# AN HARMONIC ANALYSIS FOR OPERATORS II: OPERATORS ON HILBERT SPACE AND ANALYTIC OPERATORS[1]

BY

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## 1. Introduction

We continue in this paper investigating the harmonic analysis for operators introduced in [1]. In that paper, we associated to each bounded linear operator $T$ on a homogeneous Banach space $B$ on the circle group $\mathbf{T}$ a formal Fourier series,

$$T \sim \sum_{-\infty}^{+\infty} \hat{T}(n) \tag{1.1}$$

Each $\hat{T}(n)$ is a bounded linear operator on $B$ which satisfies the functional equation

$$\hat{T}(n)R_t = e^{int}R_t\hat{T}(n), \quad t \in \mathbf{T},$$

where $R_t$ is the translation operator defined on $B$ by

$$(R_t f)(s) = f(s - t), \quad s \in \mathbf{T}.$$

Equivalently, $\hat{T}(n)$ is an operator of the form $M_n \circ U$, where $M_n$ is the operation of multiplication by $e^{in\cdot}$

$$(M_n f)(t) = e^{int}f(t), \quad t \in \mathbf{T}, \tag{1.2}$$

and $U$ is an invariant operator, that is, one which commutes with translation, so that $UR_t = R_tU$, $t \in \mathbf{T}$.

In [1] we established formal properties of the series (1.1) and showed that it is $C$-1 summable to $T$ in the strong operator topology. In addition, we obtained an extension to operators of the theorem of F. and M. Riesz which asserts that a measure whose Fourier-Stieltjes transform vanishes on the negative integers must be absolutely continuous.

In this paper we restrict ourselves to the case where $B$ is $L^2(\mathbf{T})$, the space of square integrable functions on the circle group. In Section 2 we state the formal properties of the Fourier series relative to the adjoint operation and establish an analogue for operators of the Parseval formula for functions. In Sections 3 and 4 we extend to operators the notions of support and analyticity. Finally, in Section 5, we establish an analogue for operators of the second theorem of F. and M. Riesz. This classical theorem asserts that a function having all its

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negative Fourier coefficients zero cannot vanish on a subset of **T** having positive Lebesgue measure unless it is zero almost everywhere.

## 2. Operators on Hilbert space-Parseval formula

Throughout this paper we shall be concerned with the algebra $\mathscr{L}$ of bounded linear operators on the Hilbert space $L^2(\mathbf{T})$ of square integrable functions on the unit group **T**. We shall denote by ( , ) the usual inner product on $L^2(\mathbf{T})$,

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)\overline{g(t)}\,dt, \quad f, g \in L_2^2(\mathbf{T}),$$

and by $\|\ \ \|$ the usual norm on $L^2(\mathbf{T})$,

$$\|f\| = (f, f)^{1/2}, \quad f \in L^2(\mathbf{T}).$$

In this section we first summarize the main definitions and results from [1] which are needed in what follows. We then show how the adjoint operation of $\mathscr{L}$ is related to our Fourier series for operators in $\mathscr{L}$ and finally establish a formula for operators in $\mathscr{L}$ which is an analogue of the Parseval formula for functions.

Using the notation of [1], $\mathscr{L}_0$ is the subspace of $\mathscr{L}$ consisting of operators $T$ which are *invariant*, i.e., which commute with translation. It is well known (see Chapter 16 of [2]) that the operators in $\mathscr{L}_0$ are precisely those which have the exponentials $\{e^{in\cdot}: n \in Z\}$ as eigenfunctions. They can also be characterized in terms of the Fourier transform as follows. If the Fourier transform on $L^2(\mathbf{T})$ is defined by ^ so that

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}f(t)\,dt, \quad n \in Z,$$

then an operator $T$ in $\mathscr{L}$ is in $\mathscr{L}_0$ if and only if there is a bounded function $\phi$ on $Z$ so that $\widehat{Tf}(n) = \phi(n)\hat{f}(n)$, $n \in Z$. (For this reason the operators in $\mathscr{L}_0$ are usually called *multipliers.*) In particular, $\mathscr{L}_0$ is a commutative subalgebra of $\mathscr{L}$.

For each positive integer $n$, $\mathscr{L}_n$ is defined in [1] to be the closed linear subspace of $\mathscr{L}$ consisting of all operators $T$ which satisfy the functional equation $TR_t = e^{int}R_tT$, $t \in \mathbf{T}$.

