^{r_1}\xi_1, \cdots, \lambda^{r_n}\xi_n)$$

we obtain after division by $\lambda^{\langle\beta,r\rangle}$

$$\xi^\beta = \textstyle\sum_{\alpha\in A_0} \xi^\alpha h_{\alpha,\lambda}(\xi) + \cdots, \tag{4.4}$$

where the omitted terms contain negative powers of λ. We may renumber the coordinates so that $r_i \neq 0$ for $1 \leqq i \leqq q$, and $r_i = 0$ for $q < i \leqq n$. Then by Lemma 3 there exist functions $g_\alpha \in \hat{M}$, which depend only on $\xi_{q+1}, \cdots, \xi_n$, such that

$$h_{\alpha,\lambda}(\xi) \to g_\alpha(\xi) = g_\alpha(\xi_{q+1}, \cdots, \xi_n) \quad \text{as } \lambda \to \infty.$$

Take an arbitrary $\varphi \in C_0^\infty(R^n)$ and multiply (4.4) by $\hat{\varphi}(\xi)$. Since $\xi^\alpha\hat{\varphi}(\xi)$ is the Fourier transform of a function in C_0, we obtain from (4.4) by letting λ tend to infinity

$$\int \xi^\beta\hat{\varphi}(\xi)\, d\xi = \sum_{\alpha\in A_0} \int \xi^\alpha g_\alpha(\xi_{q+1}, \cdots, \xi_n)\hat{\varphi}(\xi)\, d\xi. \tag{4.5}$$

However, since φ is arbitrary, this clearly implies that

$$\xi^\beta = \textstyle\sum_{\alpha\in A_0} \xi^\alpha g_\alpha(\xi_{q+1}, \cdots, \xi_n). \tag{4.6}$$

Now, since the monomials $\xi_1^{\alpha_1} \cdots \xi_q^{\alpha_q}$ are linearly independent over the ring of continuous functions in $\xi_{q+1}, \cdots, \xi_n$, it follows from (4.6) that there exists a subset $A_0' \subset A_0$ such that

$$(\beta_1, \cdots, \beta_q) = (\alpha_1, \cdots, \alpha_q) \quad \text{for each } \alpha \in A_0',$$

and that

$$\xi_{q+1}^{\beta_{q+1}} \cdots \xi_n^{\beta_n} = \sum_{\alpha \in A_0'} \xi_{q+1}^{\alpha_{q+1}} \cdots \xi_n^{\alpha_n} g_\alpha(\xi_{q+1}, \cdots, \xi_n).$$

But by the induction assumption this implies that

$$(\beta_{q+1}, \cdots, \beta_n) \quad \text{and} \quad \{(\alpha_{q+1}, \cdots \alpha_n); \alpha \in A_0'\}$$

satisfy (G) with $n - q$ instead of n (note that $q \geqq 1$). It is obvious that this implies that β and A_0'—and hence β and A—satisfy (G).

5. The case $1 < p < \infty$

In this section we will prove the following result, which seems to have been first proved by Il'in [4].

THEOREM 2. *Let $\beta \in N^n$, let A be a finite subset of N^n, and let $1 < p < \infty$. There exists a constant C (depending on p) such that*

$$\| D^\beta u \|_p \leqq C \sum_{\alpha \in A} \| D^\alpha u \|_p \qquad (u \in C_0^\infty(R^n)) \tag{5.1}$$

if and only if $\beta \in \operatorname{ch}(A)$.

That the condition $\beta \in \operatorname{ch}(A)$ is necessary for (5.1) to hold has already been proved twice—see the beginning of Section 4 and the introduction.

The most essential part of the proof of the sufficiency of the condition $\beta \in \operatorname{ch}(A)$ is contained in the estimate (5.3) of Lizorkin. A more general theorem containing the result of Lizorkin has been proved by other methods by Littman, McCarthy, and Rivière [6]. These results are all closely connected with the well-known multiplier theorem of Mihlin (see e.g. [3, Theorem 2.5]); they can actually be considered as extensions of the latter theorem.

To formulate Lizorkin's theorem in a convenient way one must use the language of distribution theory. As customary we denote by $\mathcal{S}'(R^n)$ the class of tempered distributions in R^n as defined by Schwartz in [13].

THEOREM 3 (Lizorkin [8]). *Let T be a distribution in $\mathcal{S}'(R^n)$ whose Fourier transform $\hat{T}(\xi)$ is a bounded function which is n times continuously differentiable in the set*

$$G = \{\xi = (\xi_1, \cdots, \xi_n); \xi_i \neq 0 \text{ for all } i\}.$$

Assume that

$$| D^\gamma \hat{T}(\xi) | \leqq C | \xi_1 |^{-\gamma_1} \cdots | \xi_n |^{-\gamma_n} \tag{5.2}$$

for all $\xi \in G$ and for all $\gamma = (\gamma_1, \cdots, \gamma_n)$ such that $\gamma_i = 0$ or 1 for all i. Then for each p, $1 < p < \infty$, there exists a constant C_p such that

