TOPOLOGIES IN LOCALLY COMPACT GROUPS II

BY

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Introduction

In this paper we study the partially ordered set of locally compact group topologies on a given abelian group. Our main interest is the cardinality of a given interval $[a, b]$ in this set. We prove that $|\,[a, b]\,| \geq c$ or is finite. This generalises the results obtained in [4] and [5] and also answers a question raised in [5]. Our methods involve delicate ways of embedding R^n in a compact group. These embedding theorems are given in Section 1. We have to study a relation $\sim$ in the set of subgroups of a given torsion free abelian group. This notion resembles that of quasi-isomorphism used by Beaumont and Pierce [1]. This is done in Section 2. We make heavy use of the results proved in [4] and [5].

Notation. All groups considered in this paper are abelian. All topological spaces considered are Hausdorff. The notions and terminologies on topological groups are as in [3] in general. T denotes the circle group with usual topology and multiplication. If G is a topological group, we say G is T-free if T is not a topological summand of G. Similarly G is said to be Z-free if G does not have Z as an algebraic summand. If G is a group and $a_1, a_2, \cdots\cdots, a_n$ are elements of G then $[a_1, \cdots, a_n]$ denotes the sub-group generated by $a_1, a_2, \cdots, a_n$ in G. Isomorphism (topological) of two groups (topological) G_1, G_2 is denoted by $\approx$.

Isomorphic embeddings of R^n into a compact abelian group G

LEMMA 1.1. *Let H be a subgroup of $R^n (n \geq 1)$. Let $\bar{H} = F \oplus V$ where V is a subspace of R^n and F is a free group different from $\{0\}$. Then Z is a summand of H.*

Proof. Follows from standard arguments and the structure of closed subgroups of R^n.

LEMMA 1.2. *Let $\hat{G}$ be a torsion free group of rank n $(n \geq 1)$. Then $\hat{G}$ can be embedded as a dense subgroup of R^n by a group isomorphism if and only if $\hat{G}$ is Z-free. When $\hat{G}$ is Z-free, we can obtain such an embedding as follows: Choose a maximal independent set $(a_1, \cdots, a_n)$ in $\hat{G}$ over the integers. Define a map*

$$\phi_0 : \{a_1, \cdots, a_n\} \rightarrow R^n$$

arbitrarily except that the set $\{\phi_0(a_1), \cdots, \phi_0(a_n)\}$ generates R^n over R. Using

Received December 28, 1970.

divisibility of R^n extend ϕ_0 to a group isomorphism $\hat{\phi}$ of $\hat{G}$ into R^n. Then $\hat{\phi}$ is one such isomorphism.

Proof. Follows from Lemma 1.1 and suitable choice of independent sets in $\hat{G}$.

COROLLARY 1.3. *Let G be a connected compact group such that*

$$\text{rank } \hat{G} = n \ (\geq 1).$$

Then there exists a continuous dense isomorphism ϕ of R^n into G if and only if G is T-free.

Proof. Now there exists a continuous dense isomorphism $\phi : R^n \rightarrow G$ of R^n into G if and only if there exists a dense group isomorphism $\hat{\phi}$ from $\hat{G}$ into R^n. So the result follows from Lemma 1.2.

LEMMA 1.4. *Let n be an integer ≥ 1. Then a free group $\hat{G}$ of rank $n + 1$ can be densely embedded in R^n by a group isomorphism. Consequently a torsion free group of countably infinite rank or of rank $k \geq n + 1$ can be embedded densely in R^n by a group isomorphism.*

Proof. Let $\{a_1, a_2, \cdots, a_{n+1}\}$ be a set of generators of $\hat{G}$. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis for R^n over R. Let $\hat{\phi}(a_i) = e_i$ for $i = 1, 2, \cdots, n$, and $\hat{\phi}(a_{n+1}) = \sqrt{2}e_1 + \sqrt{3}e_2 + \cdots + \sqrt{p_{n-1}}\, e_{n-1} + \sqrt{p_n}\, e_n$, where p_i is the i^{th} prime. Then an application of Theorem 5.1.3 of [6] gives the required result.

LEMMA 1.5. *Let G be a compact group. Let H be a closed subgroup of G. Let $\theta : R^n \rightarrow G/H$ be a continuous isomorphism from R^n into G/H. Let $\phi : G \rightarrow G/H$ be the natural map. Then there exists a continuous isomorphism ψ from R^n into G such that $\theta = \phi \circ \psi$.*

Proof. By duality, there is an algebraic homomorphism $\hat{\theta}$ from $(G/H)\hat{}$ into R^n which is the adjoint of θ. Now $(G/H)\hat{}$ is a subgroup of $\hat{G}$ and R^n is divisible. So there exists an extension $\tilde{\theta}$ of $\hat{\theta}$ which is a homomorphism of $\hat{G}$ into R^n. Then the adjoint ψ of this map from R^n into G is the required map.

LEMMA 1.6. *Let G be a compact connected group. Let $\hat{G}$ be its dual. Let $\hat{G}$ be of rank $n + 1$. Let $\hat{G}$ be Z-free. Let $\phi : R^n \rightarrow G$ be a continuous isomorphism of R^n into G. Then there exists a continuous isomorphism θ of R^{n+1} into G such that $\theta(R^{n+1}) \supset \phi(R^n)$. Moreover if ψ_1 is any other continuous isomorphism of R^{n+1} into G then $\psi_1(R^{n+1}) = \theta(R^{n+1})$.*

Proof. Let us treat R^n as a subspace of R^{n+1}. We take $e_1, e_2, \cdots, e_n, e_{n+1}$ to be the coordinate vectors in R^{n+1} so that the first n of them are in R^n and span R^n over the reals and e_{n+1} is a unit vector not lying in R^n. We treat ϕ as a map from the subspace R^n into G. Consider now the dual of R^{n+1} which is again R^{n+1}. In this dual we take unit vectors $f_1, f_2, \cdots, f_n, f_{n+1}$ so that f_{n+1} is orthogonal to $e_1, e_2, \cdots, e_n$ and $f_1, f_2, \cdots, f_n$ are orthogonal to e_{n+1} and $f_1, f_2, \cdots, f_{n+1}$ span the dual space R^{n+1}. Then the vector space spanned by $f_1, f_2, \cdots, f_n$ over R in the dual space R^{n+1} can be treated as the dual of the vector space R^n generated by $e_1, e_2, \cdots, e_n$. Then the adjoint map $\hat{\phi}$ of ϕ can be treated as a map from $\hat{G}$ into the vector space spanned by $f_1, \cdots, f_n$, in the dual R^{n+1} space. Let us call this subspace V. Let $a_1, a_2, \cdots, a_n, a_{n+1}$ be a maximal set of elements in $\hat{G}$ which are independent over the integers. Now $\hat{\phi}(\hat{G})$ is dense in V. So $\hat{\phi}(\hat{G})$ generates V as a vector space over R. So V is generated by some n among the elements $\hat{\phi}(a_1), \cdots, \hat{\phi}(a_{n+1})$. Without loss of generality we assume that $\hat{\phi}(a_1), \cdots, \hat{\phi}(a_n)$ generate V as a vector space over R. Then by choosing a different coordinate system in V if necessary and renaming we may as well assume that $\hat{\phi}(a_1) = f_1, \cdots, \hat{\phi}(a_n) = f_n$. Let the coordinates of $\hat{\phi}(a_{n+1})$ with respect to $f_1, \cdots, f_{n+1}$ be $(\lambda_1, \lambda_2, \cdots, \lambda_n, 0)$. Now we define a map $\hat{\theta}$ from $\hat{G}$ into R^{n+1} as follows:

$$\hat{\theta}(a_i) = f_i \quad \text{for} \quad i = 1, 2, \cdots, n; \qquad \hat{\theta}(a_{n+1}) = (\lambda_1, \lambda_2, \cdots, \lambda_n, a)$$

where a is a non-zero real number. We complete the definition of $\hat{\theta}$ on $\hat{G}$ by requiring it to be a homomorphism. Then it is clear that the set $\hat{\theta}(a_1), \cdots, \hat{\theta}(a_n)$ and $\hat{\theta}(a_{n+1})$ generate R^{n+1} over R. So $\hat{\theta}(G)$ is dense in R^{n+1} by Lemma 1.1 since $\hat{G}$ is Z-free and hence $\hat{\phi}(\hat{G})$ is Z-free and $\hat{\theta}$ is clearly an isomorphism. Let θ be the adjoint map of $\hat{\theta}$ from R^{n+1} into G. Then θ is a continuous isomorphism of R^{n+1} into G. Moreover, we claim that $\theta(x) = \phi(x)$ for all $x \in R^n$. For let $x \in R^n$ and $g \in \hat{G}$. Let us write $(\cdot, \cdot)$ to denote the natural inner product of a group and its dual. Then

$$(\phi(x) - \theta(x), g) = (x, \hat{\phi}(g) - \hat{\theta}(g)).$$

Now $\hat{\phi}(g)$ and $\hat{\theta}(g)$ are in the dual R^{n+1} space and their first n-coordinates are the same with respect to $f_1, f_2, \cdots, f_{n+1}$ as coordinate vectors. So the first n coordinates of $\hat{\phi}(g) - \hat{\theta}(g)$ are zeros. Since x is in the space R^n generated by $\{e_1, \cdots, e_n\}$ and f_{n+1} in the dual R^{n+1} is the annihilator of $[e_1, \cdots, e_n]$ we have that

$$(\phi(x) - \theta(x), g) = 1 \quad \text{for all} \quad g \in \hat{G}.$$

So $\phi(x) = \theta(x)$ for all $x \in R^n$. So $\theta(R^{n+1}) \supset \phi(R^n)$. So this proves the first part of Lemma 1.6. Now let $\psi_1 : R^{n+1} \to G$ be any continuous isomorphism of R^{n+1} into G. Let $\hat{\psi}_1$ be its adjoint. Then $\hat{\psi}_1(\hat{G})$ is dense in R^{n+1}. So $\hat{\psi}_1(a_1), \cdots, \hat{\psi}_1(a_{n+1})$ generate R^{n+1} over R. Then there is an invertible matrix $\hat{M}$ that acts on the dual space R^{n+1} such that $\hat{M}(\hat{\theta}(a_i)) = \hat{\psi}_1(a_i)$ for all $i = 1, 2, \cdots, n + 1$. Then it is clear that $\hat{\psi}_1 = \hat{M} \circ \hat{\theta}$. Let M be the transpose matrix of $\hat{M}$ that acts on the vector space R^{n+1} generated by $e_1, \cdots, e_{n+1}$. Then it is clear that $\psi = \theta \circ M$. Since M is non-singular it follows that $\psi(R^{n+1}) = \theta(R^{n+1})$. Hence the lemma.

