# RELATIONS BETWEEN THE COVERING HOMOTOPY AND SLICING STRUCTURE PROPERTIES

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## 1. Introduction

Maps which have the slicing structure property (SSP) as defined in [6] are a very strong type of fiber map. Unlike Serre and Hurewicz fibrations they are not defined using a covering homotopy property. However it is known [8], [15] that Hurewicz fibrations over locally equiconnected spaces have the SSP. As a matter of fact in [15] West showed that a paracompact space $B$ is locally equiconnected iff every Hurewicz fibration over $B$ has the SSP.

The SSP was used in [2], [7] and [11] in connection with local homogeneity. In [7] it was used to determine homology and homotopy groups of spaces associated with a locally homogeneous space. In [2] and [11] it was used to show that locally homogeneous spaces are like manifolds.

In [1], [3] and [13] sufficient conditions for a map to be a Hurewicz fibration are given. In all three papers it was first proven that the map had the SSP and hence if the base is paracompact then the map is a Hurewicz fibration. The fact that Hurewicz fibrations over ANR's have the SSP was used by Raymond in [10] to show the local triviality of a map. In [9] Mostert used the SSP to study light maps and quotients of topological groups.

In all of the above, except [7], it should be noted that the SSP was used to obtain topological results, not algebraic ones. Maps with the SSP are topologically easier to work with than Hurewicz fibrations.

The purpose of this paper is to study the topological structure of maps with the SSP. In particular several sufficient conditions, depending on various types of covering homotopy property, are given for a map to have the SSP. Also those Hurewicz fibrations which have the SSP are characterized by the existence of a special lifting function. An interesting corollary of the above is (3.9) which gives a very weak condition for a map to be a singular fiber map.

## 2. Definitions and notation

In the remainder of this paper $p$ will be a map from a space $E$ to a space $B$. Conditions will be placed on $E$, $p$ or $B$ as needed. The term map is used to denote a continuous function.

All function spaces will have the compact open topology and a subbasic open set will be denoted by $(C, U)$ where $C$ is compact and $U$ is open.

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A net in a space $X$ will be denoted by a pair $(D, \phi)$ where $D$ is a directed set and $\phi$ a function from $D$ to $X$.

(2.1) DEFINITION. A space $X$ is *locally equiconnected* if for each point $x$, there exists a neighborhood $U_x$ of $x$ and a map

$$N : U_x \times U_x \times I \to X$$

satisfying $N(a, b, 0) = a$, $N(a, b, 1) = b$ and $N(a, a, t) = a$. The neighborhood $U_x$ is called an *equiconnected neighborhood* of $x$.

(2.2) DEFINITION. The map $p$ has the *covering homotopy property* $(CHP)$ *for a class of spaces* if given any space $X$ in the class and maps $F : X \times I \to B$ and $g : X \to E$ such that $F(x, 0) = pg(x)$ then there exists a map $G : X \times I \to E$ such that $pG = F$ and $G(x, 0) = g(x)$. The map $p$ has the *regular CHP for the class* if $G$ can be chosen such that $G_x$ is constant whenever $F_x$ is.

(2.3) DEFINITION. A map $p$ is a *Hurewicz fibration* if the map

$$g : E^I \to \Omega_p = \{(e, w) \in E \times B^I \mid p(e) = w(0)\}$$

defined by $g(w) = (w(0), pw)$ admits a section (i.e. a map $\lambda : \Omega_p \to E^I$ such that $g\lambda =$ identity). The section is called a *lifting function.*

(2.4) DEFINITION. The map $p$ has the *slicing structure property* $(SSP)$ if for each point $b \in B$ there exists a neighborhood $U_b$ of $b$ and a map

$$\Psi_b : p^{-1}(U_b) \times U_b \to p^{-1}(U_b)$$

such that (1) $\Psi_b(e, p(e)) = e$ and (2) $p\Psi_b = \pi_2$ (the projection on $U_b$). The map $\Psi_b$ is called a *slicing function.*

## 3. Various types of CHP and the SSP

All of the results of this section are consequences of the following.

