NILPOTENT DIFFEOMORPHISM GROUPS

BY
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A group is said to be nilpotent of degree c if all commutators

$$[x_1, [x_2, [x_3, \cdots, x_{c+1}] \cdots]$$

of length $c + 1$ are 1: G is nilpotent of degree 2 iff the commutator subgroup $[G, G]$ is contained in the center. By $\mathfrak{D}^\pi(M)$, we mean the group of pseudo-isotopy classes of diffeomorphisms that are simply homotopic to the identity; i.e., the homotopy is a (relative) simple homotopy equivalence of $(M \times I, \partial(M \times I))$. If $\partial M \neq \varphi$, diffeomorphisms and homotopies are assumed to be the identity there.

THEOREM. *If M^n is compact and orientable and $n \geq 5$, then $\mathfrak{D}^\pi(M)$ is nilpotent of degree* 2.

The proof is composed of two steps, the first of which gives more insight into the structure of $[\mathfrak{D}^\pi(M), \mathfrak{D}^\pi(M)]$.

LEMMA 1. *Any commutator in $\mathfrak{D}^\pi(M)$ has a representative which is the identity outside a neighborhood of the* 1*-skeleton.*

LEMMA 2. *Every $f \in \mathfrak{D}^\pi(M)$ has a representative which is the identity on a neighborhood of the* 1*-skeleton.*

Applying Lemma 1 to $[g, h]$ and Lemma 2 to f shows that $[f, [g, h]] = 0$ and the theorem follows.

Remarks. (i) If $\pi_1(M) = 0$, this result is not new (see [T1]): in fact, if n is odd, $\mathfrak{D}^\pi(M)$ is abelian.

(ii) We point out the lack of restrictions on M; in particular, M may have boundary. $n \geq 5$ is needed for the s-cobordism theorem.

(iii) The full diffeomorphism group $\mathfrak{D}(M)$ is *not* generally nilpotent—but one would not expect this since the group of homotopy equivalences is very complicated. For a specific example, consider

$$M = (S^k \times S^l)_1 \# (S^k \times S^l)_2 \# (S^k \times S^l)_3 \quad (k < l)$$

and let d_1 and d_2 be diffeomorphisms that permute the factors according to the permutations (1 2) and (1 2 3). Then if we evaluate on $H_k(M)$, $(d_1)_*^2 = (d_2)_*^3 =$ the identity, so the subgroup of $\mathfrak{D}(M)$ generated by d_1, and d_2 maps onto a subgroup of $\mathrm{Aut}(H_k(M)) \cong \mathrm{Aut}(Z \oplus Z \oplus Z)$ which is isomorphic to the symmetric group on three letters. Since this group is not nilpotent, neither is $\mathfrak{D}(M)$.

Received October 12, 1973.
[1] Partially supported by a National Science Foundation grant.

(iv) The result is best possible in that $\mathfrak{D}(M)$ may not be abelian (= nilpotent of degree 1). An example follows the proofs of the lemmas.

(v) The condition of simple homotopies may seem somewhat unnatural, but it is very necessary for Lemma 1 (but not for Lemma 2). The fact that the techniques of Lemma 1 are so natural implies that this added restriction is justified.

(vi) Lemma 1 is valid for M non-orientable, but the proof we give for Lemma 2 does not work in this case. $\mathfrak{D}^{\pi}(M)$ can be shown to be nilpotent of degree ≤ 5 in this case, but it is unlikely that this is the best possible.

Proof of Lemma 1. As in [T1] there is an exact sequence of groups

$$L_{n+2}(\pi_1(M)) \to h\mathcal{S}[M \times I, \partial] \to [\Sigma M, G/0]$$

and a surjective map $\phi : h\mathcal{S}[M \times I, \partial] \to \mathfrak{D}^{\pi}(M)$. (The notation is slightly different here as we are using the relation of pseudo-isotopy instead of isotopy —thus P does not appear.) Since $[\Sigma M, G/0]$ is abelian, a commutator in $h\mathcal{S}[M \times I, \partial]$ lies in $\partial(L_{n+2}(\pi_1(M)))$ so it suffices, to show that all elements of $\phi \circ \partial(L_{n+2}(\pi_1(M)))$ have representatives which are the identity except on a neighborhood M_1 of the 1-skeleton of M. According to Wall [W; Theorems 5.8 and 6.5] an element of $\partial(L_{n+2}(\pi_1(M)))$ is represented by a simply homotopy equivalence obtained as follows: if $n = 2k$, then X is obtained from $M \times I$ by performing surgeries on a set of disjoint k-spheres embedded in $M_1 \times I$ (a neighborhood of the 1-skeleton of $M \times I$) each of which is isotopically trivial—then h is the "identity" on the non-surgered part and a null-homotopy on the rest; if $n = 2k + 1$ one further set of surgeries of index $k + 1$ is required, again in $M_1 \times I$. In either case, if $M_2 = M - \dot{M}_1$, then $X = X_1 \cup X_2$, $h : X_i \to M_i \times I$, $h : X_2 \to M_2 \times I$ is a diffeomorphism and h is a simple homotopy equivalence. It follows that $h : X_1 \to M_1 \times I$ is also a simple homotopy equivalence: the induced map on the homotopy exact sequence of the pair (X, X_1) shows it to be a homotopy equivalence and