For each positive integer $n$, the operator $\pi_n$ in $\mathscr{L}$ is defined in [1] in terms of vector valued integrals:

$$[\pi_n(T)](f) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}TR_tf\,dt, \quad f \in L^2(\mathbf{T})$$

If $T \in \mathscr{L}$, its *Fourier transform* is defined in [1] to be the $\mathscr{L}$-valued function $\hat{T}$ defined on $Z$ by

$$\hat{T}(n) = \pi_n(T), \quad n \in Z, \tag{2.1}$$

and the formal series

$$\sum_{-\infty}^{+\infty} \hat{T}(n) \tag{2.2}$$

is called the *Fourier series* of $T$.

The following summarizes the main facts concerning the Fourier transform which we will need in what follows.

PROPOSITION 2.1. (i) *For each $n$, $\pi_n$ is a projection of $\mathscr{L}$ onto $\mathscr{L}_n$.*

(ii) *An operator $S$ in $\mathscr{L}$ is in $\mathscr{L}_n$ if and only if it is of the form $S = M_n \circ U$, where $U$ is in an invariant operator and $M_n$ is multiplication by the function $e^{in\bullet}$, Equivalently, $S$ is in $\mathscr{L}_n$ if and only if, for each $m$, $S(e^{im\bullet})$ is a constant multiple of $e^{i(n+m)\bullet}$.*

(iii) *If $T \in \mathscr{L}$, the Fourier series of $T$ is $C$-1 summable to $T$ in the strong operator topology of $\mathscr{L}$; equivalently, for each $f \in L^2(\mathbf{T})$, the series $\sum_{-\infty}^{+\infty} \hat{T}(n)f$ is $C$-1 summable to $f$ in the norm topology of $L^2(\mathbf{T})$.*

(iv) *If $S,\ T \in \mathscr{L},\ n \in Z$, then the series $\sum_{m=-\infty}^{m=+\infty} \hat{S}(n-m)\hat{T}(m)$ is $C$-1 summable to $\widehat{ST}(n)$ in the strong operator topology of $\mathscr{L}$.*

*Proof.* (i), (ii), (iii), and (iv) follow immediately from Proposition 3.2, Corollary 2.4, Proposition 3.8, and Proposition 4.2, respectively, of [1].

We next establish the simplest formal properties of Fourier series of operators in $\mathscr{L}$ with respect to the adjoint operation. We first show that the adjoint operation preserves the graded structure we have introduced in $\mathscr{L}$, in particular, that $\mathscr{L}_n^* = \mathscr{L}_{-n}$.

LEMMA 2.2. *If $T \in \mathscr{L}_n$, then $T^* \in \mathscr{L}_{-n}$.*

*Proof.* Let $T \in \mathscr{L}_n$, $t \in \mathbf{T}$. Then

$$R_{-t}T^*R_t = R_t^*T^*R_{-t}^* = (R_{-t}TR_t)^* = (e^{int}T)^* = e^{-int}T^*.$$

Thus $T^* \in \mathscr{L}_{-n}$.

We next establish an analogue for operators of the fact that the $n$th Fourier coefficient of the complex conjugate of a function $f$ is the complex conjugate of the $(-n)$th coefficient of $f$.

PROPOSITION 2.3. *Let $T \in \mathscr{L}$, $n \in Z$. Then $\pi_n(T^*) = \pi_{-n}(T)^*$, i.e.,*

$$\widehat{T^*}(n) = [\hat{T}(-n)]^*.$$

*Proof.* Let $f, g \in L^2(\pi)$. Then

$$\begin{aligned}([\pi_n(T^*)](f), g) &= \left(\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-ins}[R_{-s}T^*R_s]f\,ds,\ g\right)\\ &= \left(\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-ins}[R_sTR_{-s}]^*f\,ds,\ g\right).\end{aligned}$$

Thus,

$$
\begin{aligned}
([\pi_n(T^*)](f), g) &= \left(\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{int}[R_{-t}TR_t]^*f\,dt,\ g\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{int}([R_{-t}TR_t]^*f, g)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{int}(f, [R_{-t}TR_t]g)\,dt \\
&= \left(f, \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{int}[R_{-t}TR_t]g\,dt\right) \\
&= (f, [\pi_{-n}(T)]g).
\end{aligned}
$$

Since $f$ and $g$ were arbitrary in $L^2(\pi)$, this shows that $\pi_n(T^*) = \pi_{-n}(T)^*$.

