FIXED POINT SETS OF CONJUGATIONS

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Abstract

An examination of the generators shows that every manifold is cobordant to the fixed point set of a conjugation on an almost complex manifold. Equivariant surgery is used to show that every 3-manifold is diffeomorphic to such a fixed point set.

I. Introduction

Let M^{2n} be an almost complex manifold with a.c. structure $J : \tau M^{2n} \to \tau M^{2n}$ on its tangent bundle. A conjugation on M^{2n} is a smooth involution $T : M^{2n} \to M^{2n}$ whose differential anti-commutes with J; i.e., $JT_* = -T_*J$. It is easy to verify (see [2]) that if x is a fixed point of T, then the linear map $T_* : \tau M_x \to \tau M_x$ defines an isomorphism between the tangent and normal bundles to the fixed point set, and hence that the fixed point set F^n is a submanifold of dimension n.

Examples. (i) Let $T : CP^n \to CP^n$ be given by conjugation of coordinates. Then T is a conjugation with fixed point set P^n.

(ii) Let $T : CP^n \times CP^n \to CP^n \times CP^n$ be given by $T([Z], [W]) = ([\bar{W}], [\bar{Z}])$, where $W = (w_0, \cdots, w_n)$, $Z = (z_0, \cdots, z_n)$, and the bar denotes complex conjugation of coordinates. Then T is a conjugation with fixed point set $F^{2n} = \{([Z], [\bar{Z}])\}$, diffeomorphic to CP^n.

(iii) Let S^7 be the unit Cayley numbers and $S^6 \subset S^7$ the Cayley numbers orthogonal to the Cayley unit. Thus, $S^6 = \{(q_1, q_2) \mid \bar{q}_1 = -q_1\}$ where q_1, q_2 are quaternions, $\| (q_1, q_2) \| = 1$. If S^6_c denotes the tangent space at $c \in S^6$, the mapping $J_c : \tau S^6_c \to \tau S^6_c$ given by $J_c(d) = dc$ (Cayley multiplication) is an almost complex structure on S^6. The involution $T : S^6 \to S^6$ given by $T(q_1, q_2) = (\bar{q}_1, q_2)$ is a conjugation of J with fixed point set $\{0, q_2)\}$, which is S^3.

We wish to determine which closed manifolds can be realized as fixed point sets of conjugations on closed, almost complex manifolds. We first recall a few facts about surgery, which is the principal tool used in the proofs.

Let $f : S^p \times D^{q+1} \to M^n$, $-1 \leq p < n$, be a smooth imbedding, with $n = p + q + 1$. $S^p \times S^q$ can be considered as the boundary of $S^p \times D^{q+1}$ or the boundary of $D^{p+1} \times S^q$. Let $\chi(M^n, f)$ be the manifold obtained by removing the interior of $S^p \times D^{q+1}$ and replacing it with the interior of $D^{p+1} \times S^q$. Then $\chi(M^n, f)$ is said to be obtained from M^n by a surgery of type

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$(p + 1, q + 1)$. M is χ-equivalent to M if it is obtained from M by a finite number of such surgeries. We will use the following results due to Milnor:

THEOREM A. *Every manifold is cobordant to the fixed point set of a conjugation.*

THEOREM B. *Two manifolds are χ-equivalent if and only if they belong to the same cobordism class.*

Theorem A is essentially a consequence of Lemma 1 of [5], where it is shown the generators for the non-oriented cobordism ring $\mathfrak{N}_*$ can be taken to be the manifolds $P^{2n}(R)$ and $H_{m,n}(R)$, which are themselves fixed point sets of conjugations.

It follows from this theorem that one might attempt to do surgery within a cobordism class in order to realize arbitrary manifolds as fixed point sets of conjugations. This is possible in dimension three.

THEOREM. *Every closed 3-manifold is the fixed point set of a conjugation on an almost complex manifold.*

The proof of this theorem will occupy the bulk of this paper. We begin with some notation and terminology from [1].

DEFINITION.

$$R^{p,q} = R^q + iR^p.$$

$$D^{p,q} = \text{unit ball in } R^{p,q}.$$

$$S^{p,q} = \text{unit sphere in } R^{p,q}.$$

An Atiyah-real vector bundle $\xi \to X$ is a complex vector bundle with involution covering an involution T on X, and such that the induced map $\xi_x \to \xi_{T(x)}$ is a conjugate linear isomorphism of complex vector spaces. An almost complex manifold with conjugation, briefly, a conjugation, is a manifold with smooth involution whose differential defines an Atiyah-real structure on the tangent bundle.

II. Equivariant surgery

Let M^n be the fixed point set of a conjugation on W^{2n} and let i be the inclusion.

LEMMA 1. *Let $f : S^p \to M^n$, $0 \leq p < n$, be an embedding. If the normal bundle of f is trivial, then the normal bundle of if is trivial.*

Proof. If $\nu(f)$ is trivial, then $f^*(\tau(M))$ is trivial, i.e., $\cong \theta^n$. Furthermore,

$$\nu(if) \oplus \tau(S^p) \cong (if)^*\tau(W) \cong f^*(\tau(M)) \oplus f^*(\tau(M)) \cong \theta^{2n},$$

since over M we have $\tau(W) \cong \tau(M) \oplus \tau(M)$. Since $p < \dim(\nu(if))$, the lemma follows.

COROLLARY. *Any embedding* $f : S^{0,p+1} \times D^{0,n-p} \to M^n$ *extends to an equivariant embedding* $F : S^{0,p+1} \times D^{n,n-p} \to W^{2n}$.

Proof. Identify $f^*(\tau(M))$ with the normal bundle of M^n in W^{2n}, obtaining an isomorphism $\nu(if) \cong \nu(f) \oplus f^*(\tau(M))$. The induced involution is the bundle involution on $f^*(\tau(M))$, hence the involution on $\nu(if)$ is as stated.

