EQUIVARIANT AND HYPEREQUIVARIANT COHOMOLOGY

BY

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0. Introduction

The notion of equivariant cohomology with supports and with coefficients in a sheaf (module bundle) is defined and studied in Section 1 (Section 2). Theorem 1.4 shows that, under certain conditions on the supports and on the coefficients, equivariant cohomology can be reduced to ordinary sheaf theoretic cohomology. In Section 3 this fact is used in the construction of an equivariant Ω-spectrum for equivariant cohomology when the coefficient module bundle and the family of supports are of a certain type (Theorem 3.2). In Section 4 hyperequivariant cohomology is introduced. Theorems 4.2 and 4.3 show that, under various assumptions (Remark 4.4), hyperequivariant cohomology can be reduced to equivariant cohomology and can be classified by a hyperequivariant Ω-spectrum. It should be noted that classically the notions of equivariant and hyperequivariant cohomology coincide due to the fact that a group of automorphisms of a space is also a group in the category of spaces. In this paper "equivariance" is based on *categorical* groups (in particular, group bundles) and "hyperequivariance" is based on *automorphism* groups (of equivariant systems).

1. Equivariant cohomology with sheaf coefficients

Let $\mathscr{A}$ be a sheaf of modules over a sheaf of rings $\mathscr{R}$ on a space X ($\mathscr{A}$ is an $\mathscr{R}$-module in the sense of [1, p. 4]). If $f\colon X \to Y$ is a continuous map and $\mathscr{A}'$ is an $\mathscr{R}'$-module on Y then any f-cohomomorphism of sheaves of modules

$$(k, r)\colon (\mathscr{A}', \mathscr{R}') \to (\mathscr{A}, \mathscr{R})$$

induces a map [1, p. 45] $k_Y\colon \Gamma(\mathscr{A}') \to \Gamma(\mathscr{A})$, the image of which has the structure of a $\Gamma(\mathscr{R}')$-module. Let γ be a compactly generated group bundle over a compactly generated space B [9, Section 1] and let $\xi \in C = (\text{Haus } CG \downarrow B)$ (see [6, pp. 46 and 181]) be a left γ-space for which $q\colon \xi \to \xi/\gamma$, the quotient map onto the space of orbits, is in C, i.e., ξ/γ is Hausdorff (in general, an object in C and the total space of that object will be denoted by the same letter). Let $\mathscr{A}$ be an $\mathscr{R}$-module on ξ. A *γ-structure* on $\mathscr{A}$, briefly denoted by $\mathscr{A}^k$, consists of an $\mathscr{R}'$-module $\mathscr{A}'$ on ξ/γ together with a q-cohomomorphism $(k, r)\colon (\mathscr{A}', \mathscr{R}') \to (\mathscr{A}, \mathscr{R})$. Define $\Gamma(\mathscr{A}^k)$, the $\Gamma(\mathscr{R}')$-module of γ-equivariant sections, by $\Gamma(\mathscr{A}^k) = \text{image } k_{\xi/\gamma}$. If ϕ is a family of supports on ξ let

$$\Gamma_\phi(\mathscr{A}^k) = \Gamma(\mathscr{A}^k) \cap \Gamma_\phi(\mathscr{A}).$$

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Let $K(k) = \bigcup_{x \in \xi} \ker k_x \subset \mathscr{A}'$ and say $\mathscr{A}^k$ is *proper* if $\ker k_x = \ker k_y$ whenever $q(x) = q(y)$.

1.1. *Remark.* If $\mathscr{A}^k$ is proper then γ "acts" on $\bar{\mathscr{A}} = \text{image } k \subset \mathscr{A}$ as follows: For $x \in \xi$, $g \in \gamma$ and $a \in \bar{\mathscr{A}}_x$ set $g * a = k_{gx}(k_x^{-1}(a)) \in \bar{\mathscr{A}}_{gx}$ whenever gx is defined. Since $\mathscr{A}^k$ is proper, $*$ is well defined. If $S \in \Gamma(\mathscr{A}^k)$ then clearly $S \in \Gamma(\bar{\mathscr{A}})$ and satisfies $S(gx) = g * S(x)$ whenever gx is defined. The converse is true if $K(k) = 0$ and q is an open map (see proof of 1.2 below).

