INTERPOLATION THEOREMS FOR THE CLASS N^+

BY
Niro Yanagihara

1. Introduction

Let D be the unit disk $\{ |z| < 1\}$. A function $f(z)$, holomorphic in D, is said to belong to the *class* H^p, $0 < p < \infty$, or H^∞, if

$$(1.1) \qquad \| f \|_p = \{\sup_{0 \leqq r < 1} \frac{1}{2\pi} \int_0^{2\pi} | f(re^{i\theta}) |^p \, d\theta\}^{1/p} < \infty$$

or

$$(1.2) \qquad \| f \|_\infty = \sup_{0 \leqq r<1} \max_{|z|=r} | f(z) | < \infty,$$

respectively.

A function $f(z)$, holomorphic in D, is said to belong to the *class N of functions of bounded characteristic* if

$$(1.3) \qquad T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ | f(re^{i\theta}) | \, d\theta \leqq M < \infty$$

for $0 \leqq r < 1$, with a constant M. A function $f(z)$ of the class N is said to belong to the *class* N^+ if

$$(1.4) \qquad \lim_{r \to 1} \int_0^{2\pi} \log^+ | f(re^{i\theta}) | \, d\theta = \int_0^{2\pi} \log^+ | f(e^{i\theta}) | \, d\theta.$$

Thus, for $0 < p < q < \infty$,

$$(1.5) \qquad H^\infty \subset H^q \subset H^p \subset N^+ \subset N,$$

and these inclusion relations are proper (see [9, p. 82], where N and N^+ are denoted as A and D, respectively).

Interpolation problems have been studied by several authors. For H^∞, by Carleson [1], Hayman [5], and Newman [8]; for H^p, $1 \leqq p < \infty$, by Shapiro and Shields [10]; for H^p, $0 < p < 1$, by Kabaila [6]; for N, by Naftalevič [7]. (The present author wishes to express his gratitude to Professor Shields for having let him know of the interesting paper [7]. See Math. Reviews, vol. 22 (1961) #11141.)

Here we consider corresponding problems for the class N^+.

2. The interpolation problems

Suppose a class X of holomorphic functions in D be given. Let $\{z_n\}$ be a point sequence in D. When a complex sequence $\{c_n\}$ is given, the problem is to seek a function $f(z) \in X$ such that

$$(2.1) \qquad f(z_n) = c_n \quad \text{for each} \quad n.$$

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Let Y be a collection of complex sequences. Suppose for any sequence $\{c_n\} \in Y$ there is a function $f(z) \in X$ which satisfies (2.1); then the sequence of points $\{z_n\}$ in D is said to be a *universal interpolation sequence for the pair* (X, Y). We write it simply as *u.i.s. for* (X, Y).

We use the following notations: For a sequence $Z = \{z_n\}$ in D, we put

$$(2.2) \qquad l_z^p = \{\{c_n\};\ \textstyle\sum_{n=1}^{\infty} (1 - |z_n|^2)\,|c_n|^p < \infty\}, \quad 0 < p < \infty.$$

In the sequel we suppose that

$$(2.3) \qquad z_n \neq 0, \qquad z_n \neq z_m \ \text{ if } \ n \neq m, \quad |z_n| \to 1 \ \text{ as } \ n \to \infty,$$

and

$$(2.4) \qquad \textstyle\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

We denote by $B_n(z)$ the infinite product

$$(2.5) \qquad B_n(z) = \textstyle\prod_{m \neq n} \{(|z_m|/z_m)((z_m - z)/(1 - \bar{z}_m z))\}.$$

Carleson [1] showed that, $\{z_n\}$ is a u.i.s. for (H^∞, l^∞) if and only if (l^∞ denotes, as usual, the set of all bounded sequences)

$$(2.6) \qquad |B_n(z_n)| = \textstyle\prod_{m \neq n} |(z_m - z_n)/(1 - \bar{z}_m z_n)| \geqq \delta > 0 \quad \text{for all} \quad n.$$

Shapiro and Shields [10] showed that (2.6) is necessary and sufficient also for $\{z_n\}$ to be a u.i.s. for (H^p, l_z^p), $1 \leqq p < \infty$. Kabaila [6] obtained analogous results for $0 < p < 1$.

Recently, Duren and Shapiro [4] showed that there is a u.i.s. for (H^p, l^∞) which does not satisfy the condition (2.6), if $0 < p < \infty$.