As a corollary we obtain the fact that an operator in $\mathscr{L}$ is self-adjoint if and only if its Fourier transform is "hermitian."

COROLLARY 2.4. *Let $T \in \mathscr{L}$. Then $T$ is self-adjoint if and only if*

$$\hat{T}(-n) = [\hat{T}(n)]^* \quad \textit{for all } n \in Z. \tag{2.3}$$

*Proof.* The necessity of (2.3) for self-adjointness follows from Proposition 2.3. Suppose now that (2.3) holds for an operator $T$ in $\mathscr{L}$. Then, as a consequence of Proposition 2.3, each $C$-1 sum of the Fourier series of $T$ will be self-adjoint. That $T$ is self-adjoint now follows from (iii) of Proposition 2.1.

We next show how Proposition 2.3 leads to an analogue for operators of the Parseval formula for functions.

THEOREM 2.5. *Let $T \in \mathscr{L}$. Then*

$$\pi_0(T^*T) = \sum_{-\infty}^{+\infty} [\pi_r(T)]^*\pi_r(T),$$

*with convergence in the strong operator topology of $\mathscr{L}$.*

*Proof.* By (iv) of Proposition 2.1, the series

$$\sum_{-\infty}^{+\infty} \pi_{-r}(T^*)\pi_r(T) \tag{2.4}$$

is $C$-1 summable to $\pi_0(T^*T)$ in the strong operator topology. Because of Proposition 2.3, the series (2.4) is the same as

$$\sum_{-\infty}^{+\infty} [\pi_r(T)]^*\pi_r(T). \tag{2.5}$$

If for each positive integer $N$ we define $S_N$ to be the $N$th partial sum

$$\sum_{-N}^{+N} [\pi_r(T)]^*\pi_r(T)$$

of the series (2.5), we have

$$(2.6) \qquad S_1 \leq S_2 \leq S_9 \leq \cdots,$$

where $\leq$ is the usual partial ordering of $\mathscr{L}$ (see Chapter VII of [4]). The sequence (2.6) is bounded from above in this ordering by $\pi_0(T^*T)$ since the series (2.5) is $C$-1 summable to $\pi_0(T^*T)$ in the strong operator topology. The theorem on p. 263 on [4] asserts that a monotonic sequence of self-adjoint operators bounded from above must converge in the strong operator topology. This shows that the series (2.5) converges to some element $V$ of $\mathscr{L}$ in the strong operator topology. But we know that it is $C$-1 summable in that topology to $\pi_0(T^*T)$ and thus $\pi_0(T^*T) = V$. This completes the proof of Theorem 2.5.

## 3. Support and cosupport

In this section we define the two notions of support and cosupport for operators in $\mathscr{L}$. These reduce to the usual notion of support when the operator is multiplication by a function. We show that a set is a support for an operator $T$ if and only if it is a cosupport for its adjoint $T^*$.

Let $M$ be a measurable subset of $\mathbf{T}$. We will denote by $M^c$ its complement in $\mathbf{T}$ and by $L^2(M)$ the subspace of $L^2(\mathbf{T})$ consisting of those functions "supported" by $M$:

$$L^2(M) = \{f\colon f \in L^2(\mathbf{T}), f = 0 \text{ a.e. on } M^c\}.$$

Let $T \in \mathscr{L}$. We would like to define the *support* of $T$ to be the smallest measurable subset $K$ of $\mathbf{T}$ satisfying:

$$f = 0 \text{ a.e. off } K \text{ implies } Tf = 0 \text{ a.e.} \quad \text{for all } f \in L^2(\mathbf{T});$$

define the *cosupport* of $T$ to be the smallest measurable subset $K$ of $\mathbf{T}$ satisfying:

$$Tf = 0 \text{ a.e. off } K \quad \text{for all } f \in L^2(\mathbf{T}).$$

Unfortunately, because of sets of measure 0, these definitions do not make sense, so we must proceed indirectly.