(3.1) THEOREM. *If $B$ is locally equiconnected and $p$ has the regular CHP for spaces of the form $p^{-1}(U) \times U$ where $U$ is a neighborhood in $B$ then $p$ has the SSP.*

*Proof.* Let $b$ belong to $B$ and let $U$ be an equiconnected neighborhood of $b$ with connecting function $N : U \times U \times I \to B$. Define

$$H : p^{-1}(U) \times U \times I \to B$$

by $H(e, b, t) = N(p(e), b, t)$. Define

$$g : p^{-1}(U) \times U \to E$$

by $g(e, b) = e$ and note that $pg(e, b) = H(e, b, 0)$. Since $p$ has the regular CHP for $p^{-1}(U) \times U$ there exists a map

$$G : p^{-1}(U) \times U \times I$$

such that $G_x$ is constant whenever $H_x$ is and such that the following diagram commutes.

$$\begin{tikzcd}
 & & E \arrow[d, "p"] \\
p^{-1}(U) \times U \times 0 \arrow[r] \arrow[urr, "g"] & p^{-1}(U) \times U \times I \arrow[r, "H"] \arrow[ur, dashed, "G"] & B
\end{tikzcd}$$

Define $\phi : p^{-1}(U) \times U \to p^{-1}(U)$ by $\phi(e, b) = G(e, b, 1)$. It is easily seen that $\phi$ is a slicing function for $p$.

The following definition will be needed for the next theorem.

(3.2) Definition. Let $P$ be a topological property of topological spaces. A space $X$ is *locally* $P$ if given any point $x$ in $X$ and any neighborhood $U$ of $x$ there exists a neighborhood $V$ of $x$ such that $V$ is $P$ and $V \subset U$. (Neighborhoods are not necessarily open.)

(3.3) Theorem. *Let $P$ be a topological property preserved by finite products, and let $B$ be a locally equiconnected space which is locally $P$. If $p$ has the regular CHP for $P$ subspaces of $E \times B$ and $p^{-1}$ of a $P$ set is a $P$ set then $p$ has the SSP.*

*Proof.* Let $b$ be a point of $B$ and $U$ an equiconnected neighborhood of $b$. Since $B$ is locally $P$ there exists a neighborhood $V$ of $b$ such that $V$ is $P$ and $V \subset U$. By the hypothesis $p^{-1}(V)$ and $p^{-1}(V) \times V$ are $P$, hence $p$ has the regular CHP for $p^{-1}(V) \times V$. To finish the proof we need only note that $V$ is an equiconnected neighborhood of $b$ and hence (3.1) can be applied to get a slicing function over $V$.

A simple modification of a theorem in [8] yields

(3.4) Theorem. *If $p$ has the CHP for a space $X$ and $B$ is metric then $p$ has the regular CHP for $X$.*

Combining (3.3) and (3.4) we get the following corollaries.

(3.5) Corollary. *If $p$ is a compact map with the CHP for compact spaces and $B$ is a locally compact, locally equiconnected metric space then $p$ has the SSP.*

(3.6) Corollary. *If $p$ is a* 0*-regular map from a Peano continuum to a locally equiconnected metric space and $p$ has the CHP for Peano continua then $p$ has the SSP.*

*Proof.* This follows from the fact that 0-regular maps preserve Peano continua under inversion [14].

(3.7) Corollary. *If $E$ is a metric space such that* $\dim E \leq n$, *$B$ a locally equiconnected metric space with* $\dim B \leq n$ *and $p$ has the CHP for spaces of dimension less than or equal to* $2n$ *then $p$ has the SSP.*

It should be noted that (3.7) contains the entire gist of Theorem 1 [13]. The next corollary follows from (3.4) and the proof of (3.1).

(3.8) COROLLARY. *Let $p$ be a map from a space $E$ to a $k$-manifold $B$. If $p$ has the CHP for the collection of spaces $\{U \times E^k\}$ where $U$ is open in $E$ then $p$ has the SSP.*

Using a theorem of Bing [4] we obtain the following interesting consequence.

(3.9) COROLLARY. *If $B$ is a Peano continuum and $p$ has the CHP for subsets of $E \times B$ then there exists a dense arcwise connected subset $C$ of $B$ such that $p \mid p^{-1}(C)$ has the SSP with one slicing function.*

*Proof.* This follows since Bing's theorem says that $B$ has a dense equiconnected subspace.

## 4. Local and almost local arcwise connectivity

In this section almost local arcwise connectivity will be defined. It is weaker than local arcwise connectivity and will be used in Section 5.