Here we put

$$(2.7) \qquad l_z^+ = \{\{c_n\};\ \textstyle\sum_{n=1}^{\infty} (1 - |z_n|^2) \log^+ |c_n| < \infty\}.$$

Then:

THEOREM 1. *In order that a sequence* $Z = \{z_n\}$ *be a u.i.s. for* (N^+, l_z^+), *it is sufficient that* (2.6) *hold, and is necessary that*

$$(2.8) \qquad (1 - |z_n|^2) \log (1/|B_n(z_n)|) \to 0 \quad \textit{as} \quad n \to \infty.$$

Remark 1. As for (2.8), we remark that if we write

$$(2.7') \qquad \hat{l}_z^+ = \{\{c_n\};\ \sup_n ((1 - |z_n|^2) \log^+ |c_n|) < \infty\},$$

then Naftalevič [7, p. 27] proved that Z is a u.i.s. for $(N, \hat{l}_z^+)$ only if

$$(2.8') \qquad \sup ((1 - |z_n|^2) \log (1/|B_n(z_n)|)) < \infty.$$

Remark 2. It is obvious that (2.6) implies (2.8), but the example

$$\{z_n\} = \{1 - n^{-2}\}$$

shows that (2.8) does not imply (2.6).

Further, we put

$$(2.7'') \qquad l_z^{\#} = \{\{c_n\};\, c_n > 0,\ \sum (1 - |z_n|^2)\,|\log c_n| < \infty\}$$

and denote by $N^{\#}$ the set of zero-free holomorphic functions such that

$$(2.9) \qquad f \in N^{\#} \quad \text{means} \quad f(0) > 0 \quad \text{and} \quad \phi(z) = \log f(z) \in H^1,$$

where we take as $\phi(0) =$ real. Obviously, $l_z^{\#} \subset l_z^+$ and $N^{\#} \subset N^+$.

THEOREM 2. *A sequence $Z = \{z_n\}$ is a u.i.s. for $(N^{\#}, l_z^{\#})$, in the sense that for any $\{c_n\} \in l_z^{\#}$ there exists $f \in N^{\#}$ with* $\log f(z_n) = \log c_n$, $n = 1, 2, \cdots$, *if and only if* (2.6) *holds.* (Note that $\log c_n =$ real.)

In [10] and [6], it is shown that if $f(z) \in H^p$, $0 < p < \infty$, then $\{f(z_n)\} \in l_z^p$, i.e.

$$\sum (1 - |z_n|^2)\,|f(z_n)|^p < \infty,$$

supposing $\{z_n\}$ satisfies (2.6). It would be natural to conjecture, as a corresponding statement, that $\{f(z_n)\} \in l_z^+$, i.e.,

$$\sum (1 - |z_n|^2) \log^+ |f(z_n)| < \infty \quad \text{for any} \quad f(z) \in N^+,$$

supposing that $\{z_n\}$ satisfies (2.6).

This is not true (Theorem 3), but a somewhat weaker result holds even for the class N (Theorem 4). That is:

THEOREM 3. *We can find a sequence $\{z_n\}$ satisfying* (2.6), *for which there is a function $f(z) \in N^+$ with*

$$(2.10) \qquad \sum_{n=1}^{\infty} (1 - |z_n|^2) \log^+ |f(z_n)| = \infty.$$

THEOREM 4. *Suppose $\{z_n\}$ satisfies* (2.6). *If $f(z) \in N$, we have*

$$(2.11) \qquad \sum_{n=1}^{\infty} (1 - |z_n|^2)(\log^+ |f(z_n)|)^{1-\delta} < \infty$$

for any δ, $0 < \delta < 1$.

On the other hand, we can find a sequence $\{z_n\}$ in D and a complex sequence $\{c_n\}$ such that $\{z_n\}$ satisfies (2.4) *as well as* (2.6), *and $\{c_n\}$ satisfies, for any δ, $0 < \delta < 1$,*

$$(2.11') \qquad \sum_{n=1}^{\infty} (1 - |z_n|^2)(\log^+ |c_n|)^{1-\delta} < \infty,$$

while there is no function $f(z) \in N$ with $f(z_n) = c_n$, $n = 1, 2, \cdots$.