If $\mathscr{A}$ has a γ-structure then the canonical resolution [1, p. 26] $\mathscr{C}^*\mathscr{A}$ of $\mathscr{A}$ inherits a γ-structure; namely, $k_* = \mathscr{C}^*k\colon \mathscr{C}^*\mathscr{A}' \to \mathscr{C}^*\mathscr{A}$ (see [1, p. 44]). Thus $C_\phi^*(\mathscr{A}^k) = \Gamma_\phi((\mathscr{C}^*\mathscr{A})^{k_*})$ is a cochain complex of $\Gamma(\mathscr{R}')$-modules. Define $H_\phi^n(\xi; \mathscr{A}^k)$, the n-dimensional γ-equivariant cohomology module of ξ with coefficients in $\mathscr{A}^k$ and supports in ϕ, by $H_\phi^n(\xi; \mathscr{A}^k) = H^n(C_\phi^*(\mathscr{A}^k))$.

1.2. Lemma. *If $K(k) = 0$ and q is open then $K(k_n) = 0$, $n \geq 0$.*

Proof. The definition of $\mathscr{C}^0$ shows $K(k_0) = 0$ if $K(k) = 0$. Let $\bar{S}$ be a serration of $\mathscr{A}'$ over an open set $U \subset \xi/\gamma$ such that $k_U(\bar{S})$ restricts to a continuous section of $\mathscr{A}$ over an open set $W \subset \xi$. If $K(k) = 0$ and q is open then clearly $\bar{S}$ restricts to a continuous section over the open set $q(W)$. This shows $K(\mathscr{Z}^1(k)) = 0$ where $\mathscr{Z}^1(k) = k_0/k\colon \mathscr{C}^0\mathscr{A}'/\mathscr{A}' \to \mathscr{C}^0\mathscr{A}/\mathscr{A}$. The lemma now follows by induction on n in view of the definition of $\mathscr{C}^n$.

Call ϕ a *γ-family* of supports on ξ if $q^{-1}\overline{q(K)} \in \phi$ whenever $K \in \phi$. In this case $q(\phi) = \{\overline{q(K)} \mid K \in \phi\}$ is a family of supports on ξ/γ.

1.3. *Remark.* If a γ-family ϕ is paracompactifying [1, p. 15] then so is $q(\phi)$ when q is open and ϕ-closed [4, 2.6, p. 165].

1.4. Theorem. *If $\mathscr{A}^k$ is proper, ϕ a γ-family of supports on ξ and q is an open map then*

(a) $H_\phi^*(\xi; \mathscr{A}^k) \simeq H_\psi^*(\xi/\gamma; \mathscr{A}'/K(k))$ *where* $\psi = q(\phi)$.

If, in addition, $K(k)$ is flabby or ψ is paracompactifying and $K(k)$ is ψ-soft then

(b) $H_\phi^*(\xi; \mathscr{A}^k) \simeq H_\psi^*(\xi/\gamma; \mathscr{A}')$.

Proof. Since q is open, $K(k)$ is a subsheaf of $\mathscr{A}'$ and $\mathscr{A}'/K(k) = \mathscr{B}$ is a well-defined $\mathscr{R}'$-module. Clearly k has the factorization

$$\mathscr{A}' \xrightarrow{p} \mathscr{B} \xrightarrow{k'} \mathscr{A}$$

where p is the quotient map. Since $0 \to \mathscr{C}^nK(k) \to \mathscr{C}^n\mathscr{A}' \to \mathscr{C}^n\mathscr{B} \to 0$ is exact with $\mathscr{C}^nK(k)$ flabby, $C_\phi^n(\mathscr{A}^k) = C_\phi^n(\mathscr{A}^{k'})$. Since $K(k') = 0$, $K(k'_*) = 0$ by 1.2. Thus

$$(k'_*)_{\xi/\gamma}\colon \Gamma_\psi(\mathscr{C}^*\mathscr{B}) \to C_\phi^*(\mathscr{A}^{k'})$$

is an isomorphism and (a) follows. Part (b) follows from (a) by a standard argument.

Theorem 1.4 together with well-known results on resolutions imply:

1.5. COROLLARY. *If $\mathscr{A}^k$ is proper, q is open, and ϕ is a γ-family with $\psi = q(\phi)$ paracompactifying then $H^n_\phi(\xi; \mathscr{A}^k)$ can be computed by using a ψ-soft resolution of $\mathscr{A}'|K(k)$, or of $\mathscr{A}'$ if $K(k)$ is ψ-soft.*

2. Equivariant cohomology with module bundle coefficients

Let Λ be a ring bundle, μ a left Λ-module bundle, and γ a group bundle, all on B and all compactly generated (see [10, Section 2]). Suppose left actions of γ on Λ and on μ, where for each $b \in B$ and for $1, g, g' \in \gamma_b$, $l, l' \in \Lambda_b$, $m, m' \in \mu_b$ (ξ_b = fiber of ξ over b), satisfy

$$(2.1)\qquad \begin{aligned} gg'(l) &= g(g'l), & gg'(m) &= g(g'm),\\ g(l + l') &= gl + gl', & 1l &= l,\\ g(m + m') &= gm + gm', & g(ll') &= (gl)(gl'),\\ g(lm) &= (gl)(gm), & 1m &= m. \end{aligned}$$