LEMMA 1.7. *The real line R can be embedded into the two-dimensional torus*

T^2 by a continuous isomorphism in a continuum many ways so that the images are all different.

Proof. Choose a Hamel base $\mathcal{C}$ of R with $1 \in \mathcal{C}$. Then the map

$$\psi_\alpha : R \to T^2 \quad \text{where} \quad \psi_\alpha(x) = (e^{2\pi i x}, e^{2\pi i \alpha x}) \quad \text{for all} \quad \alpha \in \mathcal{C}$$

are continuous isomorphisms of R into T^2 and $\psi_\alpha(R) \neq \psi_\beta(R)$ if $\alpha, \beta \in \mathcal{C}$ and $\alpha \neq \beta$.

LEMMA 1.8. *Let G be a compact connected group. Let $\phi : R^n \to G$ be a continuous isomorphism of R^n into G. Let $A = \phi(R^n)$. Let G/A have rank ≥ 2. Then there exists a set J of cardinality c and continuous isomorphisms ψ_α of R^{n+1} into G for each $\alpha \in J$ such that the following hold:*
 (i) $\psi_\alpha(R^{n+1}) \supset \phi(R^n)$ *for all* $\alpha \in J$
 (ii) $\psi_\alpha(R^{n+1}) \neq \psi_\beta(R^{n+1})$ *if* $\alpha, \beta \in J$ *and* $\alpha \neq \beta$.

Proof. Let $A^\perp$ be the annihilator of A in $\hat{G}$, where $\hat{G}$ is the dual of G. Then $A^\perp$ is of rank at least two. So there exists a closed subgroup H in G so that $H \supset A$ and G/H is the torus T^2. Let λ be the canonical map from G onto G/H. Let J be a set of cardinality c and let $(\phi_\alpha)_{\alpha \in J}$ be a collection of continuous isomorphisms of R into G/H so that $\phi_\alpha(R) \neq \phi_\beta(R)$ if $\alpha, \beta \in J$ and $\alpha \neq \beta$. This is possible in view of Lemma 1.7. Now by Lemma 1.5, given $\alpha \in J$, there is a continuous isomorphism θ_α from R into G so that $\phi_\alpha = \lambda \circ \theta_\alpha$. Now treat R^n as a subspace of R^{n+1}. Let $e_1, e_2, \cdots, e_n, e_{n+1}$ be a coordinate system for R^{n+1}, where $e_1, e_2, \cdots, e_n$ generate the subspace R^n. Given $\alpha \in J$ define ψ_α from R^{n+1} into G by

$$\psi_\alpha(e_i) = \phi(e_i) \quad \text{if} \quad 1 \leq i \leq n, \qquad \psi_\alpha(re_{n+1}) = \theta_\alpha(r) \quad \text{for} \quad r \in R$$

and require ψ_α to be a homomorphism from R^{n+1} to G. Then it is easily seen that ψ_α is an isomorphism. Thus the collection $\{\psi_\alpha\}_{\alpha \in J}$ is easily verified to satisfy the conditions of the lemma.

LEMMA 1.9. *Let F be a free subgroup of rank $n + 1$ contained in R^n. Then there is a subgroup S of F and a vector space $V \neq \{0\}$ in R^n such that S is dense and contained in V.*

Proof. We consider the vector space L generated by F over the reals. Without loss of generality we take L to be R^n. Then there are vectors $a_1, a_2, \cdots, a_n$ in F which can be taken as unit coordinate vectors in R^n. Let a_{n+1} be in F and be independent of $\{a_1, \cdots, a_n\}$ over integers. Let a_{n+1} have coordinates $(\lambda_i, \cdots, \lambda_n)$ with respect to $a_1, a_2, \cdots, a_r$. By taking suitable integral linear combinations of $a_1, a_2, \cdots, a_{n+1}$ we get an element b_{n+1} of F such that $a_1, a_2, \cdots, a_n, b_{n+1}$ are independent over integers and b_{n+1} has coordinates $(\mu_1, \mu_2, \cdots, \mu_n)$ where the μ's are irrational or 0. By renaming the coordinates $a_1, a_2, \cdots, a_n$ we might as well assume that b_{n+1} has coordinates of the form

$$(\mu_1, \mu_2, \cdots, \mu_r, 0, \cdots, 0)$$

and $r \geq 1$ and μ_i's are irrational. Now consider all the subsets E of $\{\mu_1, \cdots \mu_r\}$ which are dependent over integers. If $\{\mu_{i_1}, \cdots, \mu_{i_k}\}$ is such a set it satisfies an equation of the form

$$n_{i_1} x_{i_1} + \cdots + n_{ik} x_{ik} = 0$$

where $n_{i_1}, \cdots, n_{i_k}$ are integers. There are only a finite set of such equations. All these equations along with the system $x_{r+1} = x_{r+2} = \cdots = x_n = 0$, define a subspace V of R^n. Let $S = V \cap F$. Then adopting the argument of Lemma 1.4 we get that S is dense in V.

LEMMA 1.10. *Let $\psi : R^n \to T^{n+2}$ be a continuous isomorphism of R^n into the torus of dimension $n + 2$. Then there exists a set J of cardinality c and a continuous isomorphism ψ_α of R^{n+1} into T^{n+2} for each $\alpha \in J$ such that the following hold:*

(i) $\psi_\alpha(R^{n+1}) \neq \psi_\beta(R^{n+1})$ *for all* $\alpha, \beta \in J$ *and* $\alpha \neq \beta$.

(ii) $\psi_\alpha(R^{n+1}) \supset \psi(R^n)$ *for all* $\alpha \in J$.

Proof. Using the solutions of simultaneous equations with integer coefficients, it can be shown that the number of α's such that $\psi_\alpha(R^{n+1}) = \psi_\beta(R^{n+1})$ for a fixed real number β is countable. So (i) follows. Also ψ_α's can be so chosen that $\psi_\alpha(R^{n+1}) \supset \psi(R^n)$.

LEMMA 1.11. *Let G be a connected compact group. Let $\phi : R^n \longrightarrow G$ be a continuous isomorphism. Let $\hat{\phi} : \hat{G} \to R^n$ be the adjoint of ϕ where $\hat{G}$ is the dual of G. Let the kernel of $\hat{G}$ be of rank at least 2. Then there exists an index set J of cardinality c and continuous isomorphisms ψ_α from R^{n+1} into G so that $\psi_\alpha(R^{n+1}) \neq \psi_\beta(R^{n+1})$ for all distinct $\alpha, \beta \in J$ and $\psi_\alpha(R^{n+1}) \supset \phi(R^n)$ for all $\alpha \in J$.*

Proof. We consider R^{n+1} and take a basis $e_1, \cdots, e_{n+1}$ for R^{n+1} over R. We consider that ϕ is defined on that subspace R^n of R^{n+1} which is generated by $e_1, e_2, \cdots, e_n$. Put $A = \phi(R^n)^-$ in G. Then our hypothesis implies that the dual $\hat{A}$ of A has rank at least two. So there exists a compact subgroup B of G so that $B \supset \phi(R^n)^-$ and G/B is the torus T^2. Then by Lemma 1.8 we get the result.

LEMMA 1.12. *Let G be a compact connected group with dual $\hat{G}$. Let*

$$\phi : R^n \to G$$

be a continuous isomorphism and $\hat{\phi} : \hat{G} \to \hat{R}^n$ its adjoint. Let $\hat{G}$ have rank at least $n + 2$. Let the kernel of $\hat{\phi}$ have rank 1. Then there exists at least a continuum many isomorphisms of R^{n+1} into G so that their images contain $\phi(R^n)$.

Proof. Follows by an argument using Lemmas 1.5, 1.9, 1.10.

LEMMA 1.13. *Let G be a compact connected group with dual $\hat{G}$. Let*

$$\phi : R^n \to G$$

be a continuous isomorphism. Let $\hat{G}$ be of rank at least $n + 2$. Let $\hat{\phi}$ be the adjoint map of ϕ. Let the kernel of $\hat{\phi}$ be $\{0\}$. Then there exists a set J of cardinality c and continuous isomorphisms ψ_α from R^{n+1} into G for each $\alpha \in J$ so that $\psi_\alpha(R^{n+1}) \neq \psi_\beta(R^{n+1})$ if $\alpha, \beta \in J$ and $\alpha \neq \beta$ and $\psi_\alpha(R^{n+1}) \supset \phi(R^n)$ for all $\alpha \in J$.