The following definition, note, lemma and theorem, may be known. However, I have not been able to find any reference to them and hence am including them for completeness sake. I want to thank P. McAuley for pointing out (4.4).[1]

(4.1) DEFINITION. A map $p : E \to B$ is *open at* $e$ if the image of any neighborhood of $e$ is a neighborhood of $p(e)$ (in $B$).

(4.2) *Note.* A map $p : E \to B$ is open iff it is open at each point of $E$.

(4.3) LEMMA. *A map $p : E \to B$ is open at $e_0$ iff given any net $(D, \phi)$ in $B$ which converges to $p(e_0) = b_0$ then there exists a net $(A, \psi)$ in $E$ which converges to $e_0$ and such that $(A, p\psi)$ is a subnet of $(D, \phi)$.*

*Proof.* If $p$ is open at $e_0$, let $(D, \phi)$ converge to $b_0$. Let $M$ be the neighborhood system of $e_0$ and order $A = M \times D$ by $(m, d) \leq (n, e)$ if $m \leq n$ and $d \leq e$. Then $(A, \leq)$ is a directed set with this order. Define $\alpha : A \to D$ as follows: Let $\alpha(m, d)$ be an element of $D$ such that $\alpha(m, d) \geq d$ and if $d' \geq \alpha(m, d)$ then $\phi(d') \in p(m)$. Let $\psi : A \to E$ be defined by $\psi(m, d)$ is a point in $m \cap p^{-1}\phi\alpha(m, d)$. I now claim that $(A, \psi)$ is the desired net.

First note that $(A, \psi)$ converges to $e_0$ since $\psi(m, d) \in m$. Secondly note that $(A, p\psi) = (A, \phi\alpha)$, hence all that must be shown to complete the proof is that given any $d_0$ in $D$ there exists $(m_1, d_1)$ in $A$ such that if $(m, d) \geq (m_1, d_1)$ then $\alpha(m, d) \geq d_0$. To do this let $m_1$ be anything and let $d_1 = d_0$, then it is seen by the construction that everything works.

The proof of the converse is trivial and hence omitted.

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[1] The referee said that he has seen (4.4) in notes by E. Fadell but has not seen it in publication.

(4.4) THEOREM. *A space $B$ is locally arcwise connected iff*

$$q : B^I \to B \times B$$

*defined by* $q(w) = (w(0), w(1))$ *is open.*

*Proof.* Assume that $q$ is open. Let $b_0$ be a point of $B$ and $U$ a neighborhood of $b_0$. Then $(I, U)$ is a neighborhood of the constant path at $b_0$ and $q(I, U)$ is an open subset of $U \times U$. Hence there exists an open set $V$ containing $b_0$ such that $V \times V \subset q(I, U)$. Now if $v$ is in $V$ there exists $w \in (I, U)$ such that $w(0) = b_0$ and $w(1) = v$. Therefore $v$ can be joined by a path in $U$ to $b_0$.

For the converse assume that $B$ is locally arcwise connected.

Let $U$ be open in $B^I$, and let $w \in U$. It is easily seen that there exists a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$ and open sets $U_1, \cdots, U_k$ in $B$ such that $w \in \bigcap_{i=1}^k ([t_{i-1}, t_i], U_i) \subset U$. Since $B$ is locally arcwise connected and $w(0) \in U_1$ there exists a neighborhood $V_1 \subset U_1$ of $w(0)$ such that if $v \in V_1$ then $v$ can be joined by an arc to $w(0)$ in $U_1$. In like manner there exists a neighborhood $V_k \subset U_k$ of $w(1)$ such that if $v \in V_k$ then $v$ can be joined by an arc to $w(1)$ in $U_k$. To complete the proof it will be shown that

$$V_1 \times V_k \subset q(\textstyle\bigcap_{i=1}^k ([t_{i-1}, t_i], U_i)).$$

Let $(a, b) \in V_1 \times V_k$ and let $\alpha$ be an arc in $U_1$ from $a$ to $w(0)$ and let $\beta$ be an arc in $U_k$ from $w(1)$ to b. Define a path $\gamma$ as follows

$$\begin{aligned}
\gamma(t) &= \alpha(4s/t_1) && \text{if } 0 \le s \le \tfrac{1}{4}t_1 \\
&= w(2s - \tfrac{1}{2}t_1) && \text{if } 0 \le s \le \tfrac{1}{2}t_1 \\
&= w(s) && \text{if } \tfrac{1}{2}t_1 \le s \le (t_{k-1} + 1)/2 \\
&= w(2s - (t_{k-1} + 1)/2) && \text{if } (t_{k-1} + 1)/2 \le s \le (t_{k-1} + 3)/4 \\
&= \beta(4s/(1 - t_{k-1}) + 1 - 4/(1 - t_{k-1})) \\
& && \text{if } (t_{k-1} + 3)/4 \le s \le 1
\end{aligned}$$

