ON THE GE_n OF A RING

BY

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This paper is concerned with a subgroup of the general linear group, namely the one generated by diagonal and elementary matrices and called $GE(n, R)$. A ring for which $GL(n, R) = GE(n, R)$ is called a GE_n-ring. It will be proved (Theorem 1) that a large class of rings is not GE_2.

Let R be an associative ring with unit and let $GL(n, R)$ be the group of all invertible $n \times n$ matrices over R. Let $E(n, R)$ be the subgroup of $GL(n, R)$ generated by $E_{ij}(r)$ where $E_{ij}(r) = (a_{ij})$ such that $a_{ii} = 1$, $a_{ij} = r$, and $a_{kl} = 0$ otherwise. Let $GE(n, R)$ be the subgroup of $GL(n, R)$ generated by the group of invertible diagonal matrices and $E(n, R)$. Cohn [1] introduced this group in order to simplify his study of $GL(n, R)$. He called a ring for which $GL(n, R) = GE(n, R)$ a GE_n-ring. He then restricted his attention to $n = 2$.

Determining whether or not a ring is GE_n, or even GE_2, is a very difficult problem. Samuel suggested that the ring

$$\mathbf{R}[x, y]/(x^2 + y^2 + 1) = S$$

might be k-stage Euclidean. This would imply that S is GE_2. Murthy also wondered if this ring were GE_2 as it would be an example of a non-Euclidean GE_2-principal ideal domain. Note that S has an ascending filtration given by $S_n = \{[g]$: a minimal degree representative for $[g]$ has degree $\leq n\}$. Hence it has an associated graded ring T where $T_n = S_n/S_{n-1}$.

Let R^* denote the group of units of a ring R. For k a field, Cohn [1, p. 21, p. 24] defines a k-ring with degree function to be a ring R such that k is a subring of R, $k^* = R^*$, and there exists a map d: $R \to \mathbf{Z}$ such that:

(1) $d(a) = -\infty$ if and only if $a = 0$;
(2) $d(a) = 0$ if and only if $a \in R^*$;
(3) $d(a - b) \leq \max(d(a), d(b))$;
(4) $d(ab) = d(a) + d(b)$.

The total degree of a polynomial in $k[x_1, \ldots, x_n]$ is an example of such a degree function.

THEOREM 1. *Let S be a k-ring with an ascending filtration and let T be its associated graded ring such that T is an integral domain and $T_0 = k$. Then S has a degree function d. Moreover, if S is commutative and* $\dim_k T_1 > 1$, *then S is not a GE_2-ring.*

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Proof. Define d by $d(0) = -\infty$, and $d(a) = \min\{n: a \in S_n\}$ for $a \neq 0$. By Proposition 2.2 of [3, p. 71], this d is a degree function.

Cohn [1, p. 386] proved that if R is a k-ring with degree function which is also a GE_2-ring, then of any two elements of the same degree which form a regular row (i.e., generate the unit ideal), each is R-dependent on the other. In this case R-dependence means that given $p, q \in S$ of the same degree which generate the unit ideal, then p is R-dependent on q if there exists $a \in S$ such that $d(p - aq) < d(p)$ and $d(a) + d(q) \leq d(p)$, (See [2, p. 333] for a general discussion of R-dependence.) Clearly this last condition implies $d(a) \leq 0$, i.e., $a \in k$.

If $\dim_k T_1 > 1$, then there exists $\bar{x}, \bar{y} \in T_1$ which are linearly independent over k. Let x, y be lifts of $\bar{x}, \bar{y}$. Since S is commutative, $1 = (1 - xy)(1 + xy + (-x^2)y^2$, and hence $1 + xy$, y^2 generate the unit ideal. Since T is a graded integral domain, $\bar{x}\bar{y}$, $\bar{y}^2 \in T_2$. Thus $d(xy) = d(y^2) = 2$. If $1 + xy$ were R-dependent on y^2, there would exist $a \in k$ such that

$$d(1 + xy + ay^2) = d(1 + (x + ay)y) < 2.$$

But $d(x + ay) = 1$ for all a by the linear independence of x and y. Hence $d(1 + (x + ay)y) = 2$ for all a. Thus S is not GE_2.