We define a measurable subset $K$ of $\mathbf{T}$ to be a *supporting set* for $T$ if $T(L^2(K^c)) \subseteq \{0\}$. Let $\mu$ be Lebesgue measure on $\mathbf{T}$. We define the constant $s_T$ by

$$s_T = \inf \{\mu(K)\colon K \text{ a supporting set for } T\}.$$

PROPOSITION 3.1. *Let $T \in \mathscr{L}$. Then there is a measurable subset $K$ of $\mathbf{T}$ satisfying the following:*

(i) *$K$ is a supporting set for $T$.*
(ii) $\mu(K) = s_T$.
(iii) *If $M$ is any supporting set for $T$, $\mu\{x\colon x \in K, x \notin M\} = 0$.*

*Proof.* Let $K_1, K_2, K_3, \ldots$ be a sequence of supporting sets for $T$ with $\mu(K_n) \to s_T$. A routine exhaustion argument shows that the set $K$ defined by $K = \bigcap_{n=1}^{\infty} K_m$ satisfies (i), (ii), and (iii).

Proposition 3.1 shows that $T$ has a minimal supporting set which is unique up to a set of measure 0. Such a set will be called a *support* for $T$.

We proceed similarly to define *cosupport*. A measurable subset $K$ of $\mathbf{T}$ is called a *cosupporting* set for $T$ if $T(L^2(\mathbf{T})) \subseteq L^2(K)$. We define the constant $cs_T$ by

$$cs_T = \inf \{\mu(K)\colon K \text{ a cosupporting set for } T\}.$$

PROPOSITION 3.2. *Let $T \in \mathscr{L}$. Then there is a measurable subset $K$ of $\mathbf{T}$ satisfying the following:*

(i) *$K$ is a co-supporting set for $T$.*
(ii) $\mu(K) = cs_T$.
(iii) *If $M$ is any co-supporting set for $T$ then*

$$\mu\{x\colon x \in K, x \notin M\} = 0.$$

*Proof.* Similar to the proof of Proposition 3.1. Proposition 3.2 shows that $T$ has a minimal cosupporting set which is unique up to a set of measure 0. Such a set will be called a *cosupport* for $T$.

Simple examples show that a support for an operator need not be a cosupport for that operator. For example, if $T \in \mathscr{L}$ is defined by

$$Tf = \left[\int_0^{\pi} f(t)\, dt\right] g, \quad f \in L^2(\mathbf{T}),$$

where $g(t) = 1$, all $t \in \mathbf{T}$, then $[0, \pi]$ is a support for $T$ while $\mathbf{T}$ is a cosupport.

However, we have the following.

PROPOSITION 3.3. *Let $T \in \mathscr{L}$, $K$ a measurable subset of $\mathbf{T}$. Then the following are equivalent:*

(i) *$K$ is a support for $T$.*
(ii) *$K$ is a cosupport for $T^*$.*

This proposition is an immediate consequence of the following.

LEMMA 3.4. *Let $T \in \mathscr{L}$, $K$ a measurable subset of $\mathbf{T}$. Then the following are equivalent:*

(i) *$K$ is a supporting set for $T$.*
(ii) *$K$ is a cosupporting set for $T^*$.*

*Proof.* (i) implies (ii). Let $g \in L^2(\mathbf{T})$. We must show that $T^*g = 0$ a.e. off $K$. If this were not so there would be some $f \in L^2(K^c)$ with $(T^*g, f) \neq 0$. But $(T^*g, f) = (g, Tf) = 0$ since $Tf = 0$ for $f \in L^2(K^c)$ because of (i).

(ii) implies (i). Let $f \in L^2(K^c)$. We must show that $Tf = 0$ a.e. If this were not as so there would be some $g \in L^2(\mathbf{T})$ with $(Tf, g) \neq 0$. But $(Tf, g) = (f, T^*g) = 0$ since $T^*g \in L^2(K)$ because of (ii).

Finally, let us point out that the notion of support and cosupport reduce to the usual notion of support in the case of the operation of multiplication by a function.

PROPOSITION 3.5. *Let $\phi$ be a bounded measurable function on* $\mathbf{T}$ *and $T$ the element of $\mathscr{L}$ defined by*

$$(Tf)(x) = \phi(x)f(x) \text{ a.e. } for\ x \in \mathbf{T}.$$

*Then $\{x\colon x \in \mathbf{T},\ \phi(x) \neq 0\}$ is both a support and cosupport for $T$.*

*Proof.* Straightforward.

