ON REGULAR FUNCTIONS ON RIEMANN SURFACES II

BY

JAMES A. JENKINS[1] AND NOBUYUKI SUITA

In an earlier paper [1] the authors treated the following problem. Let Ω be an open Riemann surface with (A, B) a regular partition of its boundary into nonvoid sets. Let f be a regular function on Ω. Let $\{\Omega_\nu\}$ be a canonical exhaustion of Ω with the boundary of Ω_ν composed of cycles α_ν, negatively oriented on Ω_ν, and β_ν, positively oriented on Ω_ν, respectively bounding the complementary sets bearing A and B. Let $P_m(\alpha_\nu)$ consist of those points of the sphere about which the index of $f(\alpha_\nu)$ is at least m, $Q_n(\beta_\nu)$ those points about which the index of $f(\beta_\nu)$ is at most n $(m > n)$. Let (bar denotes closure)

$$\bar{P}_m(A) = \bigcap_{\nu=1}^{\infty} P_m(\alpha_\nu)^-, \quad \bar{Q}_n(B) = \bigcap_{\nu=1}^{\infty} Q_n(\beta_\nu)^-$$

be both nonvoid. Let $\Gamma(A, B)$ denote the family of cycles on Ω separating A and B. Let Δ denote the complement of $\bar{P}_m(A) \cup \bar{Q}_n(B)$ and let Γ_{mn} denote the family of cycles on Δ separating $\bar{P}_m(A)$ and $\bar{Q}_n(B)$. We proved that between the module $M(\Gamma(A, B))$ of $\Gamma(A, B)$ and the module $M(\Gamma_{mn})$ of Γ_{mn} subsists the inequality

$$(m - n)M(\Gamma(A, B)) \leq M(\Gamma_{mn}).$$

However we did not provide a complete description of the possibility of equality. The object of the present paper is to elucidate this matter. The result obtained is given in the following theorem.

THEOREM. *In the notation of Section* 1 *suppose that* $M(\Gamma(A, B))$ *is finite and that*

$$(m - n)M(\Gamma(A, B)) = M(\Gamma_{mn}). \tag{1}$$

Then Δ *is a domain and* f *is a* $(m - n, 1)$ *mapping of* Ω *onto* Δ *apart possibly from a relatively closed set of logarithmic capacity zero in* Δ. *Further the indexes of* $f(\alpha_\nu), f(\beta_\nu)$ *with respect to each point of* $\bar{P}_m(A)$, $\bar{Q}_n(B)$ *are respectively equal to* m *and* n *for each* ν.

The proof of this statement is broken down into a series of steps.

Let Δ' denote the subset of Δ made up of those components of Δ which have points both of $\bar{P}_m(A)$ and $\bar{Q}_n(B)$ on their boundaries. There are only a finite number of such components since $\bar{P}_m(A) \cap \bar{Q}_n(B)$ is void. Let Δ_ν denote the complement of $P_m(\alpha_\nu)^- \cup Q_n(\beta_\nu)^-$ and Δ_ν' its subset made up of those compon-

Received May 6, 1974.

[1] Research supported in part by the National Science Foundation.

ents which have points both of $P_m(\alpha_\nu)^-$ and $Q_n(\beta_\nu)^-$ on their boundaries. Clearly $\Delta' \subset \bigcup_\nu \Delta'_\nu$.

(i) *The Riemann image of* Ω *under* f *covers no point of* Δ' *more than* $m - n$ *times.*

