THE MEASURE-THEORETIC STRUCTURE GROUP IS NOT INVARIANT

BY

RUSSELL A. JOHNSON

1. Introduction

In [3], Furstenberg showed by example that a minimal distal flow (X, T) need not be uniquely ergodic. One might ask whether such flows still have "measure-theoretic invariants." For example, consider the measure-theoretic structure groups (see 2.3) $J(w)$, where w ranges over $M_T(X)$, the set of T-invariant probability measures on X. It follows from an unpublished result of Ellis that, if (X, T) is a compact group extension of a uniquely ergodic flow, then $J(w_1)$ and $J(w_2)$ are canonically isomorphic (see 2.4) for all $w_1, w_2 \in M_T(X)$. For such flows, then, $J(w)$ is independent of w. We are led to the following:

1.1 *Conjecture.* Let (X, T) be a minimal distal flow, $w_1, w_2 \in M_T(X)$. Then $J(w_1)$ and $J(w_2)$ are canonically isomorphic.

We will show by constructing a counterexample that this conjecture is false.

2. Preliminaries

2.1 DEFINITION. If (X, T) is a flow with T-invariant measure w, we can map T into the set of bounded linear operators on $L^2(X, w)$. Let $S(w)$ be the closure of (the image of) T in the weak operator topology. Let $L^2ap(w)$ (the "L^2-almost periodic functions") be $\{f \in L^2(w) \mid \{t \cdot f \mid t \in T\}$ has compact closure in $L^2(w)\}$; then $L^2ap(w)$ is a closed T-invariant subspace of $L^2(w)$.

2.2 THEOREM. *$S(w)$ is compact, and contains a unique minimal two-sided ideal, $J(w)$, which is a compact topological group. If P_w is the identity in $J(w)$, then P_w is the projection of $L^2(w)$ onto $L^2ap(w)$.*

For the proof of 2.2 see [5, 2.6, 2.31–33, 2.36, and 2.45–46].

2.3 DEFINITION. The *measure-theoretic structure group* of (X, T) with respect to w is the group $J(w)$.

2.4 DEFINITION. Let w_1, w_2 be T-invariant measures on X. Say that $J(w_1)$ and $J(w_2)$ are *canonically isomorphic* if the mapping $t \circ P_{w_1} \to t \circ P_{w_2}$ extends to an isomorphism and homeomorphism of the compact topological groups $J(w_1)$ and $J(w_2)$.

2.5 *Conventions.* From now on, all flows will be *discrete*; the notation (X, T) will refer to a compact T_2 space X together with a homeomorphism T of

Received October 7, 1975.

X. Let K be the unit circle in the complex plane, K^n the n-torus; γ will denote normalized Haar measure on K. We sometimes suppress γ and write, for instance, dw for $d\gamma(w)$. The word "measurable" will always mean *Borel* measurable.

3. The example

3.1. Let $\alpha_0 \in (0, 1)$ be irrational, $\alpha = e^{2\pi i\alpha_0}$. In Section 4, we will construct a homeomorphism T of K^3 of the form $(w, \rho_1, \rho_2) \to (w\alpha, g(w)\rho_1, \rho_1\rho_2)$, where (i) $g\colon K \to K$ is continuous; (ii) $g(w) = r(w\alpha)/r(w)$ γ-a.e. for a measurable function $r\colon K \to K$ having the property that r^m is not equal γ-a.e. to a continuous function for any integer $m \neq 0$; (iii) there is a measurable function $f\colon K \to K$ such $r(w) = f(w)/f(w\alpha^{-1})$ γ-a.e. In this section, we show that (K^3, T) is a counterexample to 1.1; in outline, the proof is as follows. We first verify minimality and distality. Then, for each $\beta \in K$, we define

$$S_\beta = \{(w, \rho_1, \rho_2) \in K^3 \mid \rho_1 = \beta_r(w)\}, \quad \text{and} \quad Q_\beta\colon K^2 \to K^2\colon (w, \rho) \to (w\alpha, \rho\beta).$$

It is shown (roughly) that each S_β is T-invariant, and supports a T-invariant probability measure w_β. Moreover, the process (S_β, T, w_β) is measure-theoretically isomorphic to $(K^2, Q_\beta, \gamma \times \gamma)$. Known results now imply that, if β_1 and β_2 differ mod $\{\alpha^n \mid n \in Z\}$, then $J(w_{\beta_1})$ and $J(w_{\beta_2})$ are not canonically isomorphic.