Remark. Naftalevič [7, p. 13 and p. 17] proved that, if $\{z_n\}$ satisfies (2.4), there is a sequence $\{z_n'\}$ with $|z_n'| = |z_n|$, such that

$$\sum_{n=1}^{\infty} (1 - |z_n'|^2) \log^+ |f(z_n')| < \infty \quad \text{for any} \quad f(z) \in N,$$

and

$$|B_n(z_n')| \geqq \delta > 0 \quad \text{for all} \quad n.$$

3. Proof of Theorem 1

(i) Suppose $\{z_n\}$ satisfies (2.6). For a sequence $\{c_n\} \in l_z^+$, let

$$(3.1) \qquad c_n' = c_n \quad \text{if} \quad |c_n| \geqq 1; \qquad c_n' = 1 \quad \text{if} \quad |c_n| < 1.$$

Then, by (3.1), (2.6) and (2.7), the function

$$g(z) = \sum_{n=1}^{\infty} (1 - |z_n|^2)^2 \log c'_n \frac{B_n(z)}{B_n(z_n)} \frac{1}{(1 - \bar{z}_n z)^2}, \tag{3.2}$$

where we take $-\pi \leqq \arg [c'_n] < \pi$, is holomorphic in D. If we put

$$f_1(z) = \exp [g(z)],$$

$f_1(z)$ is holomorphic in D and $f_1(z_n) = c'_n$, $n = 1, 2, \cdots$. Further,

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})| \, d\theta \\ &\leqq \sum_{n=1}^{\infty} (1 - |z_n|^2)^2 (\log |c'_n| + |\arg [c'_n]|)(1/|B_n(z_n)|) \\ &\quad \times \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - \bar{z}_n z|^2} \, d\theta \\ &\leqq \frac{1}{\delta} \sum_{n=1}^{\infty} (1 - |z_n|^2)(\log^+ |c_n| + \pi) \\ &< \infty; \end{aligned} \tag{3.3}$$

hence $g(z) \in H^1$, therefore $f_1(z) \in N^+$.

Put

$$c''_n = 1 \quad \text{if} \quad |c_n| \geqq 1; \qquad c''_n = c_n \quad \text{if} \quad |c_n| < 1. \tag{3.1$'$}$$

Then, by the theorem of Carleson [1], there is a bounded holomorphic function $f_2(z)$ with $f_2(z_n) = c''_n$. Thus if we put $f(z) = f_1(z) f_2(z)$ then $f(z) \in N^+$ and $f(z)$ satisfies $f(z_n) = c'_n c''_n = c_n$.

(ii) We need some lemmas to obtain the second part of the theorem.

LEMMA 1. *The class N^+ is an F-space in the sense of Banach* [2, p. 51] *with the distance function*

$$\rho(f, g) = \frac{1}{2\pi} \int_0^{2\pi} \log (1 + |f(e^{i\theta}) - g(e^{i\theta})|) \, d\theta \tag{3.4}$$

for $f, g \in N^+$. That is:

(1°) $\rho(f, g) = \rho(f - g, 0)$.

(2°) *Let f_n be functions in N^+ such that $\rho(f, f_n) \to 0$ as $n \to \infty$. Then for any complex number α, $\rho(\alpha f, \alpha f_n) \to 0$ as $n \to \infty$.*

(3°) *Let α, α_n be complex numbers such that $\alpha_n \to \alpha$ as $n \to \infty$. Then for each function $f \in N^+$, $\rho(\alpha_n f, \alpha f) \to 0$ as $n \to \infty$.*

(4°) *N^+ is complete with respect to the metric* (3.4).

LEMMA 2. *The class l_z^+ is an F-space in the sense of Banach with the distance function*

$$\sigma(u, v) = \sum_{n=1}^{\infty} (1 - |z_n|^2) \log (1 + |c_n(u) - c_n(v)|) \tag{3.5}$$

for $u = \{c_n(u)\}, v = \{c_n(v)\} \in l_z^+$.

For the proofs, see [11, Theorem 1] and [12, Theorem 1].

LEMMA 3. *We have, for* $f(z) \in N^+$,

$$(1 - |z|^2) \log (1 + |f(z)|) \leqq 4\rho(f, 0), \quad |z| < 1. \tag{3.6}$$

Proof. The function $\log (1 + |f(z)|)$ is subharmonic if $f(z)$ is holomorphic. Hence for R, $0 < R < 1$,

$$\log (1 + |f(z)|) \leqq \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr \cos (\theta - \phi)} \log (1 + |f(Re^{i\phi})|)\, d\phi$$

$z = re^{i\theta}$, $r < R$. Thus

$$\log (1+ |f(z)|) \leqq \frac{R + r}{R - r} \frac{1}{2\pi} \int_0^{2\pi} \log (1 + |f(Re^{i\phi})|)\, d\phi.$$

Letting $R \to 1$ we have, using the property (1.4) of functions of N^+,

$$(1 - |z|) \log (1 + |f(z)|) \leqq 2\rho(f, 0),$$

and hence (3.6). Q.E.D.