## 4. Analytic operators

An operator $T$ in $\mathscr{L}$ will be called *analytic* if all of its negative Fourier coefficients vanish, i.e., $\hat{T}(n) = 0$, all $n < 0$. We shall denote by $\mathscr{A}$ the set of analytic operators in $\mathscr{L}$.

In this section we establish several basic properties of analytic operators. In particular, we show that $\mathscr{A}$ is the strongly closed subalgebra of $\mathscr{L}$ generated by $\mathscr{L}_0$, the space of invariant operators, and the multiplication operator $(M_1 f)(t) = e^{it}f(t)$, $t \in \mathbf{T}$.

Suppose that $T \in \mathscr{L}$ is the operation of multiplication by a bounded measurable function $\phi$:

$$(Tf)(x) = \phi(x)f(x) \text{ a.e. for } x \in \mathbf{T}. \tag{4.1}$$

Our first result shows that analyticity of $T$ is equivalent to $\phi$ giving the boundary values of a function bounded and analytic in the unit disk. (Or what is equivalent, $\hat{\phi}(n) = 0$ for $n < 0$; see p. 39 of [3].)

PROPOSITION 4.1. *Suppose that $T \in \mathscr{L}$ is defined by* (4.1) *for a bounded measurable function on* $\mathbf{T}$. *Then the following are equivalent:*

(i) *$T$ is analytic.*
(ii) *$\hat{\phi}(n) = 0$ if $n < 0$.*

*Proof.* This is immediate from Proposition 3.3 of [1], which states that $\hat{T}(n) = \hat{\phi}(n)M_n$ for all $n \in Z$, where $M_n$ is multiplication by the function $e^{in\cdot}$, i.e., $(M_n f)(t) = e^{int}f(t)$, $t \in \mathbf{T}$.

The next few results establish that $\mathscr{A}$ is a subalgebra of $\mathscr{L}$ closed in the strong operator topology.

PROPOSITION 4.2. *$\mathscr{A}$ is the subalgebra of $\mathscr{L}$.*

*Proof.* That $\mathscr{A}$ is a linear subspace of $\mathscr{L}$ follows from the linearity of the Fourier transform. Let $S, T \in \mathscr{A}$. We will show that $ST \in \mathscr{A}$. Let $n < 0$. It suffices to show that $\widehat{ST}(n) = 0$. By (iv) of Proposition 2.1, the series $\sum_{m=-\infty}^{m=+\infty} \hat{S}(n-m)\hat{T}(m)$ is $C$-1 summable in the strong operator topology to $\widehat{ST}(n)$. But each term of this series must be 0 since $S$ and $T$ are in $\mathscr{A}$ and $n < 0$. Thus $\widehat{ST}(n) = 0$, so $ST \in \mathscr{A}$.

A lemma is needed before we are able to show that $\mathscr{A}$ is closed in the strong operator topology.

LEMMA 4.3. *Let $S \in \mathscr{L}$. Then, for each $r$ and $m$ in $Z$,*

$$(Se^{im\cdot}, e^{i(r+m)\cdot}) = (\hat{S}(r)e^{im\cdot}, e^{i(r+m)\cdot}). \tag{4.2}$$

*Proof.* By (iii) of Proposition 2.1, the series $\sum_{n=-\infty}^{n=+\infty} \hat{S}(n)e^{im\cdot}$ is $C$-1 summable to $Se^{im\cdot}$ in the norm topology of $L^2(\mathrm{T})$. Taking inner products with $e^{i(r+m)\cdot}$, we see that the series

$$\sum_{n=-\infty}^{n=+\infty} (\hat{S}(n)e^{im\cdot}, e^{i(r+m)\cdot}) \tag{4.3}$$

is $C$-1 summable to $(Se^{im\cdot}, e^{i(r+m)\cdot})$. By (ii) of Proposition 2.1, each term of the series (4.3) with $n \neq r$ must be zero, which shows that equality (4.2) holds.