From the construction it is easily seen that

$$\gamma \in \textstyle\bigcap_{i=1}^k ([t_{i-1}, t_i], U_i), \quad \gamma(0) = a \quad \text{and} \quad \gamma(1) = b.$$

Therefore $(a, b) \in q(\bigcap_{i=1}^k ([t_{i-1}, t_i], U_i))$.

(4.5) DEFINITION. A space $B$ is *almost locally arcwise connected at* $b_0$ if given any net $(D, \phi)$ in $B \times B$ which converges to $(b_0, b_0)$ then there exists a net $(A, \psi)$ in $B^I$ which converges to a path from $b_0$ to $b_0$ and such that $(A, q\psi)$ is a subnet of $(D, \phi)$. (The map $q : B^I \to B \times B$ is defined by $q(w) = (w(0), w(1))$ as in (4.4)). The space $B$ is *almost locally arcwise connected* if it is almost locally arcwise connected at each point.

(4.6) THEOREM. *If $B$ is locally arcwise connected then $B$ is almost locally arcwise connected.*

*Proof.* This follows from (4.3) and (4.4).

(4.7) LEMMA. *A space $B$ is almost locally arcwise connected iff given any point $(b_0, b_1)$ in $B \times B$ such that there exists a path from $b_0$ to $b_1$ and any net $(D, \phi)$ in $B \times B$ which converges to $(b_0, b_1)$ then there exists a net $(A, \psi)$ in $B^I$ which converges to a path from $b_0$ to $b_1$ such that $(A, q\psi)$ is a subnet of $(D, \phi)$.*

*Proof.* The sufficiency of the above condition is obvious. In order to prove the necessity let $w$ be a path from $b_0$ to $b_1$ and let $(D, \phi)$ be a net in $B \times B$ which converges to $(b_0, b_1)$. Define $\Gamma_0 : D \to B \times B$ by $\Gamma_0(d) = (\pi_1\phi(d), b_0)$ and note that $(D, \Gamma_0)$ converges to $(b_0, b_0)$. Hence there exists a net $(A_0, \psi_0)$ in $B^I$ which converges to a path from $b_0$ to $b_0$ and such that $(A_0, q\psi_0)$ is a subnet of $(D, \Gamma_0)$. Let $N : A_0 \to D$ be the function such that $q\psi_0 = \Gamma_0 N$ and for each $d$ in $D$ there is $a$ in $A_0$ such that if $a' \geq a$ then $N(a') \geq d$. Such a function exists by the definition of subnet.

Define $\Gamma_1 : A_0 \to B \times B$ by $\Gamma_1(a) = (b_1, \pi_2 \phi N(a))$ and note $(A_0, \Gamma_1)$ converges to $(b_1, b_1)$. Hence there exists a net $(A, \psi_1)$ in $B^I$ which converges to a path from $b_1$ to $b_1$ and such that $(A, q\psi_1)$ is a subnet of $(A_0, \Gamma_1)$. Let $M : A \to A_0$ be the function which exists by the definition of subnet.

Define $\psi : A \to B^I$ by $\psi(a) = \psi_0(M(a))\cdot w \cdot \psi_1(a)$. I claim that $(A, \psi)$ is the desiret net. This is easily checked and hence the proof is complete.

The following theorem together with (4.6) gives the reason for the name almost local arcwise connected.

