ON THE BEHAVIOUR OF LINEAR MAPPINGS ON ABSOLUTELY CONVEX SETS AND A. GROTHENDIECK'S COMPLETION OF LOCALLY CONVEX SPACES

BY

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The first point of this note is a simple proof of Theorem 2 which states a continuity property of linear maps with respect to absolutely convex sets and is essentially due to D. J. H. Garling [4]. In Theorem 4 the approximation theorem 16.8 of Kelley-Namioka's book [7] is sharpened by dropping the closedness assumption on the absolutely convex set on which the linear forms are to be approximated. This result is used in Theorem 5 to extend Grothendieck's well-known discussion of the completion and the completeness of the dual X' of a locally convex space X by admitting on X' topologies $\mathfrak{T}_{\mathfrak{N}}$ of uniform convergence on classes $\mathfrak{N}$ of absolutely convex subsets of X whose members need not be bounded nor closed. Finally, V. Ptak's and H. S. Collin's characterization of the completeness of $(X', \mathfrak{T}_{\mathfrak{N}})$ is carried over to these (generally not linear) topologies $\mathfrak{T}_{\mathfrak{N}}$.

The following theorem can be easily deduced from Theorem 1 of Garling [4].

THEOREM 1. *Let X and Y be locally convex spaces, f a linear mapping from X into Y, $\mathfrak{N}$ a collection of absolutely convex subsets of X, directed upwards by inclusion, whose union is absorbent. Let f be continuous on each $A \in \mathfrak{N}$. Then f is also continuous on the closure $\bar{A}$ of each $A \in \mathfrak{N}$.*

We wish to give a direct proof for the following special case of Theorem 1 to be applied later.

THEOREM 2. *Let X and Y be locally convex spaces, f a linear map from X into Y, and A an absolutely convex and absorbent subset of X such that $f \mid A$ is continuous. Then $f \mid \bar{A}$ is also continuous.*

Proof. According to a lemma of A. Grothendieck [6, p. 98] it suffices to show that $f \mid \bar{A}$ is continuous at 0. Let V be a neighbourhood of 0 in Y. Then there is an open neighbourhood U of 0 in X such that

$$f(U \cap A) \subset V. \tag{1}$$

The theorem will be proved if we show

$$f(U \cap \bar{A}) \subset V + V. \tag{2}$$

If $x \in U \cap \bar{A}$ there exists a real number ρ, $0 < \rho < 1$, such that $\rho x \in A$. Because of the continuity of $f \mid A$ at ρx there exists $y \in U \cap A$ so close to x that

$$f(\rho x) - f(\rho y) \in \rho V. \tag{3}$$

Received May 3, 1971.

(Note $U = U^\circ$ and $\rho A \subset A$). Therefore, using (1), we obtain

$$f(x) = \rho^{-1}(f(\rho x) - f(\rho y)) + f(y) \in V + V$$

which proves (2).

We mention the following variant of Theorem 2.

THEOREM 3. *Let X and Y be linear topological spaces, f a linear map from X into Y, $A \subset X$, $0 \in A$, and for every $x \in X$ let there exist a scalar $\rho \neq 0$ such that $\rho x - \rho A \subset A$. Then, if $f \mid A$ is continuous at 0, $f \mid \bar{A}$ is also continuous at 0.*

The proof is similar to that of Theorem 2; replace (3) by $f(\rho x - \rho y) \in \rho V$.

Theorems 2 and 3 have analogues for semi-norms instead of linear maps.

If, in Theorem 2, $\bar{A}$ is a neighbourhood of 0, then f is continuous on all of X. If $\bar{A}$ is not a neighbourhood of 0, it may still be true that continuity of $f \mid \bar{A}$ implies continuity of f without a restriction on Y. Namely, the following example shows that a locally convex space X with the Mackey topology may contain an absolutely convex, closed, compact subset A which is not a neighbourhood of 0 and such that any linear map from X into any locally convex space Y is continuous if it is continuous on A.