Let $\tilde{\mu}$ (respectively $\tilde{\Lambda}$) be the sheaf on ξ (a left γ-space) of germs of maps (in C) $\xi \to \mu$ (respectively $\xi \to \Lambda$) and let $\tilde{\mu}'$ (respectively $\tilde{\Lambda}'$) be the sheaf on ξ/γ generated by the presheaf $U \to \{$set of γ-equivariant maps (in C) $q^{-1}U \to \mu$ (respectively $q^{-1}U \to \Lambda$)$\}$. Conditions (2.1) imply $\tilde{\mu}'$ is a $\tilde{\Lambda}'$-module. The obvious q-cohomomorphism (k, r): $(\tilde{\mu}', \tilde{\Lambda}') \to (\tilde{\mu}, \tilde{\Lambda})$ defines a γ-structure, $\tilde{\mu}^k$, on $\tilde{\mu}$. Note that $K(k) = 0$ in case q is an open map. Define $H^*_\gamma(\xi_\phi; \mu)$, the γ-equivariant cohomology of ξ with coefficients in μ and supports in ϕ by $H^*_\gamma(\xi_\phi; \mu) = H^*_\phi(\xi; \tilde{\mu}^k)$.

3. An equivariant Ω-spectrum of module bundles

As in [11] let

$$(3.1)\qquad \begin{aligned} &0 \longrightarrow \mu \longrightarrow \nu_0 \xrightarrow{i_1} \mu_1 \longrightarrow \nu_1 \longrightarrow \cdots\\ &\longrightarrow \mu_{n-1} \longrightarrow \nu_{n-1} \xrightarrow{i_n} \mu_n \longrightarrow \cdots \end{aligned}$$

be the sequence of Λ-module bundles obtained from the sequence

$S^0 \to I \xrightarrow{p} S^1 \to I \wedge S^1 \to \cdots \to S^{n-1} \to I \wedge S^{n-1} \to S^1 \wedge S^{n-1} = S^n \to \cdots$ ($\wedge$ = smash product, I = unit interval, S^n = n-sphere, p = quotient map $I \to I/S^0 = S^1$) by letting

$$\begin{aligned} i_n &= F_B(p \wedge \mathrm{Id}_{S^{n-1}}) \otimes \mathrm{Id}_\mu\colon \nu_{n-1}\\ &= F_B(I \wedge S^{n-1}) \otimes \mu \to \mu_n\\ &= F_B(S^n) \otimes \mu \end{aligned}$$

where $F_B(X)$ is the trivial bundle on B with fiber the free abelian group generated by X with the compactly generated topology induced from that of X. By allowing γ to act on the "μ-factor" of v_n and of μ_n, the action of γ on μ of Section 2 is extended to actions on v_n and μ_n that satisfy 2.1 and relative to which i_n is equivariant. By [11], 3.1 is the sequence of Theorem 5.3 [10] when μ is an LNDR [10, Section 2]. Let C_γ be the category with objects $\xi_\phi = (\xi, \phi)$, where $\xi \in C$ is a left γ-space for which $q: \xi \to \xi/\gamma$ is an open map with $\xi/\gamma \in C$, and where ϕ is a γ-family on ξ with $q(\phi)$ paracompactifying. The morphisms of C_γ are the equivariant maps $f: \xi \to \xi'$ satisfying $f^{-1}(\phi') \subset \phi$. The "supported" equivariant analogue of the results of [10] are summed up in the following theorem. Compare with [2, Chapter III, Section 3, and Chapter IV, Section 1]. Recall that μ is a γ-LNDR means the functions (u_α, h_α) representing μ as an LNDR are γ-equivariant (γ acts trivially on I).

3.2. THEOREM. (a) *If μ is a γ-LNDR Λ-module bundle then $\{\mu_n\}$ $n \geq 1$ is a γ-spectrum for $H^*_\gamma(-;\mu)$ on C_γ, i.e., $H^n_\gamma(\xi_\phi; \mu)$ is naturally isomorphic (as $\Gamma(\Lambda')$-modules) to $[\xi_\phi, \mu_n]_\gamma$, the set of equivariant fiber homotopy classes of equivariant maps $\xi \to \mu_n$ where the homotopies $h = \{h_t\}$ have support $|h|$ in $\phi \times I$ ($|h| = \overline{\{(x, t) \mid h_t(x) \neq 0\}} \subset K \times I$ for some $K \in \phi$).*

(b) *If μ is a γ-NDR Λ-module bundle then $\{\mu_n\}$, $n \geq 1$, is a γ-Ω-spectrum, i.e., μ_n and $\Omega\mu_{n+1}$, the vertical loop space of μ_{n+1}, are of the same equivariant fiber homotopy type.*