Proof. Let $a_1, a_2, \cdots, a_{n+2}$ be $n + 2$ elements of $\hat{G}$ which are independent over integers. Then $\hat{\phi}(a_1), \hat{\phi}(a_2), \cdots, \hat{\phi}(a_{n+2})$ are independent over the integers. By using Lemma 1.9, we get that the free group generated by $\hat{\phi}(a_1), \cdots, \hat{\phi}(a_{n+1})$ contains a subgroup F which is dense in a vector space V. Again, by renaming if necessary we may assume that V is generated over the reals by $\hat{\phi}(a_1), \cdots, \hat{\phi}(a_k)$ and that F is generated by $\hat{\phi}(a_1), \cdots, \hat{\phi}(a_r)$ and $r \geq k + 1$. If we have that $r \geq k + 2$, then our construction ends here. If not we have that $r = k + 1$ and $\hat{\phi}(a_i), i = 1, \cdots k + 1$, generate F and F is dense in V and V is generated over the reals by $\hat{\phi}(a_1), \cdots, \hat{\phi}(a_r)$. Then consider the free group S generated by $\hat{\phi}(a_1), \hat{\phi}(a_2), \cdots, \hat{\phi}(a_{n+2})$ and apply Lemma 1.9 again. We get a subspace V' and a free subgroup F' contained in the group S such that F' is dense in V' and rank $F' \geq \dim V' + 1$. So by considering $F + F'$ and $V + V'$, we get a free subgroup L in $\hat{\phi}(\hat{G})$ such that L is dense in a subspace M of R^n and rank $L \geq \dim M + 2$ and L is contained in the free group generated by $\hat{\phi}(a_1), \cdots, \hat{\phi}(a_{n+2})$. So we have that in any case there is a vector subspace H of R^n and a free subgroup F_1 of $\hat{\phi}(\hat{G})$ such that F_1 is finitely generated and $F_1 \subset H$ and is dense in H and rank $F_2 \geq \dim H + 2$. From this point on we follow the proof of previous lemma and get the required result.

2. The equivalence relation $\sim$ among subgroups of an abelian group

DEFINITION 2.1. Let G be a group. Let H and K be subgroups of G. We say that H is equivalent to K and write $H \sim K$ if $H/(H \cap K)$ and $K/(H \cap K)$ are finite.

Remark 2.2. The relation $\sim$ above is an equivalence relation in the set of all subgroups of a group G.

DEFINITION 2.3. Let G be a group. A canonical collection of Z-free subgroups of G is a collection $\mathfrak{D}$ of subgroups of G with the following properties:

(a) If $H \in \mathfrak{D}$, then H is Z-free

(b) If H_1 and $H_2 \in \mathfrak{D}$, then H_1 is not equivalent to H_2 unless $H_1 = H_2$.

(c) If A is a subgroup of G and A is Z-free, then there exists a subgroup $H \in \mathfrak{D}$ such that $H \sim A$.

Remark 2.4. Let G be a group. Let $\mathfrak{L}$ be the set of all Z-free subgroups of G. Let $\mathfrak{D}$ be a canonical collection of Z-free subgroups of G. Then $\sim$ is an equivalence relation in $\mathfrak{L}$ and the quotient space of $\mathfrak{L}$ by $\sim$ and $\mathfrak{D}$ have the same cardinality.

Remark 2.5. We also note that all canonical collections have the same cardinality.

LEMMA 2.6. *Let G be torsion free countable group. Let $A \subset G$ be a subgroup and $\mathcal{A}$ the collection of all subgroups of $B \subset G$, such that $A \subset B$ and $A \sim B$. Then $\mathcal{A}$ is countable. Suppose in addition to the assumption above, that A/nA is finite for all integers $n \geq 1$. Then the collection $\mathcal{C}$ of all subgroups $C \subset G$ so that $C \subset A$ and $C \sim A$ is countable. So if all the above conditions hold then the collection of all subgroups F of G, which are equivalent to A is countable.*

Proof. Let S be the collection of all elements x in G such that $nx \in A$ for some integer $n \geq 1$. Then $\mathcal{A}$ consists of groups generated by $A \cup X$ where X is a finite subset of S. And S being countable it follows that $|\mathcal{A}| \leq \aleph_0$. We note that if $c \in \mathcal{C}$ then there exists an $n \geq 1$ such that $c \supset nA$. By an argument similar to that of $\mathcal{A}$, we again get that $|\mathcal{C}|$ is countable. Putting $\mathcal{A}$ and $\mathcal{C}$ together, the last statement in the lemma follows:

LEMMA 2.7. *Let G be a torsion free group of finite rank. Suppose that there exists an uncountable collection $\mathcal{C}_1$ of mutually inequivalent subgroups of G. Then there exists a collection $\mathcal{B}$ of mutually inequivalent subgroups of G so that $|\mathcal{B}| = |\mathcal{C}_1|$ and Z is not a summand of any group in $\mathcal{B}$. In particular if $|\mathcal{C}_1| = c$ then $|\mathcal{B}| = c$.*

Proof. Since G is of finite rank, the collection of all free subgroups of G is countable. Let the set of all free subgroups of G be written as $F_1, F_2, \cdots, F_n, \cdots$. Let F_0 denote the identity subgroup of G. Let $\mathcal{C}_n$ be the set of all those subgroups in $\mathcal{C}$ which can be written as $F_n \times X$ where X is Z-free. Then $\bigcup_{n=0}^{\infty} \mathcal{C}_n = \mathcal{C}$. Hence there exists $n_0 \geq 0$ such that $|\mathcal{C}_{n_0}| = |\mathcal{C}|$. Then each group $C \in \mathcal{C}_{n_0}$ can be expressed as $F_{n_0} \times A_C$ where A_C is Z-free. Let $\mathcal{B}$ be the collection of the groups A_C where $C \in \mathcal{C}_{n_0}$. Then $\mathcal{B}$ is easily seen to be the required collection.

LEMMA 2.8. *Let G be a group and H a subgroup of G. Let $\mathcal{S}(G)$ (respectively $\mathcal{S}(G/H)$) be the collection of all subgroups of G (respectively of G/H). Then $|\mathcal{S}(G)| \geq |\mathcal{S}(G/H)|$. If G/H has a collection $\mathcal{C}$ of mutually inequivalent subgroups then G also contains a mutually inequivalent collection $\mathcal{C}'$ of subgroups so that $|\mathcal{C}'| \geq |\mathcal{C}|$. If G/H contains a direct sum $\sum_{n=1}^{\infty} A_n$ of an infinite set of non-zero subgroups A_n, then it contains c mutually inequivalent subgroups.*

Proof. We need prove only the last statement. To show this, we note that there exists a collection $\mathfrak{F}$ of subsets of the set Z^+ of strictly positive integers so that $|\mathfrak{F}| = c$ and such that

$$A \oplus B = (A \cup B) - (A \cap B)$$

is infinite for all distinct subsets $A, B \in \mathfrak{F}$. For each $S \subset \mathfrak{F}$, let F_S be the group $\sum_{n \in S} A_n$. Then the collection $\mathcal{C}$ of all such F_S is the required collection.

THEOREM 2.9. *Let G be a torsion free group of finite rank n. Let M be a free subgroup of G of maximal rank. Then either G has at least c subgroups or G/M is of the form $\sum_{i=1}^{n} C(p_i^{\infty}) \oplus F$ where $\{p_1, \cdots, p_n\}$ is a finite set of distinct primes and F is a finite group. In the latter case, the set $\{p_1, \cdots, p_n\}$ is independent of the choice of M.*

Proof. Now G/M is a torsion group. So $G/M = \sum_{p \in \mathcal{P}} G(p)$ where $\mathcal{P}$ is a set of primes and $G(p)$ is a p-primary group; for each $p \in \mathcal{P}$. If $\mathcal{P}$ is infinite, then the previous lemma applies. Suppose $\mathcal{P}$ is finite. Let $p \in \mathcal{P}$. The $G(p)$ can be expressed as $A(p) + B(p)$ where $A(p)$ is a direct sum of a number of copies of $C(p^{\infty})$ and $B(p)$ is reduced. If there are two copies of $C(p^{\infty})$ in $A(p)$, then there are c subgroups for $A(p)$ and so for G/M and G too. If some $B(p)$ is infinite and bounded then it is a direct sum of an infinite collection of finite non-zero cyclic groups and we again get that $|\mathcal{S}(G)| \geq c$. If $B(p)$ is unbounded then it has a basic subgroup and hence $|\mathcal{S}(G)| \geq c$. So if some $B(p)$ is infinite then $|\mathcal{S}(G)| \geq c$. So we get that either $|\mathcal{L}(G)| \geq c$ or G/M is of the form stated in the theorem. Now suppose G/M is of the form stated in the theorem and M' is another maximal free subgroup of G. Then $M \cap M'$ is of finite index in M and G/M is a quotient of $G/(M \cap M')$ with the kernel $M/M \cap M'$. So it follows that $G/(M \cap M')$ is of the same form as G/M. As $M \cap M'$ is of finite index in M' and

$$G/M = (G/(M \cap M'))/(M'/(M \cap M')),$$

the last assertion follows.

Note 2.10. The above theorem remains valid even if G is not assumed to be torsion free but the torsion free rank of G is finite. We also get in view of the preceding lemmas and theorem that either $|\mathfrak{D}| \geq c$ or G/M is as in Theorem 2.9.

DEFINITION 2.11. Let G be a group. Let M be any maximal free subgroup. Then G is said to have a type if G/M is of the form $\sum_{i=1}^{n} C(p_i^{\infty}) \oplus F$ where $\{p_1, \cdots, p_n\}$ is a finite set of distinct primes and F is finite and the set $\{p_1, \cdots, p_n\}$ does not depend on M. In this case we denote by $\mathfrak{J}(G)$, the type set $\{p_1, \cdots, p_n\}$ of G.