Note that all that was needed from the assumption that S is commutative is the existence of $x, y \in S$ with $\bar{x}, \bar{y} \in T_1$ linearly independent over k such that $1 + xy$, y^2 generate the ring as an ideal. This can be shown in the case where T is commutative, k is central in S, and $\dim_k T_1 = 2$.

Let $R = k[x_1, \ldots, x_n]$, k a field, $n \geq 2$; let $f \in R$ be of degree $r \geq 2$ and let f_r be its r-th homogeneous part. Let $S = R/(f)$ and $T = R/(f_r)$. Then S has an ascending filtration given by $S_i = \operatorname{im} R_i$, where R_i is the set of polynomials of degree $\leq i$, i.e., $[h] \in S_i$ if and only if a minimal degree representative g of $[h]$ has degree $\leq i$.

PROPOSITION 2. *T is the associated graded ring of S.*

Proof. R is a graded ring with the usual grading given by the total degree. It is easily shown that there is a surjective homomorphism of graded rings θ from $R = \operatorname{gr}(R)$ to $\operatorname{gr}(S)$. If $h \in R$ is homogeneous of degree n, then $\theta(h) = [h] + S_{n-1}$ in S_n/S_{n-1}. Thus it suffices to show that $(f_r) = \ker\theta$. In S, $[f_r] = [-f_1]$ where $f = f_r + f_1$ and degree $f_1 < r$. Thus $\theta(f_r) = 0$. Suppose $\theta(h) = 0$, h homogeneous of degree n. Then $\theta(h) = 0$ implies that $[h] \in S_{n-1}$, i.e., there exists a $p \in k$ with degree $p < n$, such that $h = p + gf$, $g \in R$. Taking the nth homogeneous parts of each side gives $h = g_s f_r$ where $s + r = n$ and g_s is homogeneous of degree s. Thus $h \in (f_r)$ and $\ker\theta = (f_r)$.

COROLLARY 3. *Let R, f, f_r be as in Proposition 2. If f_r is irreducible, then $R/(f)$ is not GE_2. In particular, $\mathbf{R}[x, y]/(x^2 + y^2 + 1)$ is not GE_2.*

Proof. Since f_r is irreducible, T is an integral domain. Note that $R/(f)$ is commutative and the result follows from Theorem 1.

George Cooke (University of Maryland) observed another degree function for $S = k[x_1, \ldots, x_n]/(f)$ which can be written as

$$k[x_1, \ldots, x_{n-1}][\sqrt{a}],$$

where a is a polynomial in $x_1, \ldots, x_{n-1}$. He defined $d(p) = \deg N(p)$, where $p \in S$ and $N\colon S \to k[x_1, \ldots, x_{n-1}]$ is the norm. For a specific ring one must check that $S^* = k^*$ and that

$$d(p - q) \leq \max\,(d(p), d(q)).$$

The other properties follow easily and the proof proceeds as in Theorem 1. This shows, for example, that $S = k[x, y, z]/(x^i + y^j + z^2)$, where $i, j, 2$ are pairwise relatively prime, is not GE_2. This ring is not covered by Theorem 1. It is currently not known for what f this d is a degree function.

By [4, p. 33], $\mathbf{R}[x, y]/(y^3 - x^2)$ is not GE_2. Since $k[x, y]/(y^3)$ is GE_2, the implication "$R/(f_r)$ is GE_2 implies $R/(f)$ is GE_2" does not hold. However it seems likely to the author that the implication "$R/(f)$ is GE_2 implies $R/(f_r)$ is GE_2" holds.