Proof. First note that for $\psi = q(\phi)$, the sequence

$$0 \to \tilde{\mu}' \to \tilde{v}'_0 \to \cdots \to \tilde{v}'_n \to \cdots \tag{3.3}$$

is a ψ-soft resolution of $\tilde{\mu}'$ on ξ/γ where $\tilde{\ }'$ is as in Section 2. To see that $\tilde{v}'_n$ is ψ-soft let $s \in \Gamma(\tilde{v}'_n \mid K)$ where $K \in \psi$. Since ψ is paracompactifying, s extends to a section over an open set $U \supset K$ where $\overline{U} \in \psi$. Further there is a continuous map $\tau: \overline{U} \to I$ such that $\tau^{-1}(1) \supset \overline{U}_1$, $\tau \mid (\overline{U} - U) = 0$ where U_1 is open and $U \supset \overline{U}_1 \supset U_1 \supset K$. Viewing s as an equivariant map $q^{-1}U \to v_n$ define $\bar{s}: \xi \to v_n$ by

$$\bar{s}(x) = \begin{cases} 0 & \text{if } q(x) \notin \overline{U} \text{ or } \tau q(x) \leq \frac{1}{2} \\ H_{2\tau q(x)-1}(s(x)) & \text{if } \tau q(x) \geq \frac{1}{2} \end{cases}$$

where H_t ($H_1 = \mathrm{id}$, $H_0 = 0$) is the vertical homotopy (shrinking v_n to the 0-section) induced by contracting I in the "1st factor" of v_n. Since H_t is equivariant (γ acts on the "2nd factor" of v_n) $\bar{s}$ is seen to be an equivariant map that extends $s \mid q^{-1}(U_1)$. This shows $\tilde{v}'$ is ψ-soft. To see that 3.3 is exact recall that if μ is an LNDR then i_n has local sections s_j over elements U_j of an open cover $\{U_j\}$ of the total space of μ_n (this is essentially [9, 3.3]). If μ is a γ-LNDR then the open sets U_j and the sections s_j can be chosen to be equivariant ($x \in U_j$ implies $gx \in U_j$ and $s_j(gx) = gs_j(x)$ whenever gx is defined). This follows from an equivariant analogue of the proof (in [8]) of [9, 3.3] (essentially an application of the fibered, equivariant analogue of [7, 4.2] with E (G) replaced by the restriction of v_{n-1} (μ_{n-1}) to the open sets in B given in the definition of μ as a γ-LNDR). Clearly $\tilde{\mu}'_n$ is ker $(\tilde{v}'_n \to \tilde{v}'_{n+1})$ and $\tilde{i}'_n: \tilde{v}'_{n-1} \to \tilde{\mu}'_n$ is onto since germs of equivariant maps into μ_n ($\mu_0 = \mu$) can be lifted by the equivariant sections s_j. This shows 3.3 is exact. By 1.5, 3.3 can be used to compute $H^n_\gamma(\xi_\phi; \mu)$. Thus

$$\begin{aligned} H^n_\gamma(\xi_\phi; \mu) &\simeq \ker (\Gamma_\psi(\tilde{v}'_n) \to \Gamma_\psi(\tilde{v}'_{n+1}))/\mathrm{im}\,(\Gamma_\psi(\tilde{v}'_{n-1}) \to \Gamma_\psi(\tilde{v}'_n)) \\ &\simeq \Gamma_\psi(\tilde{\mu}'_n)/\mathrm{im}\,(\Gamma_\psi(\tilde{v}'_{n-1}) \to \Gamma_\psi(\tilde{\mu}'_n)). \end{aligned}$$

Therefore $H^n_\gamma(\xi_\phi; \mu)$ is isomorphic to the $\Gamma(\hat{\Lambda}')$-module of equivalence classes of equivariant maps $\xi \to \mu_n$ with support in ϕ, where two such maps s_0, s_1 are identified if and only if there is an equivariant map $s\colon \xi \to v_{n-1}$ with support in ϕ such that $i_n s = s_1 - s_0$. However, the existence of such an s is equivalent to the existence of a vertical, equivariant homotopy $h = \{h_t\}$ ($h_0 = s_0, h_1 = s_1$) with support in $\phi \times I$. Indeed if $i_n s = s_1 - s_0$ let $h_t(x) = s_1(x) - i_n H_t s(x)$ where H_t is the equivariant homotopy shrinking v_{n-1} ($H_0 = \mathrm{id}$, $H_1 = 0$). Clearly $h = \{h_t\}$ is equivariant and since $i_n(0) = 0 = H_t(0)$,