(4.8) THEOREM. *If $B$ is almost locally arcwise connected then*

$$q : B^I \to B \times B$$

*(defined as in (4.4)) is quasi-compact.*

*Proof.* Let $U = q^{-1}q(U)$ be an open inverse set and assume that $q(U)$ is not open. Then there exists $w$ in $U$ and a net $(D, \phi)$ in $B \times B \backslash q(U)$ which converges to $(w(0), w(1))$. Then by (4.7) there exists a net $(A, \psi)$ in $B^I$ which converges to a path $\alpha$ from $w(0)$ to $w(1)$ and such that $(A, q\psi)$ is a subnet of $(D, \phi)$. Note that since $U = q^{-1}q(U)$ and $w$ is in $U$ then $U$ contains all paths from $w(0)$ to $w(1)$. In particular $\alpha$ is in $U$. Hence $(A, \psi)$ is eventually in $U$ and so $(A, q\psi)$ is eventually in $q(U)$ but this is a contradiction to the fact that $(D, \phi)$ was a net in $B \times B - q(U)$. Therefore $q$ is quasi-compact as desired.

(4.9) THEOREM. *If $B$ is contractible then $B$ is almost locally arcwise connected.*

*Proof.* Let $H : B \times I \to B$ be a contraction. That is $H(b, 0) = b_0$ and $H(b, 1) = b$. If $(b_1, b_2) \in B \times B$ let $w_{(b_1, b_2)} : I \to B$ be defined as follows

$$w_{(b_1, b_2)}(t) = H(b_1, -2t + 1) \quad \text{if } t \leq \tfrac{1}{2}$$
$$= H(b_2, 2t - 1) \quad \text{if } t \geq \tfrac{1}{2}.$$

Note if $(D, \phi)$ is a net in $B \times B$ converging to a point $b$ then $(D, \psi)$ where $\psi(d) = w_{\phi(d)}$ is a net in $B^I$ converging to $w_b$ and hence the proof is complete.

(4.10) COROLLARY. *The cone of any space is almost locally arcwise connected.*

## 5. Really regular fiber maps

The concept of really regular, given below, combines the ideas of regular as they refer to covering maps and fiber maps. It turns out that the local really regular fiber maps are "almost" the maps with the SSP.

(5.1) DEFINITION. A Hurewicz fibration is *really regular* if there exists a lifting function $\lambda$ such that:

(a) If $\sigma \sim \tau$ (rel end pts) then $\lambda(e, \sigma)(1) = \lambda(e, \tau)(1)$

(b) If $\sigma$ is constant, then $\lambda(e, \sigma)(0) = \lambda(e, \sigma)(1)$.

Such a $\lambda$ will be called a really regular lifting function for $p$. It should be noted that any light Hurewicz fibration is really regular. This follows from Theorem 3.1 [5]. There are regular fibrations which are not really regular as is shown in a later example. The exact relation between really regular and SSP is unknown and partial answers are given below.

(5.2) DEFINITION. A map $p : E \to B$ is a local really regular fibration if for each $b$ in $B$ there exists a neighborhood $U$ of $b$ such that $p \mid p^{-1}(U)$ is a really regular fibration.

(5.3) THEOREM. *If $p$ has the SSP with one slicing function then $p$ is really regular.*

*Proof.* Let $\psi : E \times B \to E$ be the slicing function for $p$. Define $\lambda : \Omega_p \to E^I$ by $\lambda(e, w)(t) = \psi(e, w(t))$. The map $\lambda$ is a really regular lifting function.

In (5.5) we obtain a partial converse of the above; however first the following is needed.

(5.4) LEMMA. *A map $p : E \to B$ is continuous if given any net $(D, \phi)$ in $E$ converging to $e$ there exists a subnet $(A, \psi)$ converging to $e$ such that $(A, p\psi)$ converges to $p(e)$.*

*Proof.* Assume that $p$ is not continuous. Then there exists a net $(D, \phi)$ in $E$ converging to $e$ such that $(D, p\phi)$ does not converge to $p(e)$. Then there exists a neighborhood $U$ of $p(e)$ such that if $d$ is in $D$ there is a $d' \geq d$ such that $p\phi(d')$ does not belong to $U$. Let $D' = \{d \in D \mid p\phi(d) \notin U\}$. $(D', \phi \mid D')$ is a subnet of $(D, \phi)$ by the above and hence $(D', \phi \mid D')$ converges to $e$.

However there is no subnet $(A, \psi)$ of $(D', \phi \mid D')$ such that $(A, p\psi)$ converges to $p(e)$.