Note 2.12. If G is a group of finite torsion free rank and does not admit at least c subgroups then G has a type and so does every one of its subgroups and everyone of its quotient groups.

THEOREM 2.13. *Let G be a torsion free group of finite rank and let G not have at least c subgroups. Then the following hold:*

(i) *For every subgroup H of G we have that*

$$\mathfrak{J}(H) \cap \mathfrak{J}(G/H) = \emptyset \quad \textit{and} \quad \mathfrak{J}(G) = \mathfrak{J}(H) \cup \mathfrak{J}(G/H).$$

(ii) *For every subset S of $\mathfrak{J}(G)$, there exists a subgroup H_S of G so that H_S is Z-free and $\mathfrak{J}(H_S) = S$.*

(iii) *Two Z-free subgroups H_1, H_2 of G are equivalent if and only if $\mathfrak{J}(H_1) = \mathfrak{J}(H_2)$.*

(iv) *The collection $\{H_S \mid S \subset \mathfrak{J}(G)\}$ is a canonical collection of Z-free subgroups of G. So $|\mathfrak{D}|$ is finite and is of the form 2^n for some integer n.*

Proof. Let H be a subgroup of G. Let M_1 and M_2 be maximal free subgroups of H and G/H respectively. Then H/M_1 is of the form $\sum_{i\in\mathfrak{J}(H)} C(p_j^\infty) \oplus F_1$ where F_1 is finite. We can adjust M_1 and M_2 so that $F_1 = \{0\}$ and $(G/H)/M_2 = \sum_{i\in\mathfrak{J}(G/H)} C(p_i^\infty)$. Now there exists a free subgroup M_3 in G which maps onto M_2 in a 1-1 way by the canonical map from G onto G/H. Then $M_1 \oplus M_3$ is a maximal free group in G and

$$G/(M_1 \oplus M_3) = \sum_{i\in\mathfrak{J}(H)} C(p_i^\infty) + \sum_{j\in\mathfrak{J}(G/H)} C(p_j^\infty).$$

Now (i) follows using Theorem 2.9. Now let S be a given subset of $\mathfrak{J}(G)$. Let M be a maximal free subgroup of G so that $G/M = \sum_{i\in\mathfrak{P}} C(p_i^\infty)$. Let $\phi^{-1} : G \to G/M$ be the canonical map. Put $F = \sum_{i\in S} C(p_i^\infty)$. Then $\phi^{-1}(G)$ can be written as $A \oplus H_S$ where A is free and H_S is Z-free. Using (i), we note that H_S satisfies (ii). If H_1 and H_2 are two subgroups of G so that $H_1 \sim H_2$, using (i) we get that $\mathfrak{J}(H_1) = \mathfrak{J}(H_1 \cap H_2) = \mathfrak{J}(H_2)$. Conversely let H_1, H_2 be subgroups of G for which Z is not a direct summand. Let $\mathfrak{J}(H_1) = \mathfrak{J}(H_2)$. Consider $H_1 \cap H_2$. Then

$$\mathfrak{J}(H_1) = \mathfrak{J}(H_1 \cap H_2) \cup \mathfrak{J}(H_1/(H_1 \cap H_2)),$$

$$\mathfrak{J}(H_2) = \mathfrak{J}(H_1 \cap H_2) \cup \mathfrak{J}(H_2/(H_1 \cap H_2)).$$

Since $\mathfrak{J}(H_1) = \mathfrak{J}(H_2)$, from (i) we get

$$\mathfrak{J}(H_1/(H_1 \cap H_2)) = \mathfrak{J}(H_2/(H_1 \cap H_2)).$$

But

$$(H_1/(H_1 \cap H_2)) \cap (H_2/(H_1 \cap H_2)) = \{0\} \quad \text{in} \quad G/H_1 \cap H_2$$

and since G does not have c subgroups we have that $G/H_1 \cap H_2$ does not have c subgroups. So we get that

$$\mathfrak{J}(H_1/(H_1 \cap H_2)) = \mathfrak{J}(H_2/(H_1 \cap H_2)) = \emptyset.$$

But Z is not a summand of $H_1/(H_1 \cap H_2)$ or $H_2/(H_1 \cap H_2)$ since it is not a summand of either H_1 or H_2. So $H_1/(H_1 \cap H_2)$ and $H_2/(H_1 \cap H_2)$ are finite. So $H_1 \sim H_2$. This proves (iii). Now (iv) follows from (i), (ii) and (iii).

As we shall see subsequently, we will be interested in groups G which are subgroups of R^n. Then we will be interested in subgroups of G, which are dense in vector spaces. So we can ask the relation between two groups A_1 and A_2 which are equivalent and which are dense in some vector subspaces.

LEMMA 2.14. *Let F be a free group of rank $n + 1$ which is a subgroup of R^n. Let F generate R^n over the reals and be dense in R^n. Then a non-zero subgroup A*

of F can be dense in a vector subspace (i.e. $\bar{A}$ is a vector subspace) if and only if $A \sim F$. In this case $\bar{A} = R^n$.

Proof. Follows from standard structure theorems on closed subgroups of R^n.

LEMMA 2.15. *Let G be a subgroup of R^n and of rank $n + 1$. Let G generate R^n over R. Let $H \subset G$ be a subgroup, which is dense in a subspace V of R^n (i.e. $\bar{H} = V$ and V is a vector space). Then rank H is either equal to* $\dim V$ *or* $\dim V + 1$. *If Z is a summand of H and H is dense in V then* $\operatorname{rank} H = \dim V + 1$. *If S is a Z-free subgroup of G, then $\bar{S}$ is a vector subspace of R^n.*

Proof. This follows easily from Lemma 1.1 and the fact that G generates R^n over R.

LEMMA 2.16. *Let F be a free subgroup of R^n and of rank $n + 1$. Let F generate R^{n+1} over R. Then there exists one and only one non-zero vector subspace V of R^n so that $V \cap F$ is dense in V. If two subgroups A_1 and A_2 of F are dense in some non-zero vector subspaces of R^n then they are equivalent.*

Proof. Now $\{0\} \neq F \subset R^n$. So $R^n \neq \{0\}$. So the existence of V follows from Lemma 1.9. The uniqueness of V follows easily from Lemmas 2.14 and 2.15. The last statement follows easily from the first and Lemma 2.14.

LEMMA 2.17. *Let G be a subgroup of R^n with rank $n + 1$ and generating R^n over R. Let $H \subset G$ be a Z-free subgroup. Let $\bar{H} = V$. Let V_1 be any fixed complementary subspace of V in R^n. Let A be a free subgroup of G so that $A \cap H = \{0\}$. Then $A \oplus H$ is dense in a vector subspace of R^n if and only if one of the following is true.*

(a) $A = \{0\}$.

(b) *$A \subset V$ and A is infinite cyclic.*

(c) *$A \cap V = \{0\}$ and the projection $\pi(A)$ of A on V_1 along V is dense in some vector subspace.*

Proof. Let $(H \oplus A)^-$ be a vector space. Now G generates R^n over R and G is of rank $n + 1$. So $G \cap V$ can be at most of rank $((\dim V) + 1)$. Since $A \cap H = \{0\}$ and $\operatorname{rank} H = \dim V$ we get that $A \cap V$ can be at most of rank 1. So if $A \subset V$, then $A = \{0\}$ or infinite cyclic. Suppose now that $A \not\subset V$. Now

$$(H \oplus A)^- = V \oplus \pi(A)^-.$$

Since $(H \oplus A)^-$ is a vector space, it follows that $\pi(A)^-$ is a vector space. The converse is obvious.

LEMMA 2.18. *Let H be a subgroup of a group G of R^n. Let G be of rank $n + 1$ and generate R^n over R. Let H be Z-free. Let A_1 and A_2 be non-zero free subgroups of G such that $H \cap A_1 = H \cap A_2 = \{0\}$. Let $H \oplus A_1$ and $H \oplus A_2$ be dense in vector subspaces of R^n. Then $(H \oplus A_1) \sim (H \oplus A_2)$.*

Proof. Let $\bar{H} = V$. Then V is a vector space. Let V_1 be a complementary vector subspace of R^n. If $A_1 \subset V$, then rank $(G \cap V) = \dim V + 1$. Since G has to generate R^n over R, and G has rank $n + 1$, it follows from Lemmas 2.15 and 2.17 that $A_2 \subset V$. Since $G \cap V$ cannot be of rank $(\dim V + 2)$ by Lemma 2.15, we get that $H \oplus A_1 \sim H \oplus A_2$. If $A_1 \not\subset V$ then $A_2 \not\subset V$ and $\pi(A_1) \sim \pi(A_2)$ where π is the projection on V_1 along V. (This follows from Lemmas 2.17 and 2.16.) So if $b_1, b_2, \cdots, b_t$ is a maximal set of integrally independent elements of A_2 then there exists an element $x_1 \in A_1$ and integer $n_1 \neq 0$ such that

$$n_1 b_1 - x_1 \in V \cap G.$$

Then there exists $x_2 \in H$ and integer $n_2 \neq 0$ so that $n_2(n_1 b_1 - x_1) = x_2$ because

$$\text{rank } (G \cap V) = \dim V = \text{rank } H.$$

Putting $n_1 n_2 = k_1$, we get a non-zero integer k_1 so that $k_1 b_1 \in H \oplus A_1$. Similarly for $b_2, \cdots, b_t$. So we get that

$$(H \oplus A_2)/((H \oplus A_1) \cap (H \oplus A_2))$$

is finite. Similarly we get that

$$(H \oplus A_1)/((H \oplus A_1) \cap (H \oplus A_2))$$

is finite. So $H \oplus A_1 \sim H \oplus A_2$.

