COMPACT EXTREMAL OPERATORS

BY

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1. Introduction

For X a Banach space, let $\mathscr{B}(X)$ denote the space of bounded linear operators and $\mathscr{C}(X)$ the space of compact linear operators. The identity of a Banach algebra is always an extreme point of its unit ball. See [1]. As a simple consequence, any unitary element is also extreme. Kadison [4] has shown that for X Hilbert space, the extreme points of the unit ball of $\mathscr{B}(X)$ are precisely the semiunitary operators (partial isometries such that either $TT^* = I$ or $T^*T = I$).

For X an arbitrary infinite dimensional Banach space, there is no reason to suspect that $\mathscr{C}(X)$ has many, if indeed any, extreme points in its unit ball. In the first place, $\mathscr{C}(X)$ does not contain any unitary operators. Moreover, the Krein-Millman theorem cannot be readily invoked to conjure up extreme points, since there are no known examples where $\mathscr{C}(X)$ is a conjugate space, and many examples where $\mathscr{C}(X)$ is known not to be a conjugate space. See [2]. Finally, it is known that, for X either Hilbert space or c_0, $\mathscr{C}(X)$ has no extreme points. See [5] for Hilbert space.

We present two results in this paper. First, we show that the unit ball of $\mathscr{C}(l^p)$ is the norm closed convex hull of its extreme points for $1 \leq p < \infty$ and $p \neq 2$. We do so by constructing extreme points which, like unitary operators use all the coordinates. For the bizarre James' space we construct very different extremal operators, not at all analogous to unitary operators.

2. l^p spaces

LEMMA 2.1. *Let $\{e_i\}$ be the standard basis for l^p with $2 < p < \infty$. Suppose $Te_j = \sum_{i=1}^{\infty} a_i e_i$, with each $a_i \neq 0$ and $\|Te_j\| = 1$; and that Te_k is nonzero for some $k \neq j$. Then $\|T\| > 1$.*

Proof. Without loss of generality, we can assume that each $a_i > 0$, since $\|T\| = \|VT\|$ where $Ve_i = (\text{sign } a_i)e_i$. Suppose $Te_k = \sum b_i e_i$. Note that $\|e_j \pm \lambda e_k\|^p = 1 + |\lambda|^p$. We will show that for λ sufficiently small, either $\sum |a_i + \lambda b_i|^p$ or $\sum |a_i - \lambda b_i|^p$ is greater than $1 + |\lambda|^p$.

Suppose $0 < |\lambda b| < a$. By applying Taylor's theorem to

$$f(\lambda) = |a + \lambda b|^p + |a - \lambda b|^p$$

we have

$$|a + \lambda b|^p + |a - \lambda b|^p \geq 2a^p + \lambda^2 \tfrac{1}{2}(p(p-1))b^2[|a + \theta\lambda b|^{p-2} + |a - \theta\lambda b|^{p-2}]$$

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where $0 < \theta < 1$. Therefore

$$|a + \lambda b|^p + |a - \lambda b|^p \geq 2a^p + \lambda^2 \tfrac{1}{2}(p(p-1))a^{p-2}|b|^2.$$

Clearly,

$$|a + \lambda b|^p + |a - \lambda b|^p \geq 2a^p \quad \text{if } \lambda b = 0 \text{ or } |\lambda b| > a > 0.$$

Thus,

$$\sum (|a_i + \lambda b_i|^p + |a_i - \lambda b_i|^p) > 2 + \lambda^2 \sum_{|\lambda b_i| \leq a_i} \tfrac{1}{2}(p(p-1))a_i^{p-2}b_i^2.$$

As $\lambda \to 0$, the last sum (the coefficient of λ^2) increases monotonically. Thus λ can be chosen small enough so that

$$\sum (|a_i + \lambda b_i|^p + |a_i - \lambda b_i|^p) > 2(1 + |\lambda|^p).$$

LEMMA 2.2. *Let $\{e_i\}$ be the standard basis for l^p with $2 < p < \infty$. Suppose $Te_j = \sum a_i e_i$ and $Te_k = \sum b_i e_i$, where $j \neq k$, and for some i both a_i and b_i are nonzero. Then $\|T\| > \|Te_j\|$.*

Proof. The proof of Lemma 2.2 is similar to that of Lemma 2.1.