3.2 Lemma. *Suppose (Ω, S_0) is minimal with Ω compact metric. Let $S\colon X \equiv \Omega \times K \to X$ be given by $S(w, \rho) = (S_0 w, h(w)\rho)$ where $h\colon \Omega \to K$ is continuous. Then (X, T) is minimal iff the equation $h^m(w) = \xi \circ S_0(w)/\xi(w)$ has no continuous solution $\xi\colon \Omega \to K$ for any integer $m \neq 0$.*

For the proof of a more general statement, see [6, Theorem 1].

3.3 Proposition. *The flow (K^3, T) described in 3.1 is minimal and distal.*

Proof. Distality holds because (K^3, T) is constructed by means of two K-extensions of the almost periodic flow $w \to w\alpha$ on K.

Define $T_0\colon K^2 \to K^2$ by $T_0(w, \rho_1) = (w\alpha, g(w)\rho_1)$; observe that (K^3, T) is a K-extension of (K^2, T_0). We show first that (K^2, T_0) is minimal. Suppose

$$g^m(w) = r_1(w\alpha)/r_1(w) \quad \gamma\text{-a.e.}$$

where $r_1\colon K \to K$ is continuous. Then $r_1(w\alpha)/r^m(w\alpha) = r_1(w)/r^m(w)$ γ-a.e., so by ergodicity of rotation by α, $r_1(w)/r^m(w) = \text{const.}$ γ-a.e. This contradicts our assumption on r. By 3.2 (with $(\Omega, S_0) = (K, w \to w\alpha)$), (K^2, T_0) is minimal.

We now seek to apply 3.2 to (K^3, T). Suppose for contradiction that

$$\rho_1^m = \frac{\xi \circ T_0(w, \rho_1)}{\xi(w, \rho_1)}, \quad m \neq 0,$$

for a continuous $\xi\colon K^2 \to K^2$. Note that if C is the cycle $\{(1, \rho_1) \mid \rho_1 \in K\} \subset K^2$, then $T_0(C)$ and C are homologous. It follows easily that the induced map on homology $(\xi \circ T_0/\xi)_*$ takes the class of C to zero. Since $(w, \rho_1) \to \rho_1^m$ takes this class to m, a contradiction is obtained.

3.4 Definition. We fix actions of K on K^2 and K^3 as follows:

$$\beta \cdot (w, \rho_1) = (w, \beta \cdot \rho_1); \qquad \beta \cdot (w, \rho_1, \rho_2) = (w, \beta\rho_1, \rho_2).$$

There are then induced actions of K on $M(K^2)$ and $M(K^3)$, the spaces of Borel regular probabilities on K^2 and K^3. These actions are the ones referred to below.

Recall $T_0\colon K^2 \to K^2$ was defined by $(w, \rho_1) \to (w\alpha, g(w)\rho_1)$.

3.5 Lemma. *There is a measure υ_0 on K^2, ergodic with respect to T_0, such that*

$$\upsilon_0\{(w, \rho_1) \mid \rho_1 = r(w)\} = 1.$$

Proof. The function $\overline{r(w)}\rho_1$ is T_0-invariant. Let υ_1 be any T_0-ergodic probability on K^2. Then there exists $\beta_1 \in K$ such that $\upsilon_1(A_1) = 1$, where

$$A_1 = \{(w, \rho_1) \mid \overline{r(w)}\rho_1 = \beta_1\} = \{(w, \rho_1) \mid \rho_1 = \beta_1 r(w)\}.$$

Let $\upsilon_0 = \beta_1 \cdot \upsilon_1$.

Let υ be the *Haar lift* of υ_0 to K^3: $\upsilon(f) = \int_{K^2} (\int_K f(w, \rho_1, \rho_2)\, d\rho_2)\, d\upsilon_0(w, \rho_1)$. Then $\upsilon(S_1) = 1$, where $S_1 = \{(w, \rho_1, \rho_2) \mid \rho_1 = r(w)\}$. Also, if $\beta \in K$, then $\beta \cdot \upsilon$ is the Haar lift of $\beta \cdot \upsilon_0$, and $\beta\upsilon(S_\beta) = 1$, where $S_\beta = \{(w, \rho_1, \rho_2) \mid \rho_1 = \beta r(w)\} = \beta \cdot S_1$. The measures $\beta\upsilon_0$ are T_0-ergodic and the measures $\beta\upsilon$ are T-invariant ($\beta \in K$).

Recall we defined $Q_\beta\colon K^2 \to K^2$ by $(w, \rho) \to (w\alpha, \rho\beta)$ (see (3.1)).