PROPOSITION 4. *If $R = A[x_1, \ldots, x_n]$, A a commutative ring, $n \geq 3$, $f \in (x_i)$ for some i, then $R/(f)$ is not GE_2.*

Proof. Without loss of generality assume that $i = 3$. Then $\bar{x}_1\bar{x}_2$ and $\bar{x}_1^2$ are nonzero elements of $R/(f)$. Since there is a natural surjection $R/(f) \to R/(x_3)$, if

$$\begin{pmatrix} 1 + \bar{x}_1\bar{x}_2 & \bar{x}_1^2 \\ -\bar{x}_2^2 & 1 - \bar{x}_1\bar{x}_2 \end{pmatrix}$$

were elementary over $R/(f)$, it would be over $R/(x_3) \cong A[x_1, x_2, x_4, \ldots, x_n]$ and hence over $k[x_1, x_2, x_4, \ldots, x_n]$ where k is a field which is a homomorphic image of A, contradiction [1, p. 386].

Note that a GE_2-ring may be contained in a non-GE_2 ring (e.g., $Z \subsetneq Z[\sqrt{-5}]$) and a non-$GE_2$ ring may be contained in a GE_2 ring (e.g., $\mathbf{R}[x, y]/(x^2 + y^2 + 1) \subsetneq \mathbf{C}[x, y]/(x^2 + y^2 + 1)$).

Although many rings are not GE_2, it is possible to construct new GE_n-rings from known ones.

DEFINITION. A row $(a_1, \ldots, a_n)$ is called unimodular if its elements generate the ring as an ideal.

Note that for a commutative ring to be a GE_2 ring, it suffices to show that any unimodular row of length 2 can be reduced to $(1, 0)$ by transvections (i.e., by elementary operations).

PROPOSITION 5. *Let R be a commutative ring and let I be an ideal contained in $J(R)$, the Jacobian radical. If $\phi\colon GL(n, R) \to GL(n, R/I)$ is the canonical map, then*

$$\phi(GE(n, R)) = GE(n, R/I) \quad \textit{and} \quad \phi^{-1}(GE(n, R/I)) = GE(n, R).$$

In particular R is GE_n if and only if R/I is GE_n.

Proof. Clearly $\phi(E(n, R)) = E(n, R/I)$ and $\phi(GE(n, R)) \subseteq \phi(GE(n, R/I))$. Let $\bar{u} \in R/I^*$. Let u be a lift of $\bar{u}$ and u' be a lift of $\bar{u}^{-1}$. Then $uu' = 1 + v$, $v \in I$. Since $I \subseteq J(R)$, $1 + v \in R^*$. Hence $u(u'(1 + v)^{-1}) = 1$ and $u \in R^*$. Thus if $\bar{D} = \operatorname{diag}(\bar{u}_1, \ldots, \bar{u}_n)$, then

$$\operatorname{diag}(u_1, \ldots, u_n) \in GL(n, R).$$

So $\phi(GE(n, R)) = GE(n, R/I)$. Suppose $\phi(A) \in GE(n, R/I)$. By the above, there exists $B \in GE(n, R)$ such that $\phi(A) = \phi(B)$. Thus $\phi(AB^{-1}) = 1$ and $AB^{-1} = (c_{ij})$ where $c_{ij} \in I$ for $i \neq j$ and $c_{ii} = 1 + u_{ii}$, $u_{ii} \in I$. But $c_{ii} \in R^*$ so

$$AB^{-1} = \operatorname{diag}(c_{11}, \ldots, c_{nn})X, \quad X \in E(n, R).$$

Thus $A = \operatorname{diag}(c_{11}, \ldots, c_{nn})XB \in GE(n, R)$.

COROLLARY 6. (a) *$R/(f)$ is $GE_n \Leftrightarrow R/(f^r)$ is GE_n.*
(b) *$R/(f)$ is GE_n implies* inj lim *$R/(f^r)$ is GE_n.*

Proof. Since $(f) \subseteq J(R/(f^r))$, (a) follows from Proposition 5. There is a homomorphism ϕ: inj lim $R/(f^r) \to R/(f)$. Since

$$\ker (R/(f^r) \to R/(f)) \subseteq J(R/(f^r)), \quad r \geq 2,$$

$\ker \phi \subseteq J$ (inj lim $R/(f^r)$). Thus (b) follows from Proposition 5.

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