Let $(X, \mathfrak{T})$ be the Mackey dual of a non-reflexive Banach space Z, and let A be the closed unit ball of X. Then A is absolutely convex, closed, and weakly compact, but not a neighbourhood of 0. Let now f be a linear map of X into a locally convex space Y such that $f \mid A$ is continuous. We show that f is continuous. By Garling [4, p. 2] the continuity of $f \mid A$ means that f is continuous for the finest locally convex topology $\mathfrak{S}$ on X which agrees on A with $\mathfrak{T}$. ($\mathfrak{S}$ may also be described as the mixed topology $\gamma[\mathfrak{R}, \mathfrak{T}]$ determined by the strong topology $\mathfrak{R} = \beta(X, Z)$ and the Mackey topology $\mathfrak{T}$—see A. Wiweger [11, 2.1–2.2].) It is therefore sufficient to show $\mathfrak{S} = \mathfrak{T}$. The dual of $(X, \mathfrak{S})$ consists of all linear forms on X which are $\mathfrak{T}$-continuous on A. It may therefore be identified with the completion of Z with respect to the topology of uniform convergence on A (by Grothendieck's completeness theorem). Z being already complete in this topology, we obtain $(X, \mathfrak{S})' = Z$, i.e., $\mathfrak{S} \subset \mathfrak{T}$, and therefore $\mathfrak{S} = \mathfrak{T}$ which completes the proof of the continuity of f. —If we take in particular $Z = l^1$, then A, the closed unit ball of l^∞, is not only weakly compact but compact in the Mackey topology (combine Köthe [8, §21, 7.(1)] with §22, 4.(3)) which completes the desired example.

The next result is an approximation theorem.

THEOREM 4. *Let $(X, \mathfrak{T})$ be a locally convex space, $A \subset X$ absolutely convex, and v a linear form on X which is continuous on A. Then for every real number $\varepsilon > 0$, there is an element u in X' such that*

$$(4) \qquad | v(x) - u(x) | \leq \epsilon \quad \text{for all } x \in \bar{A} \cap [A]$$

where $[A]$ denotes the linear span of A.

Remarks. 1. After Grothendieck [6, p. 99, Exercise 1, (b)] the $\mathfrak{T}$-continuity of $v \mid A$ is equivalent to the weak continuity of $v \mid A$.

2. In Kelley-Namioka [7, Theorem 16.8] the approximation theorem is stated under the stronger hypothesis that A be absolutely convex and closed and $\mathfrak{T} = \sigma(X, X')$. The proof given there covers also the case that $\mathfrak{T}$ is not equal to $\sigma(X, X')$.

3. Theorem 4 implies immediately a sharpened form of the approximation theorem 5 of J. I. Nieto [9].

Proof of Theorem 4. After Grothendieck (see Remark 1 above) $v \mid A$ is weakly continuous. Without loss of generality, A may be assumed absorbent. Then, by Theorem 2, also $v \mid \bar{A}$ is weakly continuous so that the theorem is reduced to the version of Kelley-Namioka.

The following proof is more direct, starting with a variation of Kelley-Namioka's proof.

By assumption, for every $\varepsilon > 0$ there is an absolutely convex open neighbourhood U of 0 in X such that $|v|$ is less than or equal to ε on $U \cap A$, i.e., $\varepsilon^{-1}v$ lies in $(U \cap A)^\circ$ where the polar, as in the rest of the proof, is to be taken with respect to the dual pair $\langle X, X^*\rangle$, X^* denoting the algebraic dual of X. As U is open, we have $U \cap \bar{A} \subset \overline{U \cap A}$, whence

$$\varepsilon^{-1}v \in (U \cap A)^\circ = (\overline{U \cap A})^\circ \subset (U \cap \bar{A})^\circ.$$

Since $\frac{1}{2}\bar{U} \subset U$ it follows that

$$\varepsilon^{-1}v \in (\tfrac{1}{2}\bar{U} \cap \bar{A})^\circ = (2U^\circ \cup A^\circ)^{\circ\circ} \subset \overline{2U^\circ + A^\circ}^{\sigma(X^*, X)}.$$

U° being $\sigma(X^*, X)$-compact, $2U^\circ + A^\circ$ is already closed, and we obtain $v \in 2\varepsilon U^\circ + \varepsilon A^\circ$. As U was a neighbourhood of 0 in X, we have $U^\circ \subset X'$. Hence there is $u \in X'$ such that $v - u \in \varepsilon A^\circ$ which means

$$(5) \qquad |v(x) - u(x)| \leq \varepsilon \quad \text{for all } x \in A.$$

Therefore v is the limit in X^* of a Cauchy filter $\mathfrak{F}$ on X' with respect to the uniformity of uniform convergence on A. As each member of $\mathfrak{F}$ consists of *continuous* linear forms, $\mathfrak{F}$ is a Cauchy filter even with respect to uniform convergence on $\bar{A}$. As $\mathfrak{F}$ converges on $[A]$ pointwise to v, it follows that v is continuous on $\bar{A} \cap [A]$, so that (5) yields the contention (4).