$$\| T * u \|_p \leqq C_p \| u \|_p \qquad (u \in C_0^\infty(R^n)). \tag{5.3}$$

The set of Fourier transforms (in the distribution sense) of distributions

$T \in \mathcal{S}'(R^n)$ satisfying (5.3) for a given p is generally denoted M_p. It is known (see [3, p. 100]) that $M_p = M_q$ if $p^{-1} + q^{-1} = 1$ and that $M_p \subset M_r$ if $1 \leqq p \leqq r \leqq 2$, and furthermore that $M_1 = M_\infty = \hat{M}$ (= the space of Fourier transforms of measures with finite total mass), and that $M_2 = L_\infty$. It follows that for $1 \leqq p \leqq \infty$ we have $M_p \subset L_\infty$.

For $\theta \in N^n$, set

$$g_\theta(\xi) = \xi^{2\theta} / \textstyle\sum_{\alpha \in A} \xi^{2\alpha}. \tag{5.4}$$

Then

$$\xi^\beta = \textstyle\sum_{\alpha \in A} \xi^\alpha g_{(\alpha+\beta)/2}(\xi).$$

In order to prove Theorem 2 we need only show that $g_{(\alpha+\beta)/2} \in M_p$ whenever $\alpha \in A$ and $\beta \in \text{ch}(A)$, or more generally, that $g_\theta \in M_p$ whenever $\theta \in \text{ch}(A)$.

It is clear that g_θ is infinitely differentiable in all of G.

LEMMA 4. *Assume that* $\theta \in \text{ch}(A)$ *and that* $g(\xi) = g_\theta(\xi)$ *is defined by* (5.4). *Then for each* $\gamma \in N^n$ *there exists a constant* C_γ *such that*

$$|D^\gamma g(\xi)| \leqq C_\gamma |\xi_1|^{-\gamma_1} \cdots |\xi_n|^{-\gamma_n} \qquad (\xi \in G). \tag{5.5}$$

Proof. We first prove (5.5) for $|\gamma| = 0$, i.e. that $g(\xi)$ is bounded. Since $\theta \in \text{ch}(A)$ there exist non-negative real numbers t_α such that $\sum t_\alpha = 1$ and

$$\textstyle\sum_{\alpha \in A} t_\alpha \cdot \alpha = \theta.$$

By the arithmetic-geometric inequality we then obtain

$$|\xi^{2\theta}| = \textstyle\prod_{\alpha \in A} |\xi^{2\alpha}|^{t_\alpha} \leqq \sum_{\alpha \in A} t_\alpha |\xi^{2\alpha}| \leqq \sum_{\alpha \in A} |\xi^{2\alpha}|.$$

This proves that $g(\xi)$ is bounded.

Next we estimate the derivatives of $g(\xi)$. Set

$$F(\xi) = \textstyle\sum_{\alpha \in A} \xi^{2\alpha}.$$

Using the trivial estimate

$$|D_i \xi^{2\beta}| = 2\beta_i |\xi_i|^{-1} |\xi^{2\beta}| \tag{5.6}$$

we get for $k > 0$

$$\begin{aligned} \left| D_i \frac{1}{F(\xi)^k} \right| &\leqq \frac{k \sum_{\alpha \in A} |D_i \xi^{2\alpha}|}{F(\xi)^{k+1}} \\ &\leqq C \frac{\sum_{\alpha \in A} |\xi_i|^{-1} |\xi^{2\alpha}|}{F(\xi)^{k+1}} \leqq C |\xi_i|^{-1} \cdot \frac{1}{F(\xi)^k}. \end{aligned} \tag{5.7}$$

$D^\gamma g(\xi)$ can be written as a sum of terms of the form

$$\xi^\sigma / (\textstyle\sum_{\alpha \in A} \xi^{2\alpha})^k,$$

where $1 \leqq k \leqq |\gamma| + 1$. We will prove by induction over γ that each of those terms can be estimated by the right hand side of (5.2). For $k = 1$

and $|\gamma| = 0$ this has already been proved. From (5.6) and (5.7) we get

$$|D_i(\xi^\sigma F(\xi)^{-k})| \leqq |(D_i\xi^\sigma)F(\xi)^{-k}| + |\xi^\sigma(D_iF(\xi)^{-k})|$$
$$\leqq C|\xi_i|^{-1}|\xi^\sigma|F(\xi)^{-k}.$$

Combining this estimate with the induction assumption we obtain the desired estimate.

By combining Theorem 3 and Lemma 4 we see that $g_\theta \in M_p$ for each $\theta \in \mathrm{ch}(A)$ and $1 < p < \infty$. The proof of Theorem 2 is complete.

6. The case P_1 elliptic and $P_2 = 1$

In this section we use the methods of Section 3 to give a short proof of a result of de Leeuw and Mirkil [5].