DEFINITION 2.19. Let C be a subgroup of R^n and of rank n or $(n + 1)$. A collection $\mathfrak{L}^1$ of subgroups of G is called a canonically dense collection if the following hold:

(i) $A \in \mathfrak{L}^1 \Rightarrow A$ is dense in a vector subspace.

(ii) $A, B \in \mathfrak{L}^1$ and $A \neq B \Rightarrow A$ is not equivalent to B.

(iii) If $D \subset G$ is a subgroup of G which is dense in some vector subspace then $D \sim C$ for some $C \in \mathfrak{L}^1$.

This class is important for our subsequent investigations and so we discuss them below:

THEOREM 2.20. *Let G be a subgroup of R^n so that* rank G *is either n or $n + 1$ and G generates R^n over R. Let H_1 and H_2 be subgroups of G so that Z is not a summand of either H_1 or H_2 and let $H_1 \sim H_2$. Let A_1 and A_2 be non-zero free subgroups of G so that $H_1 \cap A_1 = \{0\}$ and $H_2 \cap A_2 = \{0\}$ and $H_1 \oplus A_1$ and $H_2 \oplus A_2$ are dense in vector spaces. Then $H_1 \oplus A_1 \sim H_2 \oplus A_2$. So $\mathfrak{L}^1$ is a finite set or has c elements.*

Proof. Consider $H_1 \cap H_2$. It follows from Lemma 1.2 and $H_1 \sim H_2$ that $(H_1 \cap H_2)^- = \bar{H}_1 = \bar{H}_2$. So $(H_1 \cap H_2) \oplus A_1$ and $(H_1 \cap H_2) \oplus A_2$ are dense in vector spaces. So

$$(H_1 \oplus A_1) \sim (H_2 \oplus A_2) \quad \text{and} \quad (H_1 \cap H_2) \oplus A_1 \sim (H_1 \cap H_2) \oplus A_2$$

by Lemma 2.18. Now let $\mathfrak{D}$ be a canonical collection of Z-free subgroups of G. Then $|\mathfrak{D}| \leq |\mathfrak{L}|$ which follows from Lemma 1.1. If $|\mathfrak{D}| = c$, we get the last statement. If not $|\mathfrak{D}|$ is finite by Theorem 2.13. Let $\{H_1, H_2, \cdots, H_k\}$ be such a finite canonical collection. Then for each $i = 1, 2, \cdots, k$, there can be at most one non-zero free subgroup A_i of G up to equivalence so that $H_i \oplus A_i$ is dense in a vector space, using Lemma 2.18. If we choose one such A_i for each $i = 1, 2, \cdots, k$, we get that $\mathfrak{L}^1$ is a subset of the set

$$\{H_1, \cdots, H_k, (H_1 + A_1), \cdots, (H_k + A_k)\}$$

by Lemma 2.18. So $\mathfrak{L}^1$ is finite in this case.

DEFINITION 2.21. Let G be a group and $\psi : G \to R^n$ be a homomorphism. A canonical ψ-dense collection of subgroups of G is a collection $\mathfrak{C}$ of subgroups of G such that the following hold:

(a) If $H \in \mathfrak{C}$, then $\psi(H)$ is dense in a vector subspace of R^n.

(b) If H_1 and H_2 are distinct elements of $\mathfrak{C}$, then they are not equivalent.

(c) If A is a subgroup of G so that $\psi(A)$ is dense in a vector space of R^n, then there exists $M \in \mathfrak{C}$ so that $A \sim M$.

LEMMA 2.22. *Let G be a group of torsion free rank $n + 1$ and ψ a homomorphism of G into R^n with kernel $K \neq \{0\}$. Let $\psi(G)$ be dense in R^n and let G have at least c-subgroups. Then a canonical collection of ψ-dense subgroups of G has cardinality at least c.*

Proof. By Note 2.10 there are at least c subgroups of G which do not have Z as a summand. Then a collection $\mathfrak{D}$ of canonical Z-free subgroups of G has cardinality at least c. If $H \in \mathfrak{D}$ then it is clear that Z is not a summand of $\psi(H)$ since otherwise H is not Z-free. Then from Lemma 1.1, it follows that $\psi(H)$ is dense in some vector space in R^n. So if $\mathfrak{C}$ is a canonical collection of ψ-dense subgroups of G, then $|\mathfrak{C}| \geq |\mathfrak{D}| \geq c$ as required.

THEOREM 2.23. *Let G be a group of torsion free rank $n + 1$ and ψ a homomorphism of G into R^n. Let K be the kernel of ψ. Let K be of* rank 1 *and $\psi(G)$ dense in R^n. Then $|\mathfrak{C}|$ is finite or $\geq c$ for any canonical collection $\mathfrak{C}$ of ψ-dense subgroups of G.*

Proof. If G has at least c subgroups then $|\mathfrak{C}| \geq c$ by the previous lemma. So we can assume that G has a type and a finite canonical collection $\mathfrak{D}$ of Z-free subgroups of G say $H_1, \cdots, H_k$. Now we will show that if L is a subgroup of G so that $\psi(L)$ is dense in some vector subspace of R^n and $L = A \oplus B$ where A is a free group and B is a subgroup for which Z is not a summand, then there is a free subgroup $S \subset K$ and a subgroup $T \subset G$ so that $L \sim S \oplus T$ and $T \sim H_i$ for some $i = 1, 2, 3, \cdots, k$. If $A = \{0\}$ this is clear. So let $A \neq \{0\}$. By Lemma 1.1 and the fact that $\psi(L)$ is of rank n and dense in R^n, we have that $\psi(L)$ is Z-free. So $L \cap K \neq B \cap K$. So there exists $a \in A$ and $b \in B$ such that $a + b \in L$ and $a \neq 0$. So we must have that

$B \cap L = \{0\}$. Otherwise using the fact that K is of rank 1, we will get that there exists a non-zero integer m so that $ma \in B$ which contradicts the assumption that $A \cap B = \{0\}$. Let $\{a, a_2, \cdots, a_t\}$ be a maximal integrally independent set in A. Let F be the free group generated by $\{a, a_2, \cdots, a_t\}$. Then $F \oplus B \sim A \oplus B$. So $\psi(F + B) \sim \psi(A + B)$ and hence is dense in a vector subspace of R^n. So $\psi(F + B)$ is Z-free. But we see easily that $F \oplus B = F_1 \oplus B$ where F_1 is the free group generated by $\{a + b, a_2, \cdots, a_t\}$. Moreover the cyclic subgroup C generated by $a + b$ is precisely $(F \oplus B) \cap K$ since K is of rank 1. So if N is the free group generated by $\{a_2, a_3, \cdots, a_t\}$, then we get that $\psi(F_1 + B)$ is isomorphic to $N + B$. So $N = \{0\}$. So F_1 is cyclic and we get that $L \sim S \oplus B$ where S is the free group contained in K and generated by $a + b$. Thus we have established our claim that if a subgroup L of G has Z as a summand and $\psi(L)$ is dense in a vector subspace that L is equivalent to a subgroup of the form $S \oplus H_i'$ where $S \subset K$ is an infinite cyclic group and $H_i' \sim H_i$ for some $i = 1, 2, \cdots, k$. Since $\mathfrak{D}$ is finite, we get that $\mathfrak{C}$ is finite and we get the theorem.

3. In between topologies in a group

Notation 3.1. A group topology on G means a locally compact Hausdorff topology on G, making G a topological group.

Notation 3.2. (X, τ) denotes a set X with a topology τ. If τ_1, τ_2 are two topologies on a set X, we write $\tau_1 \leq \tau_2$ or $\tau_2 \geq \tau_1$ to mean that τ_2 is stronger or finer than τ_1 (i.e. τ_2 has more open sets than τ_1).

Notation 3.3. Let G be a group with group topologies τ_1 and τ_2. Let $\tau_1 \leq \tau_2$. $[\tau_1, \tau_2]$ denotes the set of all group topologies τ on G such that $\tau_1 \leq \tau \leq \tau_2$.

Notation 3.4. If X is a set $| X |$ denotes the cardinality of X.

Notation 3.5. If (X, τ) is a topological space and $A \subset X$, then $\tau | A$ denotes the topology τ restricted to A.

DEFINITION 3.6. Let G be a group with a group topology τ. Let H be a closed subgroup of G. Then the group topology τ' on G such that $\tau' | H = \tau | H$ and such that H is open in τ' is called the group topology induced by H in G. This is denoted by τ_H. The collection $\{\tau_H | H$ is a closed subgroup of $G\}$ is denoted by τ_S. (G, τ) is used to denote the locally compact group G.

Note 3.7. If (G, τ) is a locally compact group, then each member of τ_S gives a finer group topology than τ.

DEFINITION 3.8. Let (G, τ) be a locally compact group with an open subgroup of the form $H \times A$, where (H, τ) is a compact group and (A, τ) is topologically isomorphic to some R^n. A quintet (L, ϕ, t, k, C) consists of a closed subgroup $L \subset H$, a continuous isomorphism ϕ of a vector group

R^t into H, an integer k and a closed subgroup C of G satisfying the following conditions:

(i) $\phi(R^t) \cap L = \{0\}$.

(ii) $0 \leq k \leq n$.

(iii) C is topologically isomorphic to R^k.