We now turn to an extension of well-known results by A. Grothendieck [5], V. Ptak [10], and H. S. Collins [3] on the completion and the completeness of locally convex spaces as presented in G. Köthe [8, §21, 9]. If $\langle X, Y\rangle$ is a dual pair and $\mathfrak{N}$ a collection of absolutely convex subsets of X, it is well known that on Y the topology $\mathfrak{T}_{\mathfrak{N}}$ of $\mathfrak{N}$-convergence (topology of uniform convergence on the members of $\mathfrak{N}$) is compatible with the group structure of Y as an additive group, whereas $\mathfrak{T}_{\mathfrak{N}}$ is compatible with the linear-space structure of Y if and only if the members of $\mathfrak{N}$ are weakly bounded (N. Bourbaki [2, §3, 1.]) for which reason this condition is customarily imposed on $\mathfrak{N}$. We shall see however that an essential part of the results on completeness referred to above remains true without this condition on $\mathfrak{N}$.

THEOREM 5. *Let X be a locally convex space and $\mathfrak{N}$ a collection of absolutely convex subsets of X, directed upwards by inclusion. Let Z denote the linear space of linear forms on X which are continuous*[1]) *on each $A \in \mathfrak{N}$, and let Z carry the group topology $\mathfrak{T}_{\mathfrak{N}}$ of $\mathfrak{N}$-convergence. Then*

1. *Z is complete, and X' is dense in Z.*
2. *X' is $\mathfrak{T}_{\mathfrak{N}}$-complete if and only if each $v \in Z$ is continuous on the linear space $L = \bigcup_{A \in \mathfrak{N}} [A]$.*

Proof. 1. The completeness of Z is obvious. That X' is dense in Z follows from Theorem 4, taking into account the directedness of $\mathfrak{N}$.

2. If X' is $\mathfrak{T}_{\mathfrak{N}}$-complete and $v \in Z$ there is, by 1, a filter $\mathfrak{F}$ on X' with $\mathfrak{T}_{\mathfrak{N}}$-limit v. But because of the $\mathfrak{T}_{\mathfrak{N}}$-completeness of X', $\mathfrak{F}$ has also a limit u in X'. Clearly, u and v agree on L. Hence v is continuous on L. —For the converse let now each $v \in Z$ be continuous on L and let $\mathfrak{F}$ be a $\mathfrak{T}_{\mathfrak{N}}$-Cauchy filter on X'. By 1, $\mathfrak{F}$ has a limit v in Z. By assumption, v is continuous on L, and each extension $u \in X'$ of $v \mid L$ is also a $\mathfrak{T}_{\mathfrak{N}}$-limit of $\mathfrak{F}$. Therefore X' is complete.

Remarks. 1. The linear space Z of Theorem 5 is the dual of X for the finest locally convex topology $\mathfrak{S}$ on X agrees on each $A \in \mathfrak{N}$ with the original topology of X; for this topology $\mathfrak{S}$ is the 'generalized inductive-limit topology' belonging to the canonical injections of the sets $A \in \mathfrak{N}$ into X in the sense of Garling [4], (cf. in particular p. 2).

2. In general, Z with its topology $\mathfrak{T}_{\mathfrak{N}}$, is no linear topological space. However, not only the addition $(z_1, z_2) \to z_1 + z_2$ is continuous, but also, for each scalar λ, the multiplication $z \to \lambda z$. $\mathfrak{N}$ being directed, the sets of the form

$$W_{A,\varepsilon} = \{v \in Z; \mid v(x) \mid \leq \varepsilon \text{ for all } x \in A\}$$

with $A \in \mathfrak{N}$ and $\varepsilon > 0$ constitute a $\mathfrak{T}_{\mathfrak{N}}$-neighbourhood base of 0 in Z. These sets are absolutely convex and $\sigma(Z, L)$-closed. Also we note

$$(\text{cl}\,\{0\})^{\mathfrak{T}_{\mathfrak{N}}} = \text{cl}\,\{0\})^{\sigma(Z,L)} = \{v \in X^*; v \mid L = 0\}. \tag{6}$$

3. Because of Theorem 5, part 1, the separated completion of the additive group X' with respect to $\mathfrak{T}_{\mathfrak{N}}$ (Bourbaki [1, chap. III, §3, 4.]) may be identified with $Z/L^{\perp}$, $L^{\perp}$ denoting the set in (6). If L is equal to X (which means that X is spanned by the union of the members of $\mathfrak{N}$) then Z is separated and Z itself may be considered as the $\mathfrak{T}_{\mathfrak{N}}$-completion of X'.