(5.5) THEOREM. *Let $B$ be a simply connected arcwise connected, almost locally arcwise connected space. Then any really regular fibration over $B$ has the SSP with one slicing function.*

*Proof.* Let $\lambda$ be a really regular lifting function for $p$, and define $\psi : E \times B \to E$ by $\psi(e, b) = \lambda(e, w)(1)$ where $w$ is any path from $p(e)$ to $b$. That $\psi$ is well defined and satisfies properties (1) and (2) of SSP follows from the simple connectivity of $B$ and the definition of really regular. That $\psi$ is continuous is shown as follows: Let $(D, \phi)$ be a net in $E \times B$ which converges to $(e_0, b_0)$. Define

$$\Gamma : D \to B \times B$$

by $\Gamma(d) = (p\pi_1 \phi(d), \pi_2 \phi(d))$ where $\pi_1$ and $\pi_2$ are the projections of $E \times B$ to $E$ and $B$ respectively. By the continuity of $p$, $\pi_1$ and $\pi_2$ we have that $(D, \Gamma)$ is a net in $B \times B$ which converges to $(p(e_0), b_0)$. Hence by (4.7) there exists a net $(A, \Omega)$ in $B^I$ which converges to a path $w_0$ from $p(e_0)$ to $b_0$ and such that $(A, q\Omega)$ is a subnet of $(D, \Gamma)$. Since $(A, q\Omega)$ is a subnet of $(D, \Gamma)$ there exists a function $N : A \to D$ such that $q\Omega = \Gamma N$ and if $d$ is in $D$ there exists $a$ in $A$ such that if $a' \geq a$ then $N(a') \geq d$. Let $\Delta : A \to \Omega_p$ by $\Delta(a) = (\pi_1 \phi N(a), \Omega(a))$. Then $(A, \Delta)$ is a net in $\Omega_p$ which converges to $(e_0, w_0)$, and since $\lambda$ is continuous $\lambda(A, \Delta)$ converges to $\lambda(e_0, w_0)$. Therefore $\lambda(A, \Delta)(1)$ converges to $\lambda(e_0, w_0)(1) = \psi(e_0, b_0)$. This is sufficient for the continuity of $\psi$ by (5.4) and the fact that $\lambda(\Delta(a)) = \psi(\phi(N(a)))$.

Again from Bing's theorem we obtain

(5.6) COROLLARY. *Let $B$ be a Peano continuum. If $p$ is really regular then there exists a dense arcwise connected subset $C$ of $B$ such that $p \mid p^{-1}(C)$ has the SSP with one slicing function.*

Localizing (5.3) we obtain

(5.7) THEOREM. *If $p$ has the SSP then $p$ is a local really regular fibration.*

Localizing (5.5) we obtain

(5.8) THEOREM. *Let $B$ be a semi-locally simply connected, almost locally arcwise connected space. If $p$ is a local really regular fibration then $p$ has the SSP.*

(5.9) DEFINITION. A map $p$ is a *really regular local fibration* if there exists a regular lifting function $\lambda$ such that given any $b$ in $B$ there exists a neighborhood $U$ of $b$ such that if $\sigma \sim \tau$ in $U$ (rel end points) and $p(e) = \sigma(0) = \tau(0)$ then $\lambda(e, \sigma)(1) = \lambda(e, \tau)(1)$.

From the definitions it is easily seen that a really regular local fibration is a local really regular fibration. From (5.8) and the above remark we obtain

(5.10) THEOREM. *Let $B$ be a semi-locally simply connected, almost locally arcwise connected space. Then a really regular local fibration over $B$ has the SSP.*

I do not know if local really regular fibrations (presumably over paracompact spaces) are really regular local fibrations. If they are then (5.7) and (5.11) would give a characterization of SSP in terms of lifting functions.

Really regular fibrations have the following property enjoyed by maps with the SSP with one slicing function.