DEFINITION 3.9. Let (G, τ) be a locally compact group with an open subgroup $H \times A$ where H is a compact group and (A, τ) is topologically isomorphic to a vector group R^n. Let (L, ϕ, t, k, C) be a quintet. Let τ_1 be the topology on $\phi(R^t)$ which makes the map $\phi : R^t \to \phi(R^t)$ a homeomorphism. Let τ_2 be the restriction of τ_1 to $L \oplus C$. Let τ_0 be the group topology on G in which $L \oplus \phi(R^t) \oplus C$ is open and $\tau_0 / L \oplus \phi(R^t) \oplus C$ is the same as

$$(L, \tau_2) \times (\phi(R^t), \tau_1) \times (C, \tau_2).$$

We say that τ_0 is the topology on G induced by the quintet (L, ϕ, t, k, C).

Note 3.10. It was proved in Theorem 1 of [4] that the topology τ_0 induced by the quintet (L, ϕ, t, k, C) on (G, τ) is stronger than τ and every group topology τ_1 on G which is stronger than τ is obtained in this way by a quintet.

LEMMA 3.11. *Let G be a group with group topologies τ_1 and τ_2. Let $\tau_1 \leq \tau_2$. Let H be an open subgroup in τ_1. Let τ_1^* and τ_2^* be the restrictions of τ_1 and τ_2 respectively to H. Then $| [\tau_1, \tau_2] | = | [\tau_1^*, \tau_2^*] |$.*

Proof. If $\tau_3 \in [\tau_1, \tau_2]$ then H is also open in τ_3. Moreover τ_3 is induced in G by H as in Definition 3.6. So the result follows.

Note 3.12. Let G be a group with group topologies τ_1 and τ_2 and such that $\tau_1 \leq \tau_2$. Let $H + A$ be an open subgroup of (G, τ_1) where (H, τ_1) is a compact group and (A, τ_1) is topologically isomorphic to a vector group. Then to discuss $| [\tau_1, \tau_2] |$ it is enough to discuss $| [\tau_1^*, \tau_2^*] |$ as in Lemma 3.11. So we can and also do assume hereafter that (G, τ_1) is of the form $H \oplus A$, as above.

LEMMA 3.13. *Let G be a group with group topologies τ_1 and τ_2. Let $\tau_1 \leq \tau_2$. Let (G, τ_1) be of the form $H \oplus A$ where H is compact and A is a vector group. Let τ_2 be induced by the quintet (L, ϕ, t, k, C). Then there exists a subgroup B of (G, τ_1) such that the following hold:*

(i) $B \supset C$.

(ii) $B \cap H = \{0\}$.

(iii) $B \oplus H = A \oplus H = G$.

Proof. Since τ_2 is induced by the quintet, we have that (C, τ_1) is a closed subgroup of (G, τ_1), topologically isomorphic to a vector group. Hence C is a direct summand of (G, τ_1). So $(G, \tau_1) = C \oplus M$, where M is closed in (G, τ_1). Since (H, τ_1) is compact (M, τ_1) will split as $H \oplus B_1$ where (B_1, τ_1) is a vector group. Writing $B_1 \oplus C = B$, we get the required result.

Remark 3.14. Let $(G, \tau_1) = H \oplus A$ as in Note 3.12. Let $\tau_2 \geq \tau_1$ be a group topology on (G, τ_1) induced by the quintet (L, ϕ, t, k, C). In view of the Lemma 3.13, we can assume that $C \subset A$, when discussing the $|[\tau_1, \tau_2]|$.

LEMMA 3.15. *Let τ_1 and τ_2 be two group topologies on a group G. Let $\tau_2 \geq \tau_1$. Let $(G, \tau) = H \oplus A$ as in Note 3.12. Let τ_2 be induced by a quintet (L, ϕ, t, k, C), where $C \subset A$. Let dimension of $A \geq k + 2$. Then $|[\tau_1, \tau_2]| \geq c$.*

Proof. There exist at least c distinct closed subgroups V_α of A in the τ_1 topology so that each $V_\alpha \supset C$. Then each closed subgroup $H \oplus V_\alpha$ induces a group topology τ_α on G as in Definition 3.6. It is easily verified that all these τ_α's are distinct and each $\tau_\alpha \in [\tau_1, \tau_2]$. Hence the lemma.

Remark 3.16. In view of Lemmas 3.15, 3.13 and 3.11, it is enough to consider the following: (G, τ_1) is of the form $H \oplus A$ as in 3.12 and (G, τ_2) is induced by the quintet (L, ϕ, t, k, C) where $C \subset A$ and $\dim C \leq \dim A \leq \dim C + 1$.

LEMMA 3.17. *Let τ_1 and τ_2 and G be as in Lemma 3.13. Let (G, τ_1) be of the form $H \oplus A$ and (G, τ_2) be induced by (L, ϕ, t, k, C) with $C \subset A$ and*

$$\dim C \leq \dim A \leq \dim C + 1.$$

Let τ_1^* and τ_2^* be the restrictions of τ_1 and τ_2 respectively to H. Then

$$|[\tau_1, \tau_2]| = |[\tau_1^*, \tau_2^*]| \quad or \quad 2|[\tau_1^*, \tau_2^*]|.$$

Proof. Suppose $\dim C = \dim A$. Then $C = A$. So every topology $\tau_3 \in [\tau_1, \tau_2]$ is induced by the quintet (L, ϕ, t, k, A), where $k = \dim A$. So it follows, from Definition 3.6 and Note 3.10, that $|[\tau_1, \tau_2]| = |[\tau_1^*, \tau_2^*]|$ in this case. Suppose that $\dim A = \dim C + 1$. Let $\tau_3 \in [\tau_1, \tau_2]$. Then τ_3 is induced by (L, ϕ, t, k', B) where $k' = \dim C$ or $1 + \dim C$ and $B \supset C$. If $k' = \dim C$ then $B = C$. If $k' = \dim C + 1$ then τ_3 is the same as that induced by (L, ϕ, t, k, A). So by an argument as above, it follows that $|[\tau_1, \tau_2]| = 2|[\tau_1^*, \tau_2^*]|$ in this case. Now the assertion follows.

Remark 3.18. Let (G, τ_1) be a compact group. Let τ_2 be a stronger group topology on G than τ_1. Then the quintet (L, ϕ, t, k, C) that induces τ_2 on G as in Note 3.12 has to be of the form $(L, \phi, t, 0, 0)$. So we write, (L, ϕ, t) instead of $(L, \phi, t, 0, 0)$ in this situation and say that τ_2 is induced by the triple (L, ϕ, t) instead of $(L, \phi, t, 0, 0,)$.

LEMMA 3.19. *Let (G, τ_1) be a compact group. Let τ_2, τ_3 be group topologies on G, which are stronger than τ_1. Let τ_2 be induced by the triple (L, ϕ, t) and τ_3 be induced by (K, ψ, s). Then $\tau_2 = \tau_3$ if and only if the following hold:*

(i) *$K \cap L$ is open in both K and L or equivalently $K/(K \cap L)$ and $L/(K \cap L)$ are finite.*

(ii) $\psi(R^s) \subset (K \cap L) \oplus \phi(R^t)$

(iii) $\phi(R^t) \subset (K \cap L) \oplus \psi(R^s)$

(iv) $s = t$.

Proof. Let conditions (i), (ii) and (iii), (iv) hold. Then

$$(K \cap L) \oplus \phi(R^t) = (K \cap L) \oplus \psi(R^s)$$

from (ii) and (iii). So from (i) we get that

$$(K \cap L) \oplus \phi(R^t) = (K \cap L) \oplus \psi(R^s)$$

is open in τ_2 and also τ_3. Now from the structure theorem of locally compact abelian groups, we get τ_2 and τ_3 coincide on this open subgroup. So $\tau_2 = \tau_3$. Conversely let $\tau_2 = \tau_3$. Then K and L are both compact in τ_2. So $K \cap (L + \phi(R^t))$ is compact in τ_2 and hence $\subset L$. So

$$K \cap (L + \phi(R^t)) = K \cap L.$$

So $K \cap L$ is open in K in τ_2. So $K/(K \cap L)$ is finite. Similarly $L/(K \cap L)$ is finite. Then $(K \cap L) \oplus \phi(R^t)$ is open in τ_2 and hence in τ_3. Since $\psi(R^s)$ is a connected subgroup in τ_3, we get

$$\psi(R^s) \subset (K \cap L) \oplus \phi(R^t).$$

Similarly

$$\phi(R^t) \subset (K \cap L) \oplus \psi(R^s).$$

Then clearly from the structure theorem for LCA groups, we get $s = t$.