$$|h| \subset (|s| \cup |s_1|) \times I \in \phi \times I.$$

Conversely, given h let $h' = h - s_0$. Then h' is an equivariant homotopy of 0 to $s_1 - s_0$ with

$$|h'| \subset (|h| \cup (|s_0| \times I)) \in \phi \times I,$$

i.e. $|h'| \subset q^{-1}K \times I$ for some $K \in \psi$. Since ψ is paracompactifying there is an open set U, $K \subset \bar{U} \in \psi$, $\bar{U}$ paracompact with $|h'| \subset q^{-1}\bar{U} \times I$. As in [3, p. 237, part (b)] there is an open cover $W = \{W_x\}$ $(x \in q^{-1}\bar{U})$ of $q^{-1}\bar{U}$ with

$$h'(W_x \times [(i-1)/r, i/r]) \subset U_j$$

for some U_j where $\{U_j\}$ is the equivariant cover of μ_n defined above. The equivariance of U_j and h' implies W_x can be chosen so that $q^{-1}(qW_x) = W_x$. Since $\{qW_x\}$ is an open cover of the paracompact space $\bar{U}$, $\{W_x\}$ is a numerable cover of $q^{-1}\bar{U}$. Further, the existence of the equivariant section s_j over U_j implies i_n has the stationary equivariant covering homotopy property for $h'|W_x \times [(i-1)]/r, i/r]$. For if $\bar{h}$ is an equivariant map covering $h'_{(i-1)/r}$ then $\bar{h}_t = s_j h'_t - s_j h'_{(i-1)/r} + \bar{h}$ is an equivariant covering homotopy of h' with $\bar{h}_{(i-1)/r} = \bar{h}$ and $\bar{h}_t$ is stationary with h'_t [3, Remark 4.10]. This shows that the equivariant CHPS version of [3, 4.7] applies and that h' is covered by an equivariant $\bar{h}$ (take $\bar{h}_0 = 0$) on $q^{-1}\bar{U} \times I$ that is stationary with h'. (Note that the CHPS version of [3, 4.7] is given by [3, 4.10]. The proof of the equivariant analogue of [3, 4.7] consists of redoing [3, 2.6, 2.7, 4.5, 4.6] in the case that the partitions of unity, halos, sections, etc., are all equivariant or invariant under the action of γ.) Extending $\bar{h}$ to all of $\xi \times I$ by the 0-section of v_{n-1} shows h' is covered by an equivariant $\bar{h}$ with $|\bar{h}| \in \phi \times I$. Hence $s = \bar{h}_1$ is an equivariant map, $|s| \in \phi$, and $i_n s = h'_1 = s_1 - s_0$. This shows (a). Part (b) is the equivariant analogue of [10, 6.2]. The action of γ on $\Omega\mu_n$ is given by $(g\alpha)(t) = g(\alpha(t))$ for $g \in \gamma$, $\alpha \in \Omega\mu_n$ whenever $g(\alpha(t))$ is defined. If the maps j

and h_t in the proof of [10, 6.1] are equivariant then clearly r, $\bar{r}$, k_s, $\bar{k}_s$ of that proof are also equivariant (g_t is equivariant by the equivariant covering homotopy theorem). This shows part (b).

3.4. *Remark.* If the projection of γ is open and the base space B is paracompact then 3.2 (a) implies $H_\gamma^n(\gamma; \mu) \simeq [\gamma, \mu_n]_\gamma$ where γ acts on γ by left translation. This latter set is clearly isomorphic to the set of vertical homotopy classes of sections of μ_n which, in turn, (by [9, 3.7]) is isomorphic to $H^n(B; \mu)$, $n \geq 1$. Thus $H_\gamma^n(\gamma; \mu)$ is independent of γ and by [10, 5.4 (a)] can be interpreted as the set of isomorphism classes of local principal μ_{n-1} bundles on B. In particular if B is a point then $H_\gamma^n(\gamma; \mu) = 0$.

4. Hyperequivariant cohomology

Let

$$\mathscr{A}_i^k = (\gamma_i, \xi_i, \mathscr{R}_i, \mathscr{A}_i, \mathscr{R}'_i, \mathscr{A}'_i, r_i, k_i)$$