PROPOSITION 4.4. *$\mathscr{A}$ is closed in the strong operator topology of $\mathscr{L}$.*

*Proof.* Let $T_j$ be a net in $\mathscr{A}$ converging to $T \in \mathscr{L}$ in the strong operator topology. Let $r < 0$. We must show that $\hat{T}(r) = 0$. Let $m$ be an arbitrary integer. Since linear combinations of exponentials are dense in $L^2(\mathrm{T})$, it suffices to show that $\hat{T}(r)e^{im\cdot} = 0$. And since, by (ii) of Proposition 2.1, $\hat{T}(r)e^{im\cdot}$ is a multiple of $e^{i(r+m)\cdot}$, it suffices to show that

$$(\hat{T}(r)e^{im\cdot}, e^{i(r+m)\cdot}) = 0. \tag{4.4}$$

By Lemma 4.3 and the fact that $T_j \to T$ in the strong operator topology we have

$$\begin{aligned}(\hat{T}(r)e^{im\cdot}, e^{i(r+m)\cdot}) &= (Te^{im\cdot}, e^{i(r+m)\cdot}) \\ &= \lim_j (T_je^{im\cdot}, e^{i(r+m)\cdot}) \\ &= \lim_j (\hat{T}_j(r)e^{im\cdot}, e^{i(r+m)\cdot}),\end{aligned} \tag{4.5}$$

Since $r < 0$ and each $T_j \in \mathscr{A}$, each of the terms $(\hat{T}_j(r)e^{im\cdot}, e^{i(r+m)\cdot})$ is zero. Thus (4.4) is a consequence of (4.5). This concludes the proof of Proposition 4.4. (Note that the above actually proves that $\mathscr{A}$ is closed in the weak operator topology of $\mathscr{L}$.)

The two results which follow show the extent to which $\mathscr{A}$ is generated by easily described operators.

Recall that the multiplication operator $M_1$ is defined on $L^2(\mathbf{T})$ by $(M_1 f)(t) = e^{it}f(t)$, $t \in \mathbf{T}$. We shall denote by $\mathscr{A}_0$ the (algebraic) subalgebra of $\mathscr{L}$ generated by $M_1$ and the operators in $\mathscr{L}_0$, the invariant operators.

PROPOSITION 4.5. $\mathscr{A}_0 \subseteq \mathscr{A}$.

*Proof.* Because $\mathscr{A}$ is an algebra, it suffices to show that $M_1 \in \mathscr{A}$ and $\mathscr{L}_0 \subseteq \mathscr{A}$. Proposition 3.3 of [1] shows that $\hat{M}_1(n) = 0$ unless $n = 1$, so $M_1 \in \mathscr{A}$. Let $T \in \mathscr{L}_0$. Let $n$ be an arbitrary nonzero integer, $f \in L^2(\mathbf{T})$. Then

$$\begin{aligned}[\hat{T}(n)](f) &= [\pi_n(T)](f) \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}[R_{-t}TR_t]f\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}Tf\,dt \\ &= \left(\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}\,dt\right)Tf \\ &= 0.\end{aligned}$$

Thus $\hat{T}(n) = 0$ for all $n \neq 0$ and as a consequence, $T \in \mathscr{A}$.

PROPOSITION 4.6. *$\mathscr{A}_0$ is dense in $\mathscr{A}$ in the strong operator topology.*

*Proof.* Let $T \in \mathscr{A}$. By (iii) of Proposition 2.1, the $C$-1 sums of the Fourier series of any operator in $\mathscr{L}$ converge to that operator in the strong operator topology. Thus, it suffices to show that each term $\hat{T}(n)$ of the Fourier series of $T$ is contained in $\mathscr{A}_0$. If $n < 0$, $\hat{T}(n) = 0$, and thus $\hat{T}(n) \in \mathscr{A}$. And $\hat{T}(0) \in \mathscr{L}_0$, which is contained in $\mathscr{A}_0$. Suppose now that $n > 0$. By (ii) of Proposition 2.1, $\hat{T}(n)$ is of the form $M_n \circ U$, where $U \in \mathscr{L}_0$ and $M_n$ is multiplication by the function $e^{in\cdot}$. Thus, $\hat{T}(n) = M_n \circ U$ is in $\mathscr{A}_0$, since $M_n$ is the composite of $M_1$ with itself $n$ times.

Summarizing the main results established thus far, we have:

THEOREM 4.7. *$\mathscr{A}$ is the strongly closed subalgebra of $\mathscr{L}$ generated by $\mathscr{L}_0$ and the multiplication operator $M_1$ defined by $(M_1 f)(t) = e^{it}f(t)$, $t \in \mathbf{T}$.*

As the final result in this section, we give a characterization of analytic operators in terms of invariant subspaces. Part of this result will be used in the following section.