THEOREM 3.20. *Let (G, τ_1) be a compact group and τ_2 a group topology on G stronger than τ_1. Let τ_2 be induced by (L, ϕ, n). Let τ_1^* and τ_2^* be the respective quotient topologies on G/L from τ_1 and τ_2 by the natural map $\lambda : G \to G/L$. Then $|[\tau_1, \tau_2]| = |[\tau_1^*, \tau_2^*]|$. Moreover $(G/L, \tau_1^*)$ is a compact group and and τ_2^* a group topology on G/L stronger than τ_1^* and τ_2^* is induced by $(\{0\}, \lambda \circ \phi, n)$. So to prove the main theorem when (G, τ_1) is compact, it is enough to do so with the extra assumption that $L = \{0\}$ where (L, ϕ, n) induces τ_2.*

Proof. Now $\tau_2 | L = \tau_1 | L$ since τ_2 is induced by (L, ϕ, n). So if $\tau_3 \in [\tau_1, \tau_2]$ then $\tau_3 | L = \tau_1 | L$ and hence L is compact in τ_3. So if $\lambda : G \to G/L$ is natural map from G onto G/L and τ_3^* is the quotient topology on G/L obtained from (G, τ_3), we get that $(G/L, \tau_3^*)$ is a locally compact group. Clearly $(G/L, \tau_1^*)$ is a compact group and $(G/L, \tau_2^*)$ is a locally compact group and $\tau_1^* \leq \tau_2^*$ and $\tau_3^* \in [\tau_1^*, \tau_2^*]$. So there is a map

$$* : [\tau_1, \tau_2] \to [\tau_1^*, \tau_2^*]$$

namely $\tau_3 \to \tau_3^*$ for all $\tau_3 \in [\tau_1, \tau_2]$. Now we observe that if $\tau_3 \in [\tau_1, \tau_2]$ and is induced by (H, ϕ_1, n_1), then $H \supset L$ and τ_3^* is induced by $(\lambda(H), \lambda \circ \phi, n_1)$. From this and using Lemma 3.19 we get the 1-1 nature of the map, $*$. Now let τ be an element of $[\tau_1^*, \tau_2^*]$ induced by (K, ψ, t). Then an application of Lemma 1.5 shows the existence of a continuous isomorphism

$$\theta : R^t \to G$$

such that $\theta(R^t) \cap H = \{0\}$ and $\lambda(\theta(R^t)) = \psi(R^t)$ where $H = \lambda^{-1}(K)$. Then it can be verified that $H \oplus (\theta(R^t)) \supset \phi(R^n)$. So the triple (H, θ, t) gives an element $\tau_3 \in [\tau_1, \tau_2]$. It is also easy to see that $\tau_3^* = \tau$. So $*$ is onto. So $|[\tau_1, \tau_2]| = |[\tau_1^*, \tau_2^*]|$. Now the last remark is obvious when we look at $(G/L, \tau_1^*)$.

COROLLARY 3.21. *Let (G, τ_1) be a compact group and τ_2 a stronger group topology induced by $(0, \phi, n)$. Let $H \subset G$ be a closed subgroup in τ_1 and hence in τ_2 such that $H \cap \phi(R^n) = \{0\}$. Let $\lambda : G \to G/H$ be the canonical map and τ_1^* and τ_2^* the quotient topologies of τ_1 and τ_2 respectively in G/H by λ. For every $\tau_3 \in [\tau_1, \tau_2]$, let τ_3^* be its quotient topology in G/H.*

Then $\tau_3^* \in [\tau_1^*, \tau_2^*]$ and the map $* : [\tau_1, \tau_2] \to [\tau_1^*, \tau_2^*]$ which takes τ_3 to τ_3^* is onto.

Proof. To prove this, we have to only imitate the last part of the proof of Theorem 3.20.

LEMMA 3.22. *Let (G, τ_1) be a compact group with a stronger group topology τ_2 induced by $(0, \phi, n)$. If there exists a closed subgroup H in (G, τ_1) such that $|[\tau_1^*, \tau_2^*]| \geq c$ where τ_1^*, τ_2^* are the respective quotient topologies of τ_1 and τ_2 in G/H, then $|[\tau_1, \tau_2]| \geq c$. So to prove the main theorem it is enough to do so under the assumptions (G, τ_1) is compact, τ_2 is induced by $(0, \phi, n)$ and $|[\tau_1^*, \tau_2^*]| < c$ for all closed subgroups H of (G, τ_1).*

Proof. This is an easy consequence of Corollary 3.21, Theorem 3.20 and Lemma 3.17.

DEFINITION 3.23. Let (G, τ_1) be a compact group and τ_2 a stronger group topology in G induced by $(\{0\}, \phi, n)$. We say that G satisfies condition $(*)$ if $|[\tau_1^*, \tau_2^*]| < c$ for all closed subgroups H of (G, τ_1) where τ_1^* and τ_2^* are as in Lemma 3.22.

Notation 3.24. Let (G, τ_1) be a compact group and τ_2 a stronger group topology on G induced by $(\{0\}, \phi, n)$. Then we sometimes say that τ_2 is induced by $\phi(R^n)$ instead of by $(\{0\}, \phi, n)$.

LEMMA 3.25. *Let (G, τ_1) be a compact group. Let τ_2 be a stronger group topology in G induced by $\phi(R^n)$. Let G satisfy condition $(*)$. Let G_0 be the connected component of 0 in (G, τ_1). Then there exists a closed subgroup A of (G, τ_1) so that $A \cap G_0 = \{0\}$ and $A \oplus G_0$ is open in τ_1.*

Proof. Let τ_1^*, τ_2^* respectively be the quotient topologies of τ_1 and τ_2 in G/G_0. Then τ_2^* is discrete since $\phi(R^n) \subset G_0$. So $[\tau_1^*, \tau_2^*]$ is just the set of all stronger group topologies on G/G_0, stronger than τ_1^*. Since G satisfies condition $(*)$, (and G_0 is closed) the set of stronger group topologies on G/G_0 is finite. So $(G/G_0, \tau_1^*)$ contains an open subgroup S which is topologically isomorphic to $I_{p_1}^{\#} \times \cdots \times I_{p_k}^{\#}$ where $p_1, \cdots, p_k$ are distinct primes [5].

By taking the full inverse in G, we get an open subgroup H of (G, τ_1) containing G_0 and such that H/G_0 is a finite product of p-adic integers. Now the p-adic integers is projective in the category of compact groups. So $H = A \oplus G_0$ where A is as in the lemma. Hence the result.

Note 3.26. In view of 3.25 and 3.11 to prove the main theorem, it is enough to assume the following conditions:

(i) (G, τ_1) is of the form $A \oplus G_0$ where A is a finite product of distinct p-adic integers and (G_0, τ_1) is a connected compact group.

(ii) τ_2 is induced by $\phi(R^n)$ and hence $\phi(R^n) \subset G_0$.

(iii) G satisfies $(*)$.

LEMMA 3.27. *Let (G, τ_1) be a compact group of the form $A \oplus G_0$ where A, G_0 are as in Note* 3.26. *Let τ_2 be a stronger group topology on G induced by $\phi(R^n)$. Let $\hat{G}_0$ be the dual of G_0. Let $\hat{G}_0$ have rank $\geq n + 2$. Then $|[\tau_1, \tau_2]| \geq c$.*

Proof. Now $\phi(R^n)$ is connected in (G, τ_1) and hence is contained in G_0. So by Lemmas 1.11, 1.12, 1.13 there exists at least continuum many isomorphisms ψ_α of R^{n+1} into (G, τ_1) so that $\psi_\alpha(R^{n+1}) \supset \phi(R^n)$ for all α and $\psi_\alpha(R^{n+1}) \neq \psi_\beta(R^{n+1})$ if $\alpha \neq \beta$. For each such ψ_α we get a group topology induced by $[\{0\}, \psi_\alpha, n + 1]$ in $[\tau_1, \tau_2]$. So $|[\tau_1, \tau_2]| \geq c$.

LEMMA 3.28. *Let (G, τ_1) be a compact group and τ_2 a stronger group topology on G induced by $\phi(R^n)$. Let $\tau_3 \in [\tau_1, \tau_2]$. Then τ_3 is induced by a triple (L, ψ, k) such that $L \cap \phi(R^n) = \phi(V_1)$ for some vector subspace $V_1 \subset R^n$ and $L \oplus \psi(R^k) \supset \phi(V_2)$ where V_2 is a complementary vector subspace of V_1 in R^n. If further (G, τ_1) is of the form $A \oplus G_0$ where A is a finite product of distinct p-adic integers and G_0 is connected and* rank $\hat{G}_0 = n$, *then τ_3 is induced by (L, ϕ_2, k) where ϕ_2 is the restriction of ϕ to V_2 above and* dim $V_2 = k$.

Proof. Now

$$\phi : R^n \to (G, \tau_2)$$

is a homeomorphism of R^n into G. So if $\tau_3 \in [\tau_1, \tau_2]$, then $\phi : R^n \to (G, \tau_3)$ is continuous. So if τ_3 is induced by (L, ψ, k), then $\phi(R^n) \subset L \oplus \psi(R^k)$. Let π be the projection from $L \oplus \psi(R^k)$ onto $\psi(R^k)$. Let $K \subset R^n$ be the kernel of $\pi \circ \phi$. Then K is a vector subspace of R^n, otherwise $(\psi(R^k), \tau_3)$, which is isomorphic to R^k, would contain a closed subgroup isomorphic to T. Clearly $L \cap \phi(R^n) = \phi(K)$. Hence the first part of the lemma follows. Let now (G, τ_1) be of the form $A \oplus G_0$ as in the second part of the lemma. Let rank of $\hat{G}_0 = n$. Then R^{n+1} cannot be embedded into (G_0, τ_1) by a continuous isomorphism. But τ_3 is induced by (L, ψ, k) where

$$L \cap \phi(R^n) = \phi(V_1) \quad \text{and} \quad L \oplus \psi(R^k) \supset \phi(V_2)$$

and V_1, V_2 are complementary subspaces of R^n. So dim $V_2 \leq k$ by structure theorem of locally compact abelian groups. But dim V_2 cannot be $<k$. Otherwise by looking at $\phi(V_1) \oplus \psi(R^k)$, we get that R^{n+1} can be embedded into (G_0, τ_1) by continuous isomorphism, which is not possible. So $k =$ dim V_2. So $L \oplus \psi(R^k) = L \oplus \phi(V_2)$. Hence the second part follows. Hereafter we shall follow the following:

Convention 3.29. Let (G, τ_1) be a compact group. Let τ_2 be a stronger group topology induced by $\phi(R^n)$. Let A be a vector subspace of R^n and L a compact subgroup of (G, τ_1) such that $L \cap \phi(A) = \{0\}$. Let $\dim A = k$ and ψ the restriction of ϕ to A. Then (L, ψ, k) gives a topology τ_3 in $[\tau_1, \tau_2]$; we say τ_3 is induced by $L \oplus \phi(A)$. Similarly let S be a closed subgroup of (G, τ_1) and M a subgroup of G so that $M \cap S = \{0\}$ and M is the range of a continuous isomorphism θ of a vector group R^t into G. We say that $S \oplus M$ induces the topology given by (S, θ, t) in G.