(respectively $M_i = (\gamma_i, \xi_i, \Lambda_i, \mu_i)$) ($i = 1, 2$) be systems as defined in Section 1 (respectively Section 2). A morphism $f\colon \mathscr{A}_1^{k_1} \to \mathscr{A}_2^{k_2}$ ($\bar{f}\colon M_1 \to M_2$) consists of a tuple

$$f = (f_\gamma, f_\xi, f_{\mathscr{R}}, f_{\mathscr{A}}, f_{\mathscr{R}'}, f_{\mathscr{A}'}) \quad (\bar{f} = (f_\gamma, f_\xi, \bar{f}_\Lambda, \bar{f}_\mu))$$

where $f_\gamma\colon \gamma_1 \to \gamma_2$ is a map of group bundles over B, $f_\xi\colon \xi_1 \to \xi_2$ is an f_γ-equivariant map of spaces over B, $f_{\mathscr{R}}\colon \mathscr{R}_2 \to \mathscr{R}_1$ is an f_ξ-cohomomorphism of sheaves of rings ($\bar{f}_\Lambda\colon \Lambda_1 \to \Lambda_2$ is an f_γ-equivariant map of ring bundles), $f_{\mathscr{A}}\colon \mathscr{A}_2 \to \mathscr{A}_1$ is an $f_{\mathscr{R}}$-equivariant, f_ξ-cohomomorphism of sheaves of modules ($\bar{f}_\mu\colon \mu_1 \to \mu_2$ is an $f_\gamma - \bar{f}_\Lambda$-equivariant map of module bundles). $f_{\mathscr{R}'}$ and $f_{\mathscr{A}'}$ are analogously defined (relative to $f_{\xi/\gamma}\colon \xi_1/\gamma_1 \to \xi_2/\gamma_2$ induced by f_ξ) and are required to satisfy $r_1 f_{\mathscr{R}'} = f_{\mathscr{R}} r_2$ and $k_1 f_{\mathscr{A}'} = f_{\mathscr{A}} k_2$. Since $\mathscr{C}^*$ is functorial, f induces a map

$$\mathscr{C}^*\mathscr{A}_1^{(k_1)*} \to \mathscr{C}^*\mathscr{A}_2^{(k_2)*}$$

and consequently a map $\Gamma((\mathscr{C}^*\mathscr{A}_2)^{k_2*}) \to \Gamma((\mathscr{C}^*\mathscr{A}_1)^{k_1*})$. Thus any group F of left operators on $\mathscr{A}^k$ becomes a group of right operators on $\Gamma((\mathscr{C}^*\mathscr{A})^{k*})$. Define the F-hyperequivariant cohomology $H_\Phi^n(\xi; \mathscr{A}^k)^F$ to be $H^n(\Gamma_\Phi^F((\mathscr{C}^*\mathscr{A})^{k*}))$ where $\Gamma^F((\mathscr{C}^*\mathscr{A})^{k*})$ is the subcochain complex of $\Gamma((\mathscr{C}^*\mathscr{A})^{k*})$ consisting of the elements fixed under F and

$$\Gamma_\Phi^F((\mathscr{C}^*\mathscr{A})^{k*}) = \Gamma^F((\mathscr{C}^*\mathscr{A})^{k*}) \cap \Gamma_\Phi(\mathscr{C}^*\mathscr{A}).$$

In case γ acts trivially ($gx = x$) and $(k, r) = (\mathrm{id}, \mathrm{id})$, denote $\Gamma_\Phi^F((\mathscr{C}^*\mathscr{A})^{k*})$ by $\Gamma_\Phi^F(\mathscr{C}^*\mathscr{A})$.

A morphism $\bar{f}\colon M_1 \to M_2$ clearly induces a morphism $f\colon \tilde{\mu}_1^{k_1} \to \tilde{\mu}_2^{k_2}$ in the case that $\bar{f}_\Lambda$ and $\bar{f}_\mu$ are isomorphisms and $f_{\tilde{\Lambda}}, f_{\tilde{\Lambda}'}$ ($f_{\tilde{\mu}}, f_{\tilde{\mu}'}$) are induced by $\bar{f}_\Lambda^{-1}$ ($\bar{f}_\mu^{-1}$). Thus any group F of operators on $M = (\xi, \gamma, \Lambda, \mu)$ becomes a group of operators on $\tilde{\mu}^k$. Define $H_\gamma^n(\xi_\Phi; \mu)^F$, the F-hyperequivariant cohomology of ξ_Φ with coefficients in μ to be $H_\Phi^n(\xi; \tilde{\mu}^k)^F$.

For any group F (discrete topology) of left operators on γ define $\pi = F * \gamma$, the semidirect product of γ by F, to be the fiber product of the trivial bundle on B with fiber F and γ, and with $(\bar{f}, \bar{g})(f, g) = (\bar{f}f, f^{-1}(\bar{g})g)$ for $f, \bar{f} \in F$, $g, \bar{g} \in \gamma_b$, $b \in B$ as the group bundle operation. If F is a group of left operators on $\mathscr{A}^k$ then $(f, g)(x) = f_\xi(gx)$ $(f \in F, g \in \gamma)$ defines an action of π on ξ for which the quotient map $q_\pi \colon \xi \to \xi/\pi$ has the factorization

$$\xi \xrightarrow{q} \xi/\gamma \xrightarrow{Q} \xi/\pi$$

where Q can be identified with the quotient map $\xi/\gamma \to (\xi/\gamma)/F$ (the operation of F on ξ/γ is induced from that on ξ). Note that Q is open since

$$Q^{-1}Q(V) = \bigcup_{f \in F} f_{\xi/\gamma}(V).$$

As in Section 1 it is assumed that ξ/π is in C.