LEMMA 2. $\chi(W^{2n}, F)$ *carries an involution with fixed point set* $\chi(M^n, f)$.

Proof. The map $S^{0,p+1} \times (D^{n,n-p}\setminus\{0\}) \to (D^{0,p+1}\setminus\{0\}) \times S^{n,n-p}$ defined by $(u, \theta v) \to (\theta u, v)$ is clearly equivariant. Restricting to the fixed point set gives the map $S^{0,p+1} \times (D^{0,n-p}\setminus\{0\}) \to (D^{0,p+1}\setminus\{0\}) \times S^{0,n-p}$. Hence, the fixed point set is precisely $\chi(M^n, f)$.

III. Almost complex structures

We will show that for $n = 3$ the surgery on W^{2n} can be done preserving the almost complex structure and conjugation. Hence, $\chi(M^n, f)$ will be the fixed point set of a conjugation on $\chi(W^{2n}, F)$.

LEMMA 3. *The restriction* $F^*(\tau(W))$ *is a trivial Atiyah-real vector bundle.*

Proof. The restriction of this bundle to $F(S^{0,p+1} \times \{0\})$ is the complexification of $f^*(\tau(M))$, which is trivial. Since $S^{0,p+1} \times \{0\}$ is a strong equivariant deformation retract of $S^{0,p+1} \times D^{n,n-p}$, the lemma is proved.

Now consider the natural equivariant embeddings

$$\alpha_1 : S^{0,p+1} \times D^{n,n-p} \to S^{n,n+1}, \qquad \alpha_2 : D^{0,p+1} \times S^{n,n-p} \to S^{n,n+1}$$

given by

$$\alpha_1(u, (\phi \cos \tfrac{1}{2}\pi\theta)v, i(\phi \sin \tfrac{1}{2}\pi\phi)w) = ((\sin \tfrac{1}{2}\pi\theta \cos \tfrac{1}{2}\pi\phi)u, (\cos \tfrac{1}{2}\pi\theta \cos \tfrac{1}{2}\pi\phi)v, i(\sin \tfrac{1}{2}\pi\phi)w)$$

and

$$\alpha_2(\phi u, (\cos \tfrac{1}{2}\pi\theta)v, i(\sin \tfrac{1}{2}\pi\theta)w) = ((\sin \tfrac{1}{2}\pi\theta \cos \tfrac{1}{2}\pi\phi)u, (\cos \tfrac{1}{2}\pi\theta \cos \tfrac{1}{2}\pi\phi)v, i(\sin \tfrac{1}{2}\pi\phi)w),$$

where $u \in S^P$, $v \in S^{n-p-1}$, $w \in S^{n-1}$, and $0 \leq \phi \leq 1, 0 \leq \theta \leq 1$. Then

$$\alpha_2^{-1}\alpha_1(u, (\phi \cos \tfrac{1}{2}\pi\theta)v, i(\phi \sin \tfrac{1}{2}\pi\theta)w) = (\phi u, (\cos \tfrac{1}{2}\pi\theta)v, i(\sin \tfrac{1}{2}\pi\theta)w),$$

for $0 < \phi \leq 1$, which is the map described in Lemma 2. We can now prove the following:

THEOREM. *Every closed 3-manifold is the fixed point set of a conjugation on a closed, almost complex manifold.*

Proof. Let M^3 be the fixed point set of a conjugation on W^6, and let

$$f : S^{0,p+1} \times D^{0,3-p} \to M^3$$

be an embedding, $0 \leq p < 3$. Then by the corollary to Lemma 1, there is an equivariant extension

$$F : S^{0,p+1} \times D^{3,3-p} \to W^6.$$

The restriction of τ_W to $F(S^{0,p+1} \times (D^{3,3-p} \setminus \{0\}))$ is a trivial Atiyah-real vector bundle, as are the restrictions of the tangent bundle of $S^{3,4}$ to

$$\alpha_1(S^{0,p+1} \times D^{3,3-p}) \quad \text{and} \quad \alpha_2(D^{0,p+1} \times S^{3,3-p}).$$

The differential $(\alpha_2^{-1}\alpha_1)_*$ is almost complex structure preserving, so that $\chi(W^6, F)$ is almost complex. The involution is a conjugation on $W^6 \setminus F(S^{0,p+1} \times \{0\})$ and on $(D^{0,p+1} \setminus \{0\}) \times S^{3,3-p}$, and since the identification map $\alpha_2^{-1}\alpha_1$ is equivariant, a conjugation is defined on $\chi(W^6, F)$. Clearly the fixed point set is $\chi(M^3, f)$. If follows that any manifold obtained from M^3 be a finite sequence of surgeries on embedded spheres of dimension 0, 1, or 2, will again be the fixed point set of a conjugation. Since all closed 3-manifolds can be obtained in this way, the theorem is proved.

References

1. M. Atiyah, *K-theory and reality*, Quart. J. Math., Oxford (2), vol. 17 (1966), p. 331.
2. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Erg. Mat., vol. 33, Springer-Verlag, New York.
3. P. Landweber, *Fixed point free conjugations on complex manifolds*, Ann. of Math., vol. 86 (1967), pp. 491–502.
4. J. Milnor, *A procedure for killing homotopy groups of differentiable manifolds*, Proceedings of Symposium in Pure Math. VIII, pp. 39–55.
5. ———, *Stiefel-Whitney numbers of complex manifolds and of spin manifolds*, Topology, vol. 3, (1965), p. 223.
6. N. Steenrod, *Topology of fibre bundles*, Princeton University Press, Princeton, N.J., 1951.
7. R. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, N.J., 1968.

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