If some point p in Δ' were covered at least $m - n + 1$ times, for ν sufficiently large, p would lie in Δ'_ν and be covered at least $m - n + 1$ times by the Riemann image of Ω_ν under f. $\Gamma(\alpha_\nu, \beta_\nu)$ denotes the family of cycles on Ω_ν separating α_ν and β_ν and $M(\Gamma(\alpha_\nu, \beta_\nu))$ denotes its module. We recall that as ν tends to infinity $M(\Gamma(\alpha_\nu, \beta_\nu))$ increases to $M(\Gamma(A, B))$. We may suppose that the image of Ω_ν covers a disc δ with closure in Δ'_ν with $m - n + 1$ simple discs δ_j, $j = 1, \ldots, m - n + 1$, and that the harmonic measure ω of B with respect to Ω (nondegenerate under our assumptions) has no critical point in the closures of the preimages of these discs. The extremal metric $\rho|dz|$ for $M(\Gamma(A, B))$ is given by $(D(\omega))^{-1}|\text{grad } \omega|\,|dz|$. ($D(\omega)$ denotes Dirichlet integral.) Let $\rho_\nu|dz|$ be the corresponding extremal metric for $M(\Gamma(\alpha_\nu, \beta_\nu))$. Evidently, expressed in terms of a local uniformizing parameter, ρ_ν tends pointwise to ρ, uniformly on compact subsets, as ν tends to infinity. Thus if we use the plane variable as local uniformizing parameter in each $f^{-1}(\delta_j)$ we will have ρ_ν bounded from zero on these sets, $\rho_\nu \geq \eta > 0$, for ν sufficiently large.

In [1, Section 5] we constructed by the covering method an admissible metric for $M(\Gamma(\alpha_\nu, \beta_\nu))$ which we denote now by $\sigma_\nu|dz|$ and for which we have by the considerations given there

$$M(\Gamma(\alpha_\nu, \beta_\nu)) \leq \iint_{\Omega_\nu} \sigma_\nu^2\, dA_z \leq (m - n)^{-1} M(\Gamma_{mn}^{(\nu)}) \tag{2}$$

where $\Gamma_{mn}^{(\nu)}$ denotes the family of cycles on Δ'_ν separating $P_m(\alpha_\nu)^-$ and $Q_n(\beta_\nu)^-$. We have the familiar identity

$$\tfrac{1}{2}\iint_{\Omega_\nu} \rho_\nu^2\, dA_z + \tfrac{1}{2}\iint_{\Omega_\nu} \sigma_\nu^2\, dA_z = \iint_{\Omega_\nu} \left(\frac{\rho_\nu + \sigma_\nu}{2}\right)^2 dA_z + \iint_{\Omega_\nu} \left(\frac{\rho_\nu - \sigma_\nu}{2}\right)^2 dA_z. \tag{3}$$

Now since σ_ν is zero on a subset of $\bigcup_j f^{-1}(\delta_j)$ of measure, in terms of the local uniformizing parameters, equal to $A(\delta)$, the plane area of δ (not necessarily confined to one component) and since $\frac{1}{2}(\rho_\nu + \sigma_\nu)|dz|$ is an admissible metric for $M(\Gamma(\alpha_\nu, \beta_\nu))$ we have, using (2) and (3), for ν sufficiently large

$$\begin{aligned}\tfrac{1}{2}M(\Gamma(\alpha_\nu, \beta_\nu)) &\leq \tfrac{1}{2}\iint_{\Omega_\nu} \sigma_\nu^2\, dA_z - \iint_{\bigcup_j f^{-1}(\delta_j)} \left(\frac{\rho_\nu - \sigma_\nu}{2}\right)^2 dA_z \\ &\leq \tfrac{1}{2}(m - n)^{-1} M(\Gamma_{mn}^{(\nu)}) - \tfrac{1}{4}\eta^2 A(\delta).\end{aligned}$$

Letting ν tend to infinity this contradicts (1).

(ii) *Let $\tilde{\omega}$ be the harmonic measure of $\bar{Q}_n(B)$ with respect to Δ'. There exists a set of orthogonal trajectories of the level curves of $\tilde{\omega}$ dense in Δ' each of which has respective limiting end points on $\bar{P}_m(A)$, $\bar{Q}_n(B)$ with limiting values of $\tilde{\omega}$ respectively* 0 *and* 1 *and such that for each orthogonal trajectory $l, f^{-1}(l)$ consists of $m - n$ open arcs $l^{(j)}$ each homeomorphic to $l, j = 1, \ldots, m - n$.*