Suppose, now, that ϕ is a γ-family of supports on ξ and that $\psi = q(\phi)$ is an F-family of supports on ξ/γ $(Q^{-1}\overline{Q(K)} \in \psi$ if $K \in \psi)$ where F is a group of left operators on $\mathscr{A}^k$. Define $Q_\psi^F \mathscr{C}^* \mathscr{A}'$ to be the differential sheaf on ξ/π generated by the differential presheaf $U \to \Gamma^F_{\psi \cap Q^{-1}U}(\mathscr{C}^*\mathscr{A}' \mid Q^{-1}U)$. A direct calculation shows that

$$0 \to Q_\psi^F \mathscr{A}' \to Q_\psi^F \mathscr{C}^0 \mathscr{A}' \to Q_\psi^F \mathscr{C}^1 \mathscr{A}'$$

is exact and that $Q_\psi^F \mathscr{C}^n \mathscr{A}'$ is flabby (extend serrations by 0). Thus if $H^n((Q_\psi^F \mathscr{C}^* \mathscr{A}')_y) = 0$ for $n > 0$ and all $y \in E(\theta)$, the extent of θ [1, p. 16] (note that by definition $(Q_\psi^F \mathscr{C}^* \mathscr{A}')_y = 0$ if $y \notin E(\theta)$) then $Q_\psi^F \mathscr{C}^* \mathscr{A}'$ is a flabby resolution of $\mathscr{A}'' = Q_\psi^F \mathscr{A}'$ and consequently $H_\theta^*(\xi/\pi; \mathscr{A}'') \simeq H^*(\Gamma_\theta(Q_\psi^F \mathscr{C}^* \mathscr{A}'))$ where $\theta = Q(\psi)$. Further, the first map in the sequence $\Gamma_\theta(Q_\psi^F \mathscr{C}^* \mathscr{A}') \to \Gamma_\psi^F(\mathscr{C}^* \mathscr{A}') \to \Gamma_\phi^F((\mathscr{C}^* \mathscr{A})^{k*})$ is an isomorphism by the definition of Q_ψ^F and θ, and the second map is easily seen to be an isomorphism in case $K(k) = 0$ and q is open. Under the foregone conditions, then,

$$H_\theta^*(\xi/\pi; \mathscr{A}'') \simeq H_\phi^*(\xi; \mathscr{A}^k)^F. \tag{4.1}$$

4.2. THEOREM. *If F is a group of left operators on $\mathscr{A}^k$ with q open, $K(k) = 0$, ϕ and $q(\phi)$ equivariant supports with $\theta = Q(\psi) = Qq(\phi) = q_\pi(\phi)$ and $H^n((Q_\psi^F \mathscr{C}^* \mathscr{A}')_y) = 0$ for all $y \in E(\theta)$, $n > 0$, then*

$$H_\phi^*(\xi; \mathscr{A}^k)^F \simeq H_\phi^*(\xi; \mathscr{A}^{\tilde{k}})$$
$$\text{where } \mathscr{A}^{\tilde{k}} = (\pi = F * \gamma, \xi, \mathscr{R}, \mathscr{A}, Q_\psi^F \mathscr{R}' = \mathscr{R}'', Q_\psi^F \mathscr{A}' = \mathscr{A}'', \bar{r}, \bar{k}).$$

Here $\bar{k} = kk'$ $(\bar{r} = rr')$ where $k' \colon \mathscr{A}'' \to \mathscr{A}'$ $(r' \colon \mathscr{R}'' \to \mathscr{R}')$ is the canonically induced Q-cohomomorphism.

Proof. Note that q_π is open since both q and Q are open. An easy calculation shows $K(k') = 0$, thus $K(\bar{k}) = 0$. The result now follows from 4.1 and 1.4 (b) applied to $\mathscr{A}^{\tilde{k}}$.