For each integer $m$ we denote by $L^2_m(\mathbf{T})$ the subspace of $L^2(\mathbf{T})$ consisting of all functions $f$ in $L^2(\mathbf{T})$ which satisfy $\hat{f}(n) = 0$ for $n < m$.

PROPOSITION 4.8. *Let $T \in \mathscr{L}$. Then the following are equivalent:*

(i) *$T$ is analytic.*

(ii) *$T(L^2_m(\mathbf{T})) \subseteq L^2_m(\mathbf{T})$ for all $m \in Z$.*

*Proof.* (i) implies (ii). Let $m \in Z$. Then the inclusion

$$T(L_m^2(\mathbf{T})) \subseteq L_m^2(\mathbf{T}) \tag{4.6}$$

is clear if $T = M_1$ or if $T \in \mathscr{L}_0$. Thus, (4.6) will hold for all $T \in \mathscr{A}_0$ since $\mathscr{A}_0$ is generated by $M_1$ and the operators in $\mathscr{L}_0$. That (4.6) holds for an arbitrary $T$ in $\mathscr{A}$ now follows from Proposition 4.6.

(ii) implies (i). Let $T \in \mathscr{L}$ and assume that $T$ satisfies (4.6) for all $m \in Z$. Let $r < 0$. We must show that $\hat{T}(r) = 0$. Let $m$ be an arbitrary integer. Since linear combinations of exponentials are dense in $L^2(\mathbf{T})$, it suffices to show that $\hat{T}(r)e^{im\cdot} = 0$. Since, by (ii) and (iii) of Proposition 2.1, $\hat{T}(r)e^{im\cdot}$ is a multiple of $e^{i(r+m)\cdot}$, it suffices to show that

$$(\hat{T}(r)e^{im\cdot}, e^{i(r+m)\cdot}) = 0. \tag{4.7}$$

By Lemma 4.3,

$$(Te^{im\cdot}, e^{i(r+m)\cdot}) = (\hat{T}(r)e^{im\cdot}, e^{i(r+m)\cdot}) \tag{4.8}$$

Since (ii) holds, $Te^{im\cdot} \in L_m^2(\mathbf{T})$. Since $r < 0$, $e^{i(r+m)\cdot}$ is orthogonal to $L_m^2(\mathbf{T})$, so the left side of equality (4.8) is zero. Thus (4.7) follows from (4.8).

## 5. The second F. and M. Riesz Theorem

In this section, we prove the following theorem, which is our extension to operators in $\mathscr{L}$ of a classical theorem of F. and M. Riesz.

THEOREM 5.1. *Let $T$ be an analytic operator in $\mathscr{L}$. If $T \neq 0$, then* $\mathbf{T}$ *is both a support and cosupport for $T$.*

We need two lemmas before proceeding to the proof of Theorem 5.1. The first is an immediate consequence of the classical theorem of F. and M. Riesz (this is the second corollary on p. 52 of [3]).

LEMMA 5.2. *Let $f$ be an integrable function on* $\mathbf{T}$, $m \in Z$. *Assume that either $\hat{f}(n) = 0$ if $n < m$ or $\hat{f}(n) = 0$ if $n > m$. Then $f$ cannot vanish on a subset of* $\mathbf{T}$ *having positive Lebesgue measure unless it vanishes* a.e. *on* $\mathbf{T}$.

*Proof.* The F. and M. Riesz Theorem asserts that the conclusion of Lemma 5.2 holds if $\hat{f}(n) = 0$ for $n < 0$. The other cases may be reduced to this case by consideration of the functions $e^{-im\cdot}f$ and $e^{im\cdot}\bar{f}$.

Recall that for $K$ a measurable subset of $\mathbf{T}$, $L^2(K)$ is

$$\{f: f \in L^2(\mathbf{T}), f = 0 \text{ a.e. off } K\},$$

and that for all $m \in Z$, $L_m^2(\mathbf{T})$ is

$$\{f: f \in L^2(\mathbf{T}), \hat{f}(n) = 0 \text{ if } n < m\}.$$

LEMMA 5.3. *Let $K$ be a measurable subset of* $\mathbf{T}$ *of positive Lebesgue measure and $m \in Z$. Then:*

(i) $L_m^2(\mathbf{T}) \cap L^2(K^c) = \{0\}$.
(ii) $L_m^2(\mathbf{T}) + L^2(K)$ *is dense in* $L^2(\mathbf{T})$.