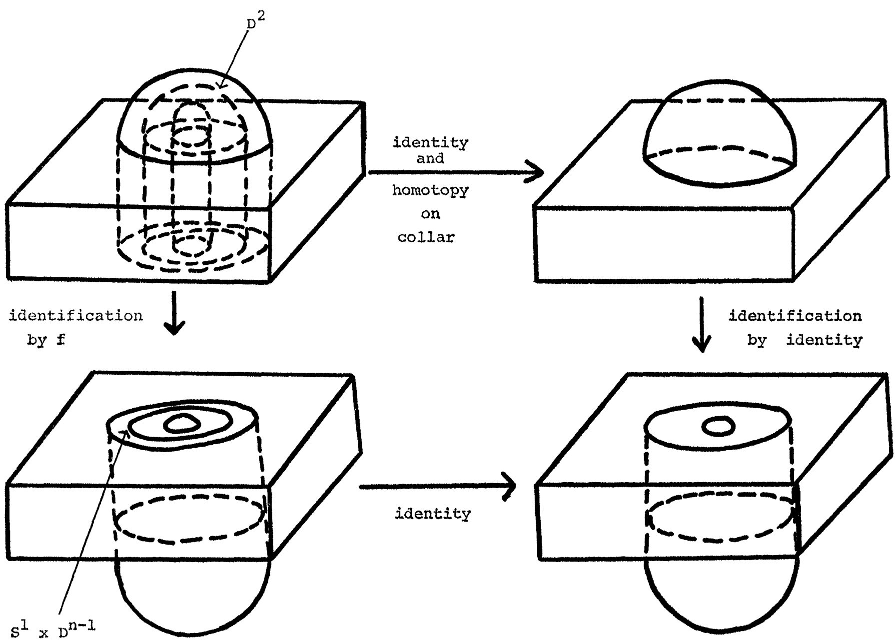
$$\tau(h \mid X_1) = \tau(h) = 0$$

so it is simple. The s-cobordism theorem then applies to show that there exists a diffeomorphism $d_1 : M_1 \times I \to X_1$, which can, in fact, be assumed to agree with the "identity" on $X_1 \cap X_2$ (since the vector field needn't be altered near $X_1 \cap X_2$). Let $d : M \times I \to X$ be d_1 on $M_1 \times I$ and the "identity" on $M_2 \times I$. Then $[h]$ and $[h \circ d]$ represent the same element in $h\mathcal{S}[M \times I, \partial]$ and $h \circ d : M \times I \to M \times I$ is the identity except on $M_1 \times I$. As ϕ is evaluation on $M \times 1$, the lemma follows.

Proof of Lemma 2. Since f is homotopic to the identity we isotope f so that it is the identity on the 1-skeleton and linear on a neighborhood—and since M is orientable, this neighborhood will be a number of $S^1 \times D^{n-1}$'s tubed together. It therefore suffices to show that if $f : M \to M$ induces the non-trivial twist (bundle map) on $S^1 \times D^{n-1} \subset M$ then f is not homotopic to the identity. Suppose it is; if M is a component of the boundary of W, the rest of whose boundary we denote by M', and there exists a disc $D^2 \subset W$ with $D^2 \cap M = S^1$, consider the manifold pair $(W \cup_f W; M' \cup M')$. This is homotopy equivalent (call the equivalence h) to $(W \cup_{\mathrm{id}} W; M' \cup M')$ via the homotopy between f and id and so defines an element δ in $h\mathcal{S}[W \cup_{\mathrm{id}} W; M' \cup M']$ with image δ' in $[W \cup_{\mathrm{id}} W/M' \cup M', G/0]$. Viewing this second set in its bundle interpretation, δ' corresponds to a homotopy trivialization of

$$\tau(W \cup_{\mathrm{id}} W, M' \cup M') - (h^{-1})_*(\tau(W \cup_f W), M' \cup M').$$

However, the restriction of this difference bundle to the S^2 in $W \cup_{\mathrm{id}} W$ obtained by joining the two copies of $D^2 \subset W$ is the non-trivial vector bundle over S^2, which is not homotopically trivial. The contradiction follows. The most natural choice for W is the trace of a surgery on the given S^1: see the accompanying diagram.



Example. $\mathfrak{D}^\pi(S^7 \times S^{11})$ *is not abelian.*

We know [T2] that $\mathfrak{D}(S^7 \times S^{11})$ is a semidirect product.

$$(\pi_{11}(SO_8) \oplus \Gamma_{19}) \times_\phi \pi_7(SO_{12})$$

and that if $\beta \in \pi_{11}(SO_8)$ and $\alpha \in \pi_7(SO_{12})$, that the commutator

$$[(0, 0, \alpha), (\beta, 0, 0)] = (\beta', \gamma', 0)$$

where β' is bilinear and γ' is the pairing described by Milnor [M]. It is straight forward to verify that $(\beta, \gamma, \alpha) \in \mathfrak{D}^\pi(S^7 \times S^{11})$ iff $J_7(\alpha) = 0$ and

$J_{11}(\beta) = 0$; these are precisely the obstructions to defining a homotopy. In these dimensions,

$$J_7 : Z \to Z_{240} \quad \text{and} \quad J_{11} : Z \oplus Z_2 \to Z_{504} \oplus Z_2$$

so $f_1 = (240, 0, 0)$ and $f_2 = (0, 504 \oplus 0, 0)$ are in $\mathfrak{D}^\pi(S^7 \times S^{11})$: to show $[f_1, f_2] \neq 0$, it suffices to check that $\gamma'(240, 504 \oplus 0, 0) \neq 0$. Calculating Milnor's invariant λ for this pair, we get

$$\lambda = \pm \frac{11 \cdot 31 \cdot 240 \cdot 504}{5 \cdot 73} \cdot 6 \cdot K \pmod 1$$

where K has all prime factors ≤ 5. Clearly λ is non-zero, and since λ is an invariant of γ', $\gamma'(240, 504 \oplus 0, 0) \neq 0$.

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