4.3. THEOREM. (a) *Let F be a group of left operators on $M = (\gamma, \xi, \Lambda, \mu)$ such that q is open and μ is both a γ and F-LNDR. If ϕ is a γ-family on ξ, $\psi = q(\phi)$ an F-family on ξ/γ with $\theta = Q(\psi)$ paracompactifying and $H^n((Q_\psi^F \mathscr{C}^* \tilde{\mu}')_y) = 0$ for all $y \in E(\theta)$, $n > 0$ then $H_\gamma^n(\xi_\phi; \mu)^F$ is isomorphic to $[\xi_\phi; \mu_n]_\gamma^F$, the set of F-$\gamma$-equivariant fiber homotopy classes of F-γ-equivariant maps $s: \xi \to \mu_n$ ($s(gx) = gs(x)$ and $sf_\xi(x) = f_{\mu_n}(s(x))$ where the homotopies have support in $\phi \times I$ and the operation of F on μ_n is induced from that on μ.*

(b) *If μ is an F-γ-NDR then $\{\mu_n\}$, $n \geq 1$, is an F-γ-Ω-spectrum.*

Proof. From the definition of $\pi = F * \gamma$ and the way in which it operates it is readily seen that a map is π-equivariant if and only if it is F-γ-equivariant. Further, if ϕ is a γ-family and $\psi = q(\phi)$ is an F-family then ϕ is a π-family. A simple calculation shows $Q_\psi^F \tilde{\mu}'$ is generated by the presheaf $U \to \{$set of π-equivariant maps $q_\pi^{-1}(U) \to \mu\}$. Since $K(k) = 0$ (Section 2) Theorems 4.2 and 3.2 (a) imply $H_\gamma^n(\xi_\phi; \mu)^F \simeq H_\pi^n[\xi_\phi, \mu_n]$, $\simeq [\xi_\phi, \mu_n]_\pi$, $n \geq 1$. A check of the definitions shows $[\xi_\phi, \mu_n]_\pi = [\xi_\phi, \mu_n]_\gamma^F$. This proves (a). Part (b) follows similarly from 3.2 (b).

4.4. *Remark.* There are many conditions on F, γ, ξ, ϕ, etc., that imply the assumptions of Theorems 4.2 and 4.3. For example, it is easily shown that q is open in case each $x \in \xi$ has a continuous local section of q passing through it or in case the projection of γ is open and either γ or ξ is locally compact or both γ and ξ satisfy the first axiom of countability (so that the fiber product $\gamma\xi$ has the product topology [4, Section 9, p. 247]). If F is finite then $\psi = q(\phi)$ is an F-family and Q is ψ-closed if and only if ψ is F-closed ($fK \in \psi$ for all $f \in F$, $K \in \psi$) since $Q^{-1}Q(K) = \bigcup_{f \in F} fK$. Thus, by 1.3, $\theta = q_\pi(\phi)$ is paracompactifying for ϕ paracompactifying, q open and ϕ-closed, F finite and $\psi = q(\phi)$ F-closed. If ψ is a paracompactifying F-family and the induced operation of F on ξ/γ is proper [5, p. 134] then $H^n(Q_\psi^F \mathscr{C}^* \mathscr{A}')_y = 0$ for all $y \in E(\theta)$ and $n > 0$. To prove this it is sufficient to show that for some $x \in Q^{-1}(y)$ the induced map

$$\begin{aligned}(Q_\psi^F \mathscr{C}^n \mathscr{A}')_y &= \varinjlim_{y \in U} \Gamma^F_{\psi \cap Q^{-1}U}(\mathscr{C}^n \mathscr{A}' \mid Q^{-1}U) \\ &\to \varinjlim_{x \in V} \Gamma_{\psi \cap V}(\mathscr{C}^n \mathscr{A}' \mid V) \\ &\to \Gamma_{\psi \cap \{x\}}(\mathscr{C}^n \mathscr{A}' \mid \{x\}) \\ &= (\mathscr{C}^n \mathscr{A}')_x\end{aligned}$$

is an isomorphism since $(\mathscr{C}^* \mathscr{A}')_x$ is acyclic. However, the second map is an isomorphism by [1, 9.15, p. 50] since ψ is paracompactifying. Thus each element in $(\mathscr{C}^n \mathscr{A}')_x$ can be represented by an $s \in \Gamma_{\psi \cap V}(\mathscr{C}^n \mathscr{A}' \mid V)$ with V a proper neighborhood of x. Extend s to an $\bar{s}$ on

$$Q^{-1}Q(V) = \bigcup_{f \in F} fV \quad \text{by } \bar{s}(w) = \mathscr{C}^n(f_{\mathscr{A}'}^{-1})_w s(f^{-1}(w))$$

for $w \in fV$. It is readily checked that

$$\bar{s} \in \Gamma^F_{\psi \cap Q^{-1}U}(\mathscr{C}^n\mathscr{A}' \mid Q^{-1}U) \quad \text{for } U = Q(V).$$

This shows that the map in question is onto. The one-to-one part is trivial and the assertion follows.

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