(5.11) THEOREM. *If $p$ is really regular then $p^* : \pi_n(E) \to \pi_n(B)$ is an epimorphism for $n \geq 2$ and if $F$ is 0-connected then it is an epimorphism for $n = 1$.*

*Proof.* Let $f : (I^n, \dot{I}^n) \to (B, b_0)$. For $x$ in $I^{n-1}$ let $\omega_x(t) = f(x, t)$. Note $\omega_x(0) = f(x, 0) = b_0 = f(x, 1) = \omega_x(1)$. Therefore $\omega_x$ is a loop and it is homotopic (rel end pts) to $b_0$. Therefore if $\lambda$ is a really regular lifting function for $p$, $\lambda(e_0, \omega_x)(1) = \lambda(e_0, b_0)(1) = e_0$. Hence define $g : (I^n, \dot{I}^n) \to (E, e_0)$ by $g(x, t) = \lambda(e_0, \omega_x)(t)$ and note $p^*[g] = [pg] = [f]$.

(5.12) COROLLARY. *If $p$ is really regular then*

$$i^* : \Pi_n(F) \to \Pi_n(E)$$

*is a monomorphism for $n \geq 1$.*

*Proof.* Trivial.

(5.13) *Example.* Let $B$ be an arcwise connected metric space such that $\Pi_n(B) \neq 0$ for some $n \geq 2$. Define $p : (B, b_0)^{(I,0)} \to B$ by $p(\omega) = \omega(1)$. Then $p$ is a regular fibration which is not really regular.

*Proof.* $(B, b_0)^{(I,0)}$ is contractible and therefore

$$p^* : \Pi_n(B, b_0)^{(I,0)} \to \Pi_n(B, b_0)$$

is not onto.

In the case of light maps a large number of those with the SSP are locally trivial. This follows from the following two theorems.

(5.14) THEOREM. *If $p$ has the SSP then $p$ has strong local sections.*

*Proof.* If $b \in B$ there is a neighborhood $U_b$ of $b$ and a slicing function

$$\psi_b : p^{-1}(U_b) \times U_b \to U_b .$$

Let $e \in p^{-1}(U_b)$ and define $\phi_e : U_b \to E$ by $\phi_e(\omega) = \psi(e, \omega)$. Then $p\phi_e(\omega) = p\psi(e, \omega) = \omega$ and $\phi_e(p(e)) = \psi(e, p(e)) = e$.

(5.15) THEOREM. *If $p$ is an a-light map with the SSP and if $B$ is an arcwise* *connected, uniformly locally arcwise connected metric space and if* $p^{-1}(b)$ *is compact for all* $b$ *in* $B$, *then* $p$ *has the BP.*

*Proof.* This follows directly from (5.14) and Theorem 4.2 [12].

BIBLIOGRAPHY


1. D. Addis, *A generalization of completely regular mapping*, to appear.
2. T. Akasaki, *On a kind of homotopy manifold*, Canad. J. Math., vol. 19 (1967), pp. 361–365.
3. J. E. Arnold, *Local to global theorems in the theory of Hurewicz fibrations*, Trans. Amer. Math. Soc., vol. 164 (1972), pp. 179–188.
4. R. H. Bing, *A convex metric with unique segments*, Proc. Amer. Math. Soc., vol. 4 (1953), pp. 167–174.
5. M. S. Curtis, *The covering homotopy theorem*, Proc. Amer. Math. Soc., vol. 7 (1956), p. 682–684.
6. S. T. Hu, *On generalizing the notion of fiber spaces to include the fibre bundles*, Proc. Amer. Math. Soc., vol. 1 (1950), pp. 756–762.
7. ———, *Fiberings of enveloping spaces*, Proc. London Math. Soc. Ser. III, vol. 11 (1961), pp. 691–707.
8. W. Hurewicz, *On the concept of a fiber space*, Proc. Nat. Acad. Sci. U.S.A., vol. 41 (1955), pp. 956–961.
9. P. S. Mostert, *Fibre spaces with totally disconnected fibres*, Duke Math. J., vol. 21 (1954), pp. 67–74.
10. F. Raymond, *Local triviality for Hurewicz fiberings of manifolds*, Topology, vol. 3 (1965), pp. 43–57.
11. G. S. Ungar, *Local homogeneity*, Duke Math. J., vol. 34 (1967), pp. 693–700.
12. ———, *Light fiber maps*. Fund. Math, vol. 62 (1968), pp. 31–45.
13. ———, *Conditions for a mapping to have the slicing structure property*. Pacific J. Math., vol. 30 (1969), pp. 549–553.
14. ———, *Pseudoregular mappings*, Colloq. Math., vol. 19 (1968), pp. 225–229.
15. D. West, Thesis, University of Wisconsin, 1965.


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