Now we prove the second part of Theorem 1. Let K be the set of functions $f(z) \in N^+$ such that $f(z_n) = 0$, $n = 1, 2, \cdots$. K is easily seen to be a closed subspace of N^+. Put

$$\bar{N}^+ = N^+/K, \qquad \bar{f} = f + K \in \bar{N}^+ \quad \text{for} \quad f \in N^+,$$

and

$$\bar{\rho}(\bar{f}, \bar{0}) = \inf_{f \in \bar{f}} \rho(f, 0), \qquad \bar{\rho}(\bar{f}, \bar{g}) = \bar{\rho}((f - g)^-, \bar{0}).$$

Then $\bar{\rho}$ is a distance function in $\bar{N}^+$, and $\bar{N}^+$ becomes an F-space in the sense of Banach.

For each $u = \{c_n(u)\} = \{c_n\} \in l_z^+$ there corresponds a unique $\bar{f} \in \bar{N}^+$ such that

$$f(z_n) = c_n, \qquad n = 1, 2, \cdots \quad \text{for each} \quad f \in \bar{f}.$$

Write this correspondence as $\bar{T}: \bar{f} = \bar{T}u$. Obviously $\bar{T}$ is linear. We will show that $\bar{T}$ is a closed operator. Suppose $u_n \in l_z^+$, $\sigma(u_n, 0) \to 0$, and

$$\bar{\rho}(\bar{T}u_n, \bar{f}^*) \to 0.$$

We have only to prove that $\bar{f}^* = \bar{0}$, i.e.,

$$f^*(z_k) = 0, \qquad k = 1, 2, \cdots \quad \text{for} \quad f^* \in \bar{f}^*.$$

Put $\bar{T}u_n = \bar{f}_n$. Then, from $u_n = \{c_k(u_n)\} \to 0$, we have

$$f_n(z_k) = c_k(u_n) \to 0 \quad \text{for each} \quad k, \quad \text{as} \quad n \to \infty.$$

Put $f^*(z_k) = c_k$ and $g_n(z) = f_n(z) - f^*(z)$; then

$$g_n(z_k) = c_k(u_n) - c_k, \qquad k = 1, 2, \cdots \quad \text{for each} \quad g_n \in \bar{g}_n.$$

Since $\bar{\rho}(\bar{g}_n, \bar{0}) \to 0$, for any given $\epsilon > 0$ there is an n_0 such that, if $n \geqq n_0$, we can find a $g_n \in \bar{g}_n$ with $\rho(g_n, 0) < \epsilon/4$. Then, by Lemma 3,

$$(1 - |z_k|^2) \log (1 + |c_k(u_n) - c_k|) < \epsilon \quad \text{for each} \quad k.$$

Letting $n \to \infty$ and $\epsilon \to 0$, we get $c_k = 0$, which proves that $\bar{T}$ is closed.

By the closed graph theorem [2, p. 57], we know that $\bar{T}$ is continuous.

Let $u_n = \{c_k(u_n)\}_{k=1}^{\infty}$ be the sequence such that

$$c_k(u_n) = 0 \quad \text{if} \quad k \neq n; \qquad c_n(u_n) = 1.$$

Obviously $\sigma(u_n, 0) \to 0$, hence $\bar{\rho}(\bar{f}_n, \bar{0}) \to 0$, where $\bar{f}_n = \bar{T}u_n$. There are $f_n \in \bar{f}_n$ such that $\rho(f_n, 0) \to 0$, Put $F_n(z) = f_n(z)/B_n(z)$; then $F_n(z) \in N^+$ and $|F_n(e^{i\theta})| = |f_n(e^{i\theta})|$, almost every θ.

Thus, since $f_n(z_n) = 1$ and $\rho(F_n, 0) = \rho(f_n, 0)$,

$$\begin{aligned}(1 - |z_n|^2) \log (1/|B_n(z_n)|) &\leqq (1 - |z_n|^2) \log (1 + |f_n(z_n)/B_n(z_n)|) \\ &= (1 - |z_n|^2) \log (1 + |F_n(z_n)|) \\ &\leqq 4\rho(F_n, 0) \\ &= 4\rho(f_n, 0) \to 0,\end{aligned}$$

which proves (2.8).