DEFINITION. An operator S in $\mathscr{C}(l^p)$ is said to be concentrated on $[e_1, \ldots, e_n]$ if range $S \subseteq [e_1, \ldots, e_n]$ and kernel $S \supseteq [e_{n+1}, \ldots]$.

PROPOSITION 2.3. *For $2 < p < \infty$, the unit ball of $\mathscr{C}(l^p)$ is the norm closed convex hull of its extreme points.*

Proof. For any positive integer n, let S be any operator in $\mathscr{C}(1^p)$ which is concentrated on $[e_1, \ldots, e_n]$ and which is extremal in the unit ball of $\mathscr{B}([e_1, \ldots, e_n])$. We will show that for each such S there exists an operator $V + T$ such that both $S + V + T$ and $S - V - T$ are extremal in the unit ball of $\mathscr{C}(l^p)$. Then, since the unit ball of $\mathscr{B}([e_1, \ldots, e_n])$ is the closed convex hull of its extreme points, and since the set of all operators which are concentrated on some $[e_1, \ldots, e_n]$ are dense in $\mathscr{C}(1^p)$, it follows that the unit ball of $\mathscr{C}(1^p)$ is the norm closed convex hull of its extreme points.

For S as described above, we now give the construction of V. Consider

$$\{W\colon We_{n+1} \in [e_1, \ldots, e_n],\ We_i = 0 \text{ for all other } i, \text{ and } \|S + W\| = 1\}.$$

Let V_1 be an operator of maximum norm from that set of W. Suppose $V_1, \ldots, V_k$ have been defined, where $k < n$. Consider

$$\{W\colon We_{n+k+1} \in [e_1, \ldots, e_n],\ We_i = 0 \text{ for all other } i, \text{ and } \|S + V_1 + \cdots + V_k + W\| = 1\}.$$

Let V_{k+1} be an operator of maximum norm from that set of W. This defines $V_1, \ldots, V_n$. Let $V = V_1 + \cdots + V_n$.

Note that the V_j must map onto disjoint coordinates. That is, for $j < k$, if $V_j e_{n+j}$ has nonzero ith coordinate, then $V_k e_{n+k}$ must have ith coordinate zero.

Suppose the contrary. Then by Lemma 2.2, $\|V_j + V_k\| > \|V_j\|$, and there would exist an element z of unit norm such that $\|(V_j + V_k)z\| > \|V_j\|$. Then W, defined by $We_{n+j} = (V_j + V_k)z$ and $We_m = 0$ for all other m, would contradict the maximality property of V_j.

We claim that $S + V$ is extremal as an element of the unit ball of

$$\mathscr{B}(l^p, [e_1, \ldots, e_n]).$$

Suppose $S + V + A$ and $S + V - A$ both have norm one. Consider the following three cases.

For $1 \leq j \leq n$, $Ae_j = 0$, since S was extremal in the unit ball of

$$\mathscr{B}([e_1, \ldots, e_n]).$$

For $1 \leq j \leq n$, $Ae_{n+j} = 0$, by a simple induction argument using the fact that l^p is strictly convex and the maximality property of each V_j.

For $m > 2n$, $Ae_m = 0$. To see this, consider these two cases: If $V_n = 0$, then Ae_m nonzero would contradict the maximality property of V_n. If $V_1, \ldots, V_n$ are all nonzero, and $Ae_m \neq 0$, then, for some i and j, Ae_m and V_je_{n+j} would both have nonzero ith coordinate, and this too would contradict the maximality of V_j. Thus, we have finished the proof that $S + V$ is extreme in the unit ball of $\mathscr{B}(1^p, [e_1, \ldots, e_n])$.

Next, define the operator T by $Te_{2n+1} = \sum_{i=n+1}^{\infty} a_ie_i$, with each $a_i \neq 0$ and $\|Te_{2n+1}\| = 1$, and $Te_j = 0$ all other j.

We claim that $S + V + T$ is extremal in the unit ball of $\mathscr{C}(l^p)$. Suppose that $S + V + T \pm B$ both have norm 1. By the strict convexity of l^p, $Be_{2n+1} = 0$. For $m \neq 2n + 1$, Be_m cannot have nonzero ith coordinate for $1 \leq i \leq n$, by the extremality of $S + V$, and for $i > n$, by Lemma 2.2 applied to $T \pm B$. Thus, $B = 0$, and $S + V + T$ is extreme. Of course, $S - V - T$ is also extreme, and this concludes the proof of the proposition.