3.6 Proposition. *For each $\beta \in K$, the set S_β contains a T-invariant Borel set S'_β such that $\beta\upsilon(S'_\beta) = 1$. The processes $(S'_\beta, T, \beta\upsilon)$ and $(K^2, Q_\beta, \gamma \times \gamma)$ are measure-theoretically isomorphic.*

Proof. It is convenient to prove the two statements simultaneously. Fix $\beta \in K$. We will show that there are Borel sets $B \subset K^2$ and $S'_\beta \subset S_\beta$ and a map $\psi_1\colon B \to S'_\beta$ such that: (i) $\gamma \times \gamma(B) = 1$ and $Q_\beta B = B$; (ii) $\beta\upsilon(S'_\beta) = 1$ and $TS'_\beta = S'_\beta$; (iii) ψ_1 is a Borel isomorphism; (iv) $\psi_1 \circ Q_\beta \circ \psi_1^{-1} = T$; (v) $\psi_1(\gamma \times \gamma) = \beta\upsilon$. Observe that, if (i)–(v) are satisfied, then automatically $\psi_1^{-1}(\beta\upsilon) = \gamma \times \gamma$.

Begin by defining $\psi\colon K^2 \to S_\beta\colon (w, \rho) \to (w, \beta r(w), f(w\alpha^{-1})\rho)$. Then ψ is measurable and bijective. Let h be a bounded Borel function on K^3. Since $\beta\upsilon_0$ is concentrated on $\{(w, \rho_1) \mid \rho_1 = \beta r(w)\}$, one has

$$(\beta\upsilon)(h) = \int_{K^2} \left(\int_K h(w, \rho_1, \rho_2)\, d\rho_2\right) d(\beta\upsilon_0)$$
$$= \int_{K^2} \left(\int_K h(w, \beta r(w), \rho_2)\, d\rho_2\right) d(\beta\upsilon_0).$$

Now, since $\beta\upsilon_0$ is T_0-invariant, one has $\pi_*(\beta\upsilon_0) = \gamma$, where $\pi: K^2 \to K$: $(w, \rho_1) \to w$. Also, $\int_K h(w, \beta r(w), \rho_2)\, d\rho_2$ depends only on w. Thus the last multiple integral equals

$$\int_K \int_K h(w, \beta r(w), \rho_2)\, d\rho_2\, dw,$$

which equals $\int_{K^2} h(w, \beta r(w), f(w\alpha^{-1})\rho_2)\, d\rho_2\, dw = (\gamma \times \gamma)(h)$. Hence $\psi(\gamma \times \gamma) = \beta\upsilon$.

We now find the Borel sets B and S'_β. Consider $A_1 = \{(w, \beta r(w)) \mid w \in K\}$. There is a Borel set $A_2 \subset A_1$ such that $\upsilon_0(A_2) = 1$ and $T_0 \cdot A_2 = A_2$. By the Kuratowski theorem [2, 2.2.10], the projection $A_3 = \{w \in K \mid (w, r(w)) \in A_2\}$ is Borel. Let $B = A_3 \times K$, $S'_\beta = \psi(B) = \beta \cdot A_2 \times K$, $\psi_1 = \psi|_B$. Clearly B and S'_β are Borel. Also, $Q_\beta B = B$, $\gamma \times \gamma(B) = 1$, and $\beta\upsilon(S'_\beta) = 1$. We prove (iv):

$$T_0\psi_1(w, \rho) = (w\alpha, \beta r(w\alpha), \beta r(w) f(w\alpha^{-1})\rho) = (w\alpha, \beta r(w\alpha), f(w) \cdot (\rho\beta)),$$

so $\psi_1^{-1} T_0 \psi_1(w, \rho) = (w\alpha, \rho\beta) = Q_\beta(w, \rho)$. The equality $TS'_\beta = S'_\beta$ now follows from the definition of S'_β, completing (ii). Part (v) follows from the preceding paragraph.

Part (iii) remains; we must show that ψ_1 takes Borel sets to Borel sets. So, let $A_4 \subset A_3$ be Borel, $V \subset K$ *open*. Write

$$\psi_1(x) = (\psi'(x), \psi''(x)) \quad \text{where } \psi'(x) \in \beta \cdot A_2,\ \psi''(x) \in K.$$

Since $\psi'(w, \rho) = (w, r(w))$, $\psi'(A_4 \times V)$ is Borel (use the Kuratowski theorem). Also,

$$\psi''(A_4 \times V) = \bigcup \{f(w\alpha^{-1}) \cdot V \mid w \in A_4\}$$

is open. Thus $\psi_1(A_4 \times V)$ is Borel; hence $\psi_1(B')$ is Borel whenever $B' \subset B$ is.