4. Proof of Theorems 2

Sufficiency. Take a sequence $\{c_n\} \in l_z^{\#}$. Then $\{b_n\}$, $b_n = \log c_n$ ($\arg [c_n] = 0$), belongs to l_z^1, and by the theorem of Shapiro and Shields [10], there is a function $g(z) \in H^1$ with $g(0) = 0$ and $g(z_n) = b_n$, $n = 1, 2, \cdots$. Hence we put $f(z) = \exp [g(z)]$, we have that $f(z) \in N^{\#}$ and $f(z_n) = c_n$, $n = 1, 2, \cdots$.

Necessity. $l_z^{\#}$ can be considered as a real Banach space with addition and scalar multiplication defined as follows:

(4.1$_1$) $\{c_n\} + \{b_n\}$ is defined to be the sequence $\{c_n b_n\}$.
(4.1$_2$) For a real number λ, $\lambda\{c_n\}$ is defined to be the sequence $\{(c_n)^{\lambda}\}$.
(4.2) $\| \{c_n\} \| = \sum_{n=1}^{\infty} (1 - |z_n|^2) |\log c_n|$.

$N^{\#}$ can also be considered as a real Banach space with addition and scalar multiplication defined as follows:

(4.3$_1$) $f + g$ is defined to be the function whose value at z equals $f(z)g(z)$, i.e., $(f + g)(z) = f(z)g(z)$,

(4.3$_2$) For a real number λ, λf is defined to be the function whose value at z equals $(f(z))^{\lambda}$, i.e., $(\lambda f)(z) = (f(z))^{\lambda}$, $(\lambda f)(0) > 0$,

(4.4) $\|f\| = \sup_{0 \leqq r \leqq 1} (1/2\pi) \int_0^{2\pi} |\log f(re^{i\theta})| \, d\theta = (1/2\pi) \int_0^{2\pi} |\log f(e^{i\theta})| \, d\theta$ where the logarithm is determined by $\arg [f(0)] = 0$.

Now, let P be the set of functions $f(z) \in N^{\#}$ such that $\log f(z_n) = 0$, $n = 1, 2, \cdots$. P is obviously a closed subspace of $N^{\#}$. Let $\bar{N}^{\#} = N^{\#}/P$, $\bar{f} = f + P$. Then $\bar{N}^{\#}$ is a real Banach space with the norm $\|\bar{f}\| = \inf_{f \in \bar{f}} \|f\|$. For each

$u = \{c_n(u)\} = \{c_n\} \in l_z^\#$ there corresponds a unique $\bar{f} \in \bar{N}^\#$ such that

$$\log f(z_n) = \log c_n(u), \qquad n = 1, 2, \cdots, \quad \text{for each} \quad f \in \bar{f}.$$

Write this correspondence as $\bar{S}$, i.e., $\bar{f} = \bar{S}[u]$. Obviously $\bar{S}$ is linear. $\bar{S}$ is shown to be a closed operator, as in (ii) of the proof of Theorem 1. Thus $\bar{S}$ is continuous by the closed graph theorem. Hence we have

$$\| f \| \leqq M' \| u \| \tag{4.5}$$

with a constant M', for an $f \in \bar{f} = \bar{S}[u]$. Obviously

$$(1 - | z |^2) \log f(z) \mid \leqq M'' \| f \| \tag{4.6}$$

with a constant M''.

Let $u_n = \{c_k(u_n)\}_{k=1}^{\infty}$ be a positive sequence such that

$$c_k(u_n) = 1 \quad \text{if} \quad k \neq n; \qquad c_n(u_n) = e.$$

Then $\| u_n \| = (1 - | z_n |^2)$.

Let f_n be a function of $\bar{S}[u_n]$ satisfying (4.5). Put $\arg [B_n(0)] = \alpha_n$ and

$$F_n(z) = \exp [(\log f_n(z))/(e^{-i\alpha_n} B_n(z))].$$

Then $F_n(z) \in N^\#$ and $| \log F_n(e^{i\theta}) | = | \log f_n(e^{i\theta}) |$, a.e. Thus

$$(1 - | z_n |^2) \mid \log F_n(z_n) \mid \leqq M'' \| F_n \| = M'' \| f_n \| \leqq M'M''(1 - | z_n |^2).$$

On the other hand

$$| \log F_n(z_n) | = | \log f_n(z_n) | / | B_n(z_n) | = 1/ | B_n(z_n) |.$$

Hence $| B_n(z_n) | \geqq 1/M'M''$, which proves (2.6).