THEOREM 4. *Assume that P is elliptic and that the degree of Q is strictly lower than that of P. Then for each p, $1 \leqq p \leqq \infty$,*

$$\|Q(D)u\|_p \leqq C(\|P(D)u\|_p + \|u\|_p) \qquad (u \in C_0^\infty(R^n)). \tag{6.1}$$

In the same article de Leeuw and Mirkil showed that if P is elliptic and Q and P have the same degree, then the homogeneous parts of highest degree in Q and P must be proportional in order that (6.1) be valid for $p = \infty$. This together with Theorem 4 easily gives a characterization of the operators Q satisfying (6.1) for a given elliptic operator P and $p = \infty$.

Proof of Theorem 4. Write $P = P_0 + P_1$, where P_0 is the homogeneous part of highest degree d, and the degree of P_1 is $< d$. The fact that P is elliptic means that $P_0(\xi) \neq 0$ for each non-zero real ξ. We will prove that for any $\varepsilon > 0$ there exists a constant C_ε such that

$$\|Q(D)u\|_p \leqq \varepsilon\|P_0(D)u\|_p + C_\varepsilon\|u\|p \qquad (u \in C_0^\infty(R^n)). \tag{6.2}$$

Using (6.2) we easily prove (6.1) as follows. Writing $P_0 = P - P_1$ and using Minkowski's inequality we obtain

$$\|Q(D)u\|_p \leqq \varepsilon\|P(D)u\|_p + \varepsilon\|P_1(D)u\|_p + C_\varepsilon\|u\|_p. \tag{6.3}$$

The same inequality for $P_1(D)u$ instead of $Q(D)u$ can be written

$$(1-\varepsilon)\|P_1(D)u\|_p \leqq \varepsilon\|P(D)u\|_p + C_\varepsilon\|u\|_p. \tag{6.4}$$

If $\varepsilon < 1$, the result follows from (6.3) and (6.4).

We will prove (6.2) by constructing h_1 and h_2 in $\hat{L}_1$ such that

$$Q(\xi) = P_0(\xi)h_1(\xi) + 1\cdot h_2(\xi). \tag{6.5}$$

We may clearly assume that Q is a homogeneous polynomial of degree $e > 0$. Take an infinitely differentiable function φ defined for $t \geqq 0$ such that

$\varphi(t) = 0$ for $t > A$ and for $0 \leqq t < a$, $a > 0$, and such that

$$\int_0^\infty \varphi\left(\frac{1}{t}\right)\frac{dt}{t} = 1.$$

Set

$$h_1(\xi) = \int_b^\infty \frac{Q(\xi)}{P_0(\xi)}\varphi\left(\frac{|\xi|}{t}\right)\frac{dt}{t} \quad \text{and} \quad h_2(\xi) = \int_0^b Q(\xi)\varphi\left(\frac{|\xi|}{t}\right)\frac{dt}{t}.$$

Then (6.5) holds and $h_j \in \hat{L}_1$ since for each $t > 0$,

$$\frac{Q(\xi)}{P_0(\xi)}\varphi\left(\frac{|\xi|}{t}\right) \in C_0^\infty(R^n) \subset \hat{L}_1,$$

and

$$\left\|\frac{Q(\xi)}{P_0(\xi)}\varphi\left(\frac{|\xi|}{t}\right)\right\|_{\hat{L}_1} = t^{e-d}\left\|\frac{Q(\xi)}{P_0(\xi)}\varphi(|\xi|)\right\|_{\hat{L}_1} = C_1 t^{e-d},$$

and

$$\left\|Q(\xi)\varphi\left(\frac{|\xi|}{t}\right)\right\|_{\hat{L}_1} = t^e \,\| Q(\xi)\varphi(|\xi|)\|_{\hat{L}_1} = C_2 t^e.$$

Since $0 < e < d$ we conclude that $\| h_j \|_{\hat{L}_1}$ is finite for $j = 1, 2$. By choosing b large enough we can make $\| h_1 \|_{\hat{L}_1}$ as small as we please. This completes the proof.

It seems worth pointing out that the same method can be used to prove that a given function is an $\hat{L}_1$-transform in a somewhat more general situation, as follows.

THEOREM 5. *Assume that $k(\xi)$ is an infinitely differentiable function defined in R^n, and that $k(\xi)$ is positive-homogeneous of degree $r < 0$ (r real) outside some compact set. Then $k \in \hat{L}_1$.*

Proof. The assumption means that there exists an A such that

$$k(t\xi) = t^r k(\xi) \quad \text{for } \xi \in R^n, \quad |\xi| > A, \quad t > 1. \tag{6.6}$$

Let φ denote the same function as in the proof of Theorem 4, and set

$$g(\xi) = \int_b^\infty k(\xi)\varphi\left(\frac{|\xi|}{t}\right)\frac{dt}{t}.$$

If b is large enough, $k(\xi)$ satisfies (6.6) for all ξ such that the integrand is different from zero, and hence the proof of Theorem 4 shows that $g \in \hat{L}_1$. Since $k - g \in C_0^\infty$, it follows that $k \in \hat{L}_1$.

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