To see that (K^3, T) does not satisfy 1.1, define the *spectrum* of $(K^3, T, \beta\upsilon)$ by

$$\operatorname{Sp}(\beta\upsilon) = \{\lambda \in K \mid \text{there exists a Borel function } h \text{ such that } h(Tx) = \lambda h(x)\ \beta\upsilon\text{-a.e.}\}.$$

Then [4, Theorem 2.17] $J(\beta_1\upsilon)$ and $J(\beta_2\upsilon)$ are canonically isomorphic iff $\operatorname{Sp}(\beta_1\upsilon) = \operatorname{Sp}(\beta_2\upsilon)$ (set equality). Now, by 3.6 and standard properties of Q, one obtains $\operatorname{Sp}(\beta\upsilon) = \{\alpha^n\beta^m \mid n, m \in Z\}$. Hence if $\beta_1 \neq \beta_2 \bmod \{\alpha^n \mid n \in Z\}$, then $\beta_1\upsilon$ and $\beta_2\upsilon$ have distinct J's.

4. Construction of f

4.1 LEMMA. *There is a sequence $(n_l)_{l=1}^\infty$ of positive integers such that:*

(i) $n_{l+1} > n_l$,

(ii) $n_l \equiv 1 \bmod 4$,

(iii) $$\frac{1}{2\pi\sqrt[4]{l}} < n_l\alpha_0 - [n_l\alpha_0] < \frac{2}{2\pi\sqrt[4]{l}}, \quad l > 1.$$

Here α_0 is as in 3.1, *and* [ ] *refers to the greatest integer function.*

Proof. An easy consequence of the irrationality of α_0.

Fix such a sequence $(n_l)_{l=1}^{\infty}$.

Let F be a square-integrable Borel function on $[0, 1)$ such that

$$F \sim \sum_{l=1}^{\infty} \frac{1}{l^{3/4}} \cos 2\pi n_l \theta.$$

Let $R(\theta) = F(\theta) - F(\theta - \alpha_0)$ (F, and all other functions defined on $[0, 1)$, are assumed extended to $\mathbf{R}$ by periodicity). Then

$$R(\theta) \sim \sum_{l=1}^{\infty} \frac{1}{l^{3/4}} \{[1 - \cos 2\pi n_l \alpha_0] \cos 2\pi n_l \theta - \sin 2\pi n_l \alpha_0 \sin 2\pi n_l \theta\}.$$

We agree that a function defined on $[0, 1)$ is continuous iff its periodic extension to $\mathbf{R}$ is continuous.

4.2 Proposition. *$R(\theta)$ is Borel, but is not equal μ-a.e. to a continuous function.*

Proof. Fix ε in $(0, 1)$. Let $X = 2\pi(n_l\alpha_0 - [n_l\alpha_0])$; by 4.1(iii), $1/\sqrt[4]{l} < X_l < 2/\sqrt[4]{l}$. Hence we can find an l_0 such that

$$l > l_0 \Rightarrow \frac{1 - \varepsilon}{2} < \frac{1 - \cos X_l}{X_l^2} < \frac{1}{2};$$

one has

$$\frac{1 - \varepsilon}{2\sqrt{l}} < \frac{1 - \varepsilon}{2} \cdot X_l^2 < 1 - \cos X_l < \tfrac{1}{2} X_l^2 < \frac{2}{\sqrt{l}}.$$

Let

$$\delta_l = \frac{1}{l^{3/4}} [1 - \cos 2\pi n_l \alpha_0] = \frac{1}{l^{3/4}} [1 - \cos X_l];$$

then $(1 - \varepsilon)/2l^{5/4} < \delta_l < 2/l^{5/4}$. These inequalities imply that

$$\sum_{l=1}^{\infty} \delta_l \cos 2\pi n_l \theta$$

converges uniformly to a continuous function $h(\theta)$. Now

$$R - h \sim - \sum_{l=1}^{\infty} \sin 2\pi n_l \alpha_0 \sin 2\pi n_l \theta.$$

Changing l_0 if necessary, we may assume $l > l_0 \Rightarrow 1 - \varepsilon < (\sin Xl)/X_l$. So if

$$\rho_l = \frac{1}{l^{3/4}} \sin 2\pi n_l \alpha_0 = \frac{1}{l^{3/4}} \sin X_l,$$

then $(1 - \varepsilon)/l < \rho_l$. Since $n_l \equiv 1 \bmod 4$, $\rho_l \sin 2\pi n_l \theta|_{\theta = 1/4} = \rho_l$; we see that

$$\sum_{l=1}^{\infty} \rho_l \sin 2\pi n_l \theta$$

is not Césaro summable at $\theta = 1/4$, hence [7, Theorem 8.1, p. 57] the series cannot be that of a continuous function. Thus $R - h$ is not equal μ-a.e. to a continuous function, so R is not.