4. In the situation of Theorem 5, let $\mathfrak{N}_1$ be another collection of absolutely convex subsets of X, directed upwards, and suppose $L_1 = \bigcup_{A \in \mathfrak{N}_1} [A]$ equal to L. Then the topology $\mathfrak{T}_{\mathfrak{N}_1} \mid X'$ induced by $\mathfrak{T}_{\mathfrak{N}_1}$ on X' has a neighbourhood base of 0 which is $\sigma(X', L)$-closed and therefore $\mathfrak{T}_{\mathfrak{N}}$-closed. Consequently

[1] Cf. Remark 1 to Theorem 4.

there are similarly related neighbourhood bases of 0 for the associated separated spaces (quotients mod $L^{\perp} \cap X'$). If $\mathfrak{T}_{\mathfrak{N}_1} \mid X'$ is finer than $\mathfrak{T}_{\mathfrak{N}} \mid X'$, the last remark implies certain completeness properties[2]) of which we mention only that the separated $\mathfrak{T}_{\mathfrak{N}_1}$-completion of X' may be imbedded in the $\mathfrak{T}_{\mathfrak{N}}$-completion of X'. This can be made more precise as follows. The condition that $\mathfrak{T}_{\mathfrak{N}_1} \mid X'$ is finer than $\mathfrak{T}_{\mathfrak{N}} \mid X'$ implies $Z_1 \subset Z$, as can easily be seen, and this implies the relation $Z_1/L^{\perp} \subset Z/L^{\perp}$ between the separated $\mathfrak{T}_{\mathfrak{N}_1}$- and $\mathfrak{T}_{\mathfrak{N}}$-completions.

5. On X' the topology $\mathfrak{T}_{\mathfrak{N}}$ of $\mathfrak{N}$-convergence is equal to the topology of $\hat{\mathfrak{N}}$-convergence where $\hat{\mathfrak{N}} = \{\bar{A};\ A \in \mathfrak{N}\}$. Therefore the corresponding separated completions of X' are also the same, so that for their discussion it is no essential restriction to assume that the sets $A \in \mathfrak{N}$ are closed.

From Theorem 5 we obtain the following generalization of results of Ptak [10] and Collins [3] as presented in Köthe [8, §21, 9].

THEOREM 6. *Let $(X, \mathfrak{T})$ be a locally convex space and $\mathfrak{N}$ a collection of absolutely convex subsets of X, directed upwards by inclusion, and such that $\bigcup_{A \in \mathfrak{N}} A$ spans X. Let $\mathfrak{R}$ (respectively $\mathfrak{S}$) denote the finest general (respectively the finest locally convex) topology on X that agrees with $\mathfrak{T}$ on the sets $A \in \mathfrak{N}$. Then the following are true.*

1. *X' is complete in the topology $\mathfrak{T}_{\mathfrak{N}}$ of $\mathfrak{N}$-convergence, if and only if every $\mathfrak{R}$-closed linear hyperplane H in X is $\mathfrak{T}$-closed, i.e.*[3]), *if and only if the fact that that $H \cap A$ is $\mathfrak{T}$-closed in A for each $A \in \mathfrak{N}$ implies that H is $\mathfrak{T}$-closed.*
2. *A linear hyperplane H in X is $\mathfrak{R}$-closed if and only if it is $\mathfrak{S}$-closed.*

Proof. By Remark 3 after Theorem 5 we may identify the $\mathfrak{T}_{\mathfrak{N}}$-completion of X' with Z of Theorem 5. Completeness of X' then means $X' = Z$. A hyperplane $H = v^{-1}(0)$ in X, where $v \in X^*$, is $\mathfrak{T}$-closed if and only if $v \in X'$. Furthermore, H is $\mathfrak{R}$-closed if and only if $H \cap A$ is $\mathfrak{T}$-closed in A for each $A \in \mathfrak{N}$, i.e. (by Kelley-Namioka [7, Theorem 13.5 (III)]) if and only if v is $\mathfrak{T}$-continuous on each $A \in \mathfrak{N}$, which means $v \in Z$. From this follows contention 1 of the theorem. Part 2 follows from 1.

If in Theorem 6, $\mathfrak{N}$ is not only directed upwards, but if for every two sets $A, B \in \mathfrak{N}$ there is $C \in \mathfrak{N}$ such that $A + B \subset C$, then Theorem 6 may also be proved on the lines of Köthe [8, §21, 9.(6)], and "linear hyperplane" may be replaced by "affine hyperplane" in that theorem. For this, the following three auxiliary statements have to be proved (in analogy to Köthe [8, §21, 9.(1)]).

1. $\mathfrak{R}$ is invariant under translations and multiplication by nonzero scalars.
2. $\mathfrak{R}$ has a basis of circled absorbing neighbourhoods of 0.

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[2] See Bourbaki [1, Chapter III, 3, 5., Proposition 9 and corollaries], as well as Grothendieck [5, Corollary 2] (where the assumption $E_0(S) = E_0(T)$ is missing).

[3] Since it can readily be seen that the R-closed sets are exactly the sets whose intersections with the sets $A \in N$ are T-closed.

3. The absolutely convex $\mathfrak{R}$-neighbourhoods of 0 form an $\mathfrak{S}$-neighbourhood base of 0.

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