*Proof.* (i) is immediate from Lemma 5.2, since $K$ has positive Lebesgue measure. We prove (ii) by contradiction. Assume that $L_m^2(\mathbf{T}) + L^2(K)$ is not dense in $L^2(\mathbf{T})$. Then there is a nonzero element $f$ of $L^2(\mathbf{T})$ orthogonal to both $L_m^2(\mathbf{T})$ and $L^2(K)$. Since $f$ is orthogonal to $L_m^2(\mathbf{T})$,

$$\hat{f}(n) = 0 \quad \text{if } n \geq m. \tag{5.1}$$

Since $f$ is orthogonal to $L^2(K)$,

$$f(x) = 0 \text{ a.e. on } K. \tag{5.2}$$

Lemma 5.2 shows that (5.1) and (5.2) cannot both hold unless $f$ is the zero element of $L^2(\mathbf{T})$.

We can now proceed to the proof of Theorem 5.1. We show first that $\mathbf{T}$ is a support for $T$. We argue by contradiction. If $\mathbf{T}$ were not a support for $T$, there would be a set $K$ of positive Lebesgue measure so that $T(L^2(K)) \subseteq \{0\}$. Then, because of the analyticity of $T$ and Proposition 4.8,

$$T(L_m^2(\mathbf{T}) + L^2(K)) \subseteq L_m^2(\mathbf{T}) \quad \text{for all } m \in Z. \tag{5.3}$$

By (5.3) and (ii) of Proposition 5.2,

$$T(L^2(\mathbf{T})) \subseteq L_m^2(\mathbf{T}) \quad \text{for all } m \in Z. \tag{5.4}$$

Since $\bigcap_{m \in Z} L_m^2(\mathbf{T}) = \{0\}$, (5.4) shows that $T$ must have been the zero operator. This completes the proof that $\mathbf{T}$ is a support for $T$.

We show next that $\mathbf{T}$ is a cosupport for $T$. If $\mathbf{T}$ were not a cosupport, there would be a subset $K$ of $\mathbf{T}$ whose complement $K^c$ has positive Lebesgue measure and which satisfies

$$T(L^2(\mathbf{T})) \subseteq L^2(K). \tag{5.5}$$

By (5.5), the analyticity of $T$ and Proposition (4.8),

$$T(L_m^2(\mathbf{T})) \subseteq L^2(K) \cap L_m^2(\mathbf{T}) \quad \text{for all } m \in Z. \tag{5.6}$$

Part (i) of Lemma 5.3 shows that $L^2(K) \cap L_m^2(\mathbf{T}) = \{0\}$, since $K^c$ has positive Lebesgue measure, so (5.6) becomes

$$T(L_m^2(\mathbf{T})) \subseteq \{0\} \quad \text{for all } m \in Z. \tag{5.7}$$

Let $P$ be the set of trigonometric polynomials on $\mathbf{T}$. Since $P \subseteq \bigcup_{-\infty}^{+\infty} L_m^2(\mathbf{T})$, we have

$$T(P) \subseteq \{0\} \tag{5.8}$$

as a consequence of (5.7). Since $P$ is dense in $L^2(\mathbf{T})$, (5.8) shows that $T$ must have been the zero operator. This completes the proof that $\mathbf{T}$ is a cosupport for $T$.

Finally, we point out that if $T$ is the operation of multiplication by a bounded measurable function $\phi$, Theorem 5.1 reduces to the classical F. and M. Riesz Theorem for $\phi$. This is immediate from Proposition 3.5 and Proposition 4.1.

## BIBLIOGRAPHY

1. K. deLeeuw, *An harmonic analysis for operators I: Formal properties*, Illinois J. Math., vol. 19 (1975), pp. 593–606.
2. R. E. Edwards, *Fourier series—A modern introduction*, Rinehart and Winston, New York, 1967.
3. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
4. F. Riesz and B. SzNagy, *Functional analysis*, F. Ungar, New York, 1955.

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