Let $G(\theta) = R(\theta + \alpha_0) - R(\theta)$. Then

$$G(\theta) \sim -\sum_{l=1}^{\infty} \frac{1}{l^{3/4}} [2 - 2\cos 2\pi n_l \alpha_0] \cos 2\pi n_k \theta = -\sum_{l=1}^{\infty} 2\,\delta_l \cos 2\pi n_l \theta$$

where δ_l is as in the proof of 4.2. The bounds on δ_l stated there imply that this series is that of a continuous function, so:

4.3 PROPOSITION. *$G(\theta)$ is equal μ-a.e. to a continuous function.*

Let $m \in Z$. The proof of 4.2 applies equally well to $mR(\theta)$. By [1, Proposition A1, p. 83], the set $\Lambda_m = \{\lambda \in \mathbf{R} \mid e^{2\pi i\theta} \to e^{2\pi i\lambda mR(\theta)}$ is not equal γ-a.e. to a continuous function$\}$ is residual in $\mathbf{R}$. Pick $\lambda \in \bigcap_{m=-\infty}^{\infty} \Lambda_m$, and let $f(e^{2\pi i\theta}) = e^{2\pi i\lambda F(\theta)}$. Define

$$r(w) = \frac{f(w)}{f(w\alpha^{-1})}, \qquad g(e^{2\pi i\theta}) = e^{2\pi i\lambda G_1(\theta)}$$

where $G_1(\theta) = G(\theta)$ μ-a.e. and G_1 is continuous; the corresponding flow (K^3, T) meets all requirements of 3.1.

4.4 *Remarks.* (1) Observe that $\alpha_0 \in (0, 1)$ may be any irrational number.

(2) Let $\tilde{f}(w, \rho_1, \rho_2) = \overline{f(w)}\rho_1\rho_2$. It may be checked that, on S_β,

$$\tilde{f} \circ T(w, \rho_1, \rho_2) = \beta\tilde{f}(w, \rho_1, \rho_2).$$

For each β, then, the class of $\tilde{f}$ in $L^2(K^3, \beta\upsilon)$ is a T-eigenfunction with eigenvalue β.

4.5 *Questions.* (1) The map T constructed here is continuous. Are there examples (K^3, T) with T C^1? C^∞? analytic?

(2) Can anything general be said about the "map" from invariant measures μ on a minimal distal flow to the corresponding groups $J(\mu)$? Specifically, let (K^3, T) be given by

$$T\colon (w, \rho_1, \rho_2) \to (w\alpha, g(w)\rho_1, h(w, \rho_1)\rho_2)$$

where $g(w) = r(w\alpha)/r(w)$ γ-a.e. and r is Borel but not equal γ-a.e. to a continuous function. Define the measures $\beta\upsilon(\beta \in K)$ as above. How does $J(\beta\upsilon)$ vary with β?

(3) What are some other candidates for measure-theoretic invariants of minimal distal flows?

Acknowledgments. The author wishes to thank Prof. Robert Ellis for many helpful conversations. He is also grateful for assistance from Profs. Harvey Keynes and Leonard Shapiro.

References

1. E. Effros and F. Hahn, *Transformation groups and C*-algebras*, Mem. Amer. Math. Soc., no. 75, 1967.
2. H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.
3. H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math., vol. 83 (1961), pp. 573–601.
4. J. Johnson, *A measure-theoretic study of group extensions of dynamical systems*, Dissertation, University of Minnesota, 1973.
5. G. Laison, *A semigroup associated with a measure on a transformation group*, Dissertation, University of Pennsylvania, 1969.
6. W. Parry, *Compact abelian group extensions of discrete dynamical systems*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, vol. 13 (1969), pp. 95–113.
7. A. Zygmund, *Trigonometric series I*, 2nd ed., Cambridge, London, 1959.

University of Wisconsin-Parkside
Kenosha, Wisconsin