By [2; Theorem 2.32] almost all (in the sense there indicated) orthogonal trajectories of $\tilde{\omega}$ have limiting end points on $\bar{P}_m(A)$, $\bar{Q}_n(B)$ with limiting values of $\tilde{\omega}$ respectively 0 and 1. Let such orthogonal trajectories be denoted by l_λ, where λ is indexed by a set Λ. We can apply the argument of [1; Section 5] to the orthogonal trajectories of $\tilde{\omega}$ rather than those of ω_ν on Δ_ν and see that over l_λ in the Riemann image of Ω_ν by f there will be $m - n$ open arcs which are the images of arcs $l_{\lambda,\nu}^{(j)}$ $j = 1, \ldots, m - n$, $\nu = 1, 2, \ldots$, joining α_ν and β_ν. Let $\tau_\nu|dz|$ now denote the metric corresponding to $\rho_1|dz|$ in [1; Section 5] on Ω_ν. Then

$$M(\Gamma(\alpha_\nu, \beta_\nu)) \leq \iint_{\bigcup_{\lambda,j} l_{\lambda,\nu}^{(j)}} \tau_\nu^2 \, dA_z \quad \text{and} \quad (m - n) \iint_{\bigcup_{\lambda,j} l_{\lambda,\nu}^{(j)}} \tau_\nu^2 \, dA_z \leq M(\Gamma_{mn}).$$

Since $M(\Gamma_{mn}) = (D(\tilde{\omega}))^{-1}$ we must have by (1) that for almost all l_λ there will be $m - n$ covering arcs in the Riemann image of Ω_ν on which the variation of $\tilde{\omega}$ tends to 1 as ν becomes large. Thus by (i) there can be only $m - n$ covering arcs altogether and each of them is homeomorphic to l_λ.

(iii) *For every choice of $\alpha_\nu(\beta_\nu)$, $f(\alpha_\nu)(f(\beta_\nu))$ has index exactly $m(n)$, about each point of $\bar{P}_m(A)(\bar{Q}_n(B))$, providing this index is defined.*

Let q be a point of $\bar{P}_m(A)$ not on $f(\alpha_\nu)$. Then in the notation of [1; Section 2], $I(\alpha_\nu; q) = m'$, $m' \geq m$. $f(\alpha_\nu)$ divides the sphere into a finite number of domains one of which, D, will contain q so that $f(\alpha_\nu)$ will have index m' about every point of D. Some orthogonal trajectory l of the set described in (ii) will penetrate into D. Followed in a suitable sense it will tend to a point r of $\bar{Q}_n(B)$. $f(\beta_\nu)$ divides the sphere into a finite number of domains and r lies in one such domain or on the boundary of several such domains and the index of $f(\beta_\nu)$ about the points of any such domain is at most n. Thus we can apply the argument of [1; Section 5] to a subarc $\hat{l}$ of l with endpoints w_0 and w_1 where $I(\alpha; w_0) = m'$ and $I(\beta; w_1) = n'$, $n' \leq n$. In the notation employed there

$$I(\alpha; w_0) - I(\beta; w_1) \geq l_2 - l_2'.$$

Taking l and the $l^{(j)}$ sensed by $\tilde{\omega}$ increasing and the subarcs $\hat{l}^{(j)}$ on $l^{(j)}$ covering $\hat{l}$ we see that l_2 is the number of arcs of intersection of the $\hat{l}^{(j)}$ with Ω_ν which run from α_ν to β_ν, l_2' is the number of such arcs which run from β_ν to α_ν. On a given $\hat{l}^{(j)}$ these occur alternately thus its contribution to $l_2 - l_2'$ is at most 1 and $l_2 - l_2' \leq m - n$. On the other hand $l_2 - l_2' \geq m' - n'$. Thus $m' = m$. The result for $\bar{Q}_n(B)$ is proved analogously.