As we have already mentioned, the unit ball of $\mathscr{C}(1^2)$ has no extreme points. The previous proposition, however, can be extended to $\mathscr{C}(1^p)$, for $1 < p < 2$, by using the fact that $\mathscr{C}(1^p)$ is isometrically isomorphic to $\mathscr{C}(1^q)$, where $1/p + 1/q = 2$; also it can easily be proved directly for $p = 1$. Thus we have:

THEOREM 2.4. *For $1 \leq p < \infty$ and $p \neq 2$, the unit ball of $\mathscr{C}(l^p)$ is the norm closed convex hull of its extreme points.*

Remark. The extremal operators that we constructed in proving Proposition 2.3 are analogous to unitary operators in that they map onto all coordinates of l^p. That is, the matrix for such an operator has at least one nonzero entry in each row. The adjoint of such an operator, which would be extreme in $(l^p)^*$, has at least one nonzero entry in each column. In the next section, we give compact extremal operators which are not analogous to unitary operators in this sense.

PROPOSITION 2.5. *Let T be an isometry from l^p to l^p, for $2 < p < \infty$. Then for each j, there exists $\sigma_j \subset N$ such that $Te_j = \sum_{i \in \sigma_j} \lambda_i e_i$ and $\sigma_j \cap \sigma_k = \emptyset$ when $k \neq j$.*

Proof. T must achieve its norm on each e_j, since T is an isometry. By a proof similar to that of Lemma 2.1, if for $j \neq k$, Te_j and Te_k both had a nonzero coefficient for some e_i, then the norm of T would be greater than one.

3. A space of James

Let X be the normed space of all those sequences x in R^ω such that (1) $\lim x(i) = 0$ and (2) $\|x\|$ is finite where

$$\|x\| = \sup \left\{ (x(p_n) - x(p_1))^2 + \sum_1^{n-1} (x(p_{j+1}) - x(p_j))^2 \right.$$
$$\left. \text{such that } \{p_j\} \text{ is a finite increasing subset of the positive integers} \right\}.$$

James has shown [3] that X is a Banach space and the standard vectors $\{e_i\}$ form a monotone, shrinking basis. The following three facts are easily verified:

(i) For each $x \in X$ and $k \in \omega$, $\|x\|^2 \geq 2|x(k)|^2$,

(ii) Any element in X with n consecutive ones (where $n \geq 1$), and all other coordinates zero, has norm $\sqrt{2}$.

(iii) Any $x \in X$ with $x(j) = 1$ and $x(k)$ negative, for some j and k, has norm greater than $\sqrt{2}$.

PROPOSITION 3.1. *Suppose E sends e_j to $\pm e_k$ and all other basis vectors to 0. Then E is extremal in the unit ball of $\mathscr{B}$.*

Proof. We will give the proof for E which sends e_j to e_k. Clearly, E has norm 1. Suppose there exists an operator A such that both $E + A$ and $E - A$ have norm 1. Note that

$$(E \pm A)\left(\sum_p^q e_i\right) = \pm A\left(\sum_p^q e_i\right) + e_k \quad \text{if } p \leq j \leq q.$$

Then, if $p \leq j \leq q$ we have $\|\sum_p^q e_i\| = \sqrt{2}$ and also $\| \pm A(\sum_p^q e_i) + e_k\| > \sqrt{2}$ for at least one choice of sign if $A(\sum_p^q e_i) \neq 0$. Hence $A(\sum_p^q e_i) = 0$ whenever $p \leq j \leq q$. This implies that $A = 0$ and thus E is extreme.

Question 1. For a Banach space X, what is a sufficient condition for the unit ball of $\mathscr{C}(X)$ to be the norm closed convex hull of its extreme points?

Question 2. Which is more typical with regard to extreme points, the behavior of $\mathscr{C}(c_0)$ and $\mathscr{C}(l^2)$, or $\mathscr{C}(l^p)$ with $p \neq 2$?

Question 3. Are there any X for which $\mathscr{C}(X)$ is a conjugate space? (It is a theorem of Bessaga and Pelczynski that the unit ball of any separable conjugate space is the *norm* closed convex hull of its extreme points.)

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