LEMMA 3.30. *Let (G, τ_1) be a compact group of the form $A \oplus G_0$, where A is a finite product of distinct p-adic integers, and G_0 is a connected group. Let the rank of $\hat{G}_0$ be $n + 1$. Let τ_2 be a stronger group topology induced by $\phi(R^n)$. Let there be a continuous isomorphism θ from R^{n+1} into G. Then every group topology $\tau_3 \in [\tau_1, \tau_2]$ is induced by $L \oplus M$ where L is a closed subgroup of (G, τ_1) and M is a subgroup of G so that $M \cap L = \{0\}$ and one of the following holds:*

(1) *$L \cap \phi(R^n) = \phi(V_1)$ and $M = \phi(V_2)$ where V_1 and V_2 are complementary subspaces of R^n.*

(2) *$L \cap \theta(R^{n+1}) = L \cap \phi(R^n) = \theta(V_1)$ and $M = \theta(V_2)$ where V_2 and V_1 are complementary subspaces of R^{n+1}.*

Proof. If $\tau_3 \in [\tau_1, \tau_2]$, then τ_3 is induced by (L, ψ, k). Then

$$L \cap \phi(R^n) = \phi(V_1)$$

for some vector subspace of R^n, as in Lemma 3.28. Suppose V_2 is a complementary vector subspace of V_1 in R^n and $k = \dim V_2$. Then following the proof of Lemma 3.28, we get that $L + \phi(V_2) = L + \psi(R^k)$. Taking $M = \phi(V_2)$ we see that case (i) occurs this time. Suppose $k \neq \dim V_2$. Then we have that

$$\phi(V_2) \subset \phi(R^n) \subset L \oplus \psi(R^k)$$

since $\tau_3 \leq \tau_2$. But $L \cap \phi(V_2) = \{0\}$. Hence $\dim k \geq \dim V_2$. Since rank $\hat{G}_0 = n + 1$, we have that R^{n+2} cannot be continuously embedded in (G, τ_1). But $\phi(V_1) \oplus \psi(R^k)$ is clearly the image of a continuous isomorphism of R^{n-l+k} where $l = \dim V_2$. So $n - l + k \leq n \pm 1$. So $k \leq \dim V_2 + 1$. Since $k \neq \dim V_2$, we must have that $k = \dim V_2 + 1$. Then $\phi(V_1) \oplus \psi(R^k)$ is a continuous isomorphic image of R^{n+1}. So by Lemma 1.5 and 1.2, we have that

$$\theta(R^{n+1}) = \phi(V_1) \oplus \psi(R^k).$$

Following the proof of Lemma 3.28, we see that the second case arises this time.

LEMMA 3.31. *Let (G, τ_1) be a compact group of the form $A + G_0$ and τ_2 a stronger group topology induced by $\phi(R^n)$, as in the previous lemma. Let the dual $\hat{G}_0$ of G_0 have rank n or $n + 1$. Let $\hat{G}$ be the dual of (G, τ_1) and $\hat{\phi} : \hat{G} \to R^n$*

be the adjoint of ϕ. *Let* L *be a compact subgroup of* (G, τ_1) *and* $L^\perp$ *its annihilator. Then* $L \cap \phi(R^n) = \phi(V)$ *for some vector subspace* V *of* R^n *if and only if* $\hat{\phi}(L^\perp)$ *is dense in a vector subspace. If* (L_1, ψ_1, k_1) *induces a group topology* τ_3 *in* $[\tau_1, \tau_2]$ *and* (L_2, ψ_2, k_2) *induces* $\tau_4 \in [\tau_1, \tau_2]$ *then* $\tau_3 = \tau_4$ *if and only if* $L_1 \cap L_2$ *is of finite index in both* L_1 *and* L_2, *and* $k_1 = k_2$. *Let* L *be a compact subgroup of* (G, τ_1). *Then there exists a topology* $\tau \in [\tau_1, \tau_2]$ *induced by a triple of the form* (L, θ, t) *if and only if* $L \cap \phi(R^n) = \phi(V)$, *for some vector subspace* V *of* R^n. *In this case there can be at most two such topologies.*

Proof. The first statement is an easy consequence of Pontrjagin's duality theorems. The second follows easily from Lemmas 3.30 and 3.19. The last statement follows again by the same lemmas.

LEMMA 3.32. *Let* (G, τ_1) *be a compact group of the form* $A \oplus G_0$ *and* τ_2 *a stronger group topology induced by* $\phi(R^n)$ *as in Lemma* 3.31. *Let* τ_3 *and* τ_4 *be topologies in* $[\tau_1, \tau_2]$ *induced respectively by* (L_1, ϕ_1, t_1) *and* (L_2, ϕ_2, t_2). *Let* $L_1^\perp$ *and* $L_2^\perp$ *be the annihilators of* L_1 *and* L_2 *respectively in* $\hat{G}$. *Then* $\tau_3 = \tau_4$ *if and only if* $L_1^\perp \sim L_2^\perp$ *in the sense of Definition* 2.1 *and* $t_1 = t_2$.

Proof. From Lemmas 3.19 and 3.31, we have that $\tau_3 = \tau_4$ if and only if $L_1 \cap L_2$ is of finite index in L_1 and L_2 and $t_1 = t_2$. Then by duality theorems it follows that $L_1 \cap L_2$ is of finite index in both L_1 and L_2; if and only if $L_1^\perp \cap L_2^\perp$ is of finite index in both $L_1^\perp$ and $L_2^\perp$. Hence the lemma.

LEMMA 3.33. *Let* (G, τ_1) *be a compact group of the form* $A \oplus G_0$ *and* τ_2 *a stronger group topology on* G *induced by* $\phi(R^n)$. *Let the rank of* $\hat{G}_0$ *be* n *or* $n + 1$. *Let* $\mathfrak{C}$ *be a canonical collection of* $\hat{\phi}$ *dense subgroups of* $\hat{G}$ *as in Definition* 2.21. *Then* $|\mathfrak{C}| \leq |[\tau_1, \tau_2]| \leq 2|\mathfrak{C}|$.

Proof. This follows easily from Lemmas 3.30 and 3.31 and 3.32.

THEOREM 3.34. *Let* G *be an abelian group with group topologies* τ_1 *and* τ_2 *such that* $\tau_1 \leq \tau_2$. *Then* $|[\tau_1, \tau_2]|$ *is either finite or* $\geq c$.

Proof. From Note 3.26 it is enough to prove our theorem when

$$(G, \tau_1) = A \oplus G_0,$$

where A is a finite product of distinct p-adic integers, and G_0 is a connected compact group and τ_2 is induced by $\phi(R^n)$ as in Lemma 3.30. Let $\hat{A}$ and $\hat{G}_0$ be the duals of A and G_0. From Lemma 3.27, we get if rank $\hat{G}_0 \geq n + 2$ then $|[\tau_1, \tau_2]| \geq c$. Suppose that rank $\hat{G}_0 = n$ or $n + 1$. Now the dual $\hat{G}$ of G is $\hat{A} \oplus \hat{G}_0$ and $\hat{A} = \sum C(p_i^\infty)$; where $p_1, p_2, \cdots, p_n$ are distinct primes. For every subset S of $\{p_1, p_2, \cdots, p_n\}$ put $H_S = \sum_{p_i \in S} C(p_i^\infty)$ if $S \neq \emptyset$ and $= \{0\}$ if $S = \emptyset$. Let $\mathfrak{C}_1$ be the canonical collection of $\hat{\phi}$ dense subgroups of G_0. Then from Theorems 2.23 and 2.20 we have that $|\mathfrak{C}_1|$ is finite or $\geq c$. If $|\mathfrak{C}_1| \geq c$ then Lemma 3.33 gives that $|[\tau_1, \tau_2]| \geq c$. So let $|\mathfrak{C}_1|$ be finite. Let $\{F_1, \cdots, F_k\}$ be such a collection $\mathfrak{C}_1$. Then $\hat{G}_0$ has a type. If

$$\mathfrak{J}(\hat{G}_0)) \cap \{p_1, p_2, \cdots, p_n\} \neq \emptyset$$

then there are c subgroups for $\hat{G}$ and following the proof of Lemma 2.22, we get that a canonical collection of $\hat{\phi}$ dense subgroups of $\hat{G}$ has at least c elements. So we will get that $|[\tau_1, \tau_2]| \geq c$ by Lemma 3.33 if

$$\mathfrak{J}(\hat{G}_0) \cap \{p_1, p_2, \cdots, p_n\} \neq \emptyset.$$

Suppose now that $\mathfrak{J}(\hat{G}_0) \cap \{p_1, p_2, \cdots, p_n\} = \emptyset$. Then by standard structure theorems in abelian groups we get that if $M \subset \hat{G}$ is a subgroup then M is equivalent in the sense of Definition 2.1 to a subgroup $H_S \oplus M_1$ where $S \subset \{p_1, \cdots, p_n\}$ and M_1 is a subgroup of $\hat{G}_0$. So if we put

$$\mathfrak{C} = \{H_S + F_i \mid S \subset \{p_1, \cdots, p_n\} \quad \text{and} \quad i = 1, 2, \cdots, k\}$$

then $\mathfrak{C}$ is a canonical collection of $\hat{\phi}$ dense subgroups of $\hat{G}$. So $|\mathfrak{C}| = 2^n |\mathfrak{C}_1|$ and hence is finite. So $|[\tau_1, \tau_2]|$ is finite by Lemma 3.33. Thus we get the theorem.

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