(iv) *$f(\Omega)$ contains no point of $\bar{P}_m(A)$ or $\bar{Q}_n(B)$. Δ' is a domain. Δ' coincides with Δ.*

If $f(\Omega)$ contained a point of $\bar{P}_m(A)$ or $\bar{Q}_n(B)$ by choosing α_ν or β_ν suitably we would obtain a contradiction to (iii). Further $f(\Omega)$ is connected so it can contain no point in the complement of Δ' since it contains points in Δ' and no points of its boundary. Since $f(\Omega)$ contains points in each component of Δ', Δ' is a domain. If Δ contained a component Ξ not in Δ', the boundary of Ξ would lie in $\bar{P}_m(A)$ or $\bar{Q}_n(B)$ thus by the definition of the latter sets Ξ itself would lie in $\bar{P}_m(A)$ or $\bar{Q}_n(B)$, a contradiction.

(v) *Let γ be a cycle on Ω represented by a finite number of disjoint Jordan curves forming the common boundary of two open sets bearing respectively A and B neither containing any relatively compact component, γ sensed with A to its left. Let c be a cycle on Δ represented by a finite number of disjoint Jordan curves forming the common boundary of two open subsets of the sphere containing respectively $\bar{P}_m(A)$ and $\bar{Q}_n(B)$ neither containing any component disjoint from $\bar{P}_m(A) \cup \bar{Q}_n(B)$, c sensed with $\bar{P}_m(A)$ to its left. Then $f(\gamma)$ is homologous to $(m - n)c$ in Δ.*

Suppose that $\bar{P}_m(A)$ and $\bar{Q}_n(B)$ are both compact plane sets. Then necessarily $m > 0 > n$. Let c_1 be a separating cycle for $\bar{P}_m(A)$, $\bar{Q}_n(B)$ represented by a finite number of disjoint Jordan curves such that $\bar{P}_m(A)$ lies in their collective (disjoint) interiors and $\bar{Q}_n(B)$ is exterior to all, sensed as above. Let c_2 be a separating cycle for $\bar{P}_m(A)$, $\bar{Q}_n(B)$ represented by a finite number of disjoint Jordan curves such that $\bar{Q}_n(B)$ lies in their collective (disjoint) interiors and $\bar{P}_m(A)$ is exterior to all, sensed as above. Then $f(\gamma)$ and $mc_1 - nc_2$ have the same index about every boundary point of Δ, thus are homologous in Δ while $mc_1 - nc_2$ is homologous to $(m - n)c$.

If $\bar{P}_m(A)$ contains the point at infinity, $m = 0$, $n < 0$. If $\bar{Q}_n(B)$ contains the point at infinity, $m > 0$, $n = 0$. The result in these cases is proved as above, indeed even more simply.

(vi) *If $h_\Omega(\gamma)$ denotes the harmonic length of γ on Ω and $h_\Delta(f(\gamma))$ denotes the harmonic length of $f(\gamma)$ on Δ, we have*

$$h_\Omega(\gamma) = h_\Delta(f(\gamma)).$$

In terms of the harmonic measures defined above we have $h_\Omega(\gamma) = D(\omega)$, $h_\Delta(c) = D(\tilde{\omega})$, $M(\Gamma(A, B)) = (D(\omega))^{-1}$, $M(\Gamma_{mn}) = (D(\tilde{\omega}))^{-1}$. By (v), $h_\Delta(f(\gamma)) = (m - n)D(\tilde{\omega})$. Thus by (1), $h_\Omega(\gamma) = h_\Delta(f(\gamma))$.

(vii) *f maps Ω in a $(m - n, 1)$ manner onto Δ with the possible exception of a relatively closed set of (logarithmic) capacity zero on Δ.*

This follows at once from (vi) and [3; Theorem 2].

Bibliography

1. James A. Jenkins and Nobuyuki Suita, *On regular functions on Riemann surfaces*, Illinois J. Math., vol. 17 (1973), pp. 563–570.
2. Makoto Ohtsuka, *Dirichlet problem, extremal length and prime ends*, Van Nostrand Reinhold, New York, 1970.
3. Nobuyuki Suita, *Analytic mapping and harmonic length*, Kōdai Mathematical Seminar Reports, vol. 23 (1971), pp. 351–356.

Washington University
St. Louis, Missouri
The Institute for Advanced Study
Princeton, New Jersey
Tokyo Institute of Technology
Tokyo