5. Proof of Theorems 3 and 4.

We say that $\{z_n\}$ is an *exponential sequence* if

$$\lim_{n\to\infty} \sup ((1 - | z_{n+1} |)/(1 - | z_n |)) < 1. \tag{5.1}$$

Such a sequence is easily seen to satisfy (2.6). Further, if $\{z_n\}$ lies on a radius, (5.1) is equivalent to (2.6) [3, p. 155, Theorem 9.2].

Proof of Theorem 3. Take a number b, $0 < b < 1$. Let $\{z_n\}$ be defined by

$$z_n = 1 - b^n, \qquad n = 1, 2, \cdots. \tag{5.2}$$

Put

$$f(z) = \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} h(t)\, dt \right] \tag{5.3}$$

where

$$\begin{aligned} h(t) &= (1/ | t |)(\log (1/ | t |))^{-2}, \qquad &\text{if} \quad | t | \leqq \pi/4, \\ &= 0, &\text{if} \quad | t | > \pi/4. \end{aligned} \tag{5.4}$$

Then $f(z) \in N^+$, and, if $z = re^{i\theta}$,

$$\log^+ | f(z) | = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 2r \cos (\theta - t)} h(t)\, dt.$$

Thus, writing $1 - r_n = \delta_n$, $r_n = | z_n | = z_n$,

$$\begin{aligned} \log^+ | f(z_n) | &\geqq \frac{1}{2\pi} \int_{-\delta_n}^{\delta_n} \frac{1 = r_n^2}{(1 - r_n)^2 + 4r_n \sin^2 (t/2)} h(t)\, dt \\ &\geqq \frac{1}{2\pi} \frac{1 + r_n}{2(1 - r_n)} \int_{-\delta_n}^{\delta_n} h(t)\, dt \\ &\geqq \frac{1}{2\pi} \frac{1}{1 - r_n} \int_0^{\delta_n} h(t)\, dt \\ &= \frac{1}{2\pi} (1 - r_n)^{-1} (\log(1/\delta_n))^{-1}. \end{aligned}$$

Since $\delta_n = b^n$, we have

$$\sum_{n=1}^{\infty} (1 - | z_n |^2) \log^+ | f(z_n) | \geqq \frac{1}{2\pi} \frac{1}{\log (1/b)} \sum_{n=1}^{\infty} \frac{1}{n} = \infty \qquad \text{Q.E.D.}$$

Proof of Theorem 4. Let $f(z) \in N$ and $B(z)$ be the Blaschke product with respect to zero points of $f(z)$. If we write $g(z) = f(z)/B(z)$, $\log | g(z) |$ is easily seen to be represented by a Poisson-Stieltjes integral, hence $\log g(z)$ belongs to H^p for any p, $0 < p < 1$ [3, p. 35, Corollary]. Hence by [6, Theorem 2],

$$\sum_{n=1}^{\infty} (1 - | z_n |^2) | \log g(z_n) |^p < \infty, \qquad 0 < p < 1;$$

therefore for any δ, $0 < \delta < 1$,

$$\sum_{n=1}^{\infty} (1 - | z_n |^2)(\log^+ | f(z_n) |)^{1-\delta} \leqq \sum_{n=1}^{\infty} (1 - | z_n |^2) | \log g(z_n) |^{1-\delta} < \infty,$$

which proves the first part of the Theorem 4.

For the second part, let b be a number, $0 < b < 1$, and put $z_n = 1 - b^n$; $c_n = \exp [n/b^n]$. Then $\{z_n\}$ satisfies (2.4) as well as (2.6), $\{c_n\}$ satisfies (2.11′) for any δ, $0 < \delta < 1$, and

$$(1 - | z_n |) \log^+ | c_n | \uparrow \infty. \tag{5.5}$$

Since for any $f(z) \in N$ there must hold $\log^+ | f(z) | = O(1/(1 - | z |))$ [9, p. 106, where N is denoted as A], (5.5) shows that there is no $f(z) \in N$ with $f(z_n) = c_n$. Q.E.D.

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Chiba University
Chiba City, Chiba-Ken, Japan.