ZETA FUNCTIONS OF SELBERG'S TYPE FOR COMPACT SPACE FORMS OF SYMMETRIC SPACES OF RANK ONE

BY

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0. Introduction

Let M be a compact Riemann surface of genus $g \geq 2$. Then $M = \Gamma\backslash H$ where H is the upper half plane, and Γ is a discrete subgroup of $SL(2, \mathbf{R})$, acting freely on H via fractional linear transformations. Let T be a finite dimensional unitary representation of Γ with character χ. In a well-known paper [21], A. Selberg showed how we may attach a zeta function $Z_\Gamma(s, \chi)$ (of a complex variable s) to this data, and showed how the location and the orders of the zeros of Z_Γ give us information about the spectrum of M on the one hand and about the topology of M (via its Euler characteristic) on the other hand.

Now let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup, and H the symmetric space G/K; We endow H with a G-invariant metric. Let Γ be a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ is compact. Then the manifold $\Gamma\backslash H(\Gamma\backslash G/K)$ which we will call M, is a compact Riemannian manifold, whose simply connected covering manifold is H, and we have $\Gamma \cong \pi_1(M)$. M is a compact space form of H.

We assume throughout this paper that rank $(G/K) = 1$.

Let T be a finite-dimensional unitary representation of Γ, and let χ be its character. The object of this paper is to study a certain zeta function $Z_\Gamma(s, \chi)$ attached to the data (G, K, Γ, χ). We shall see that this zeta function has all of the properties possessed by Selberg's zeta function. The following properties will be discovered:

(1) Z_Γ is holomorphic in a half plane $\operatorname{Re} s > 2\rho_0$ where ρ_0 is a positive real number depending only on (G, K).

(2) Z_Γ has a meromorphic continuation to the whole complex plane.

(3) Z_Γ satisfies the functional equation

$$Z_\Gamma(2\rho_0 - s, \chi) = \left\{\exp\left(\kappa\chi(1) \operatorname{vol}(\Gamma\backslash G) \int_0^{s-\rho_0} c(it)^{-1}c(-it)^{-1}\, dt\right)\right\} Z_\Gamma(s, \chi).$$

Here, vol $(\Gamma\backslash G)$ is the volume of $\Gamma\backslash G$ in a suitable normalization, κ is a positive integer depending only on (G, K), and $c(\cdot)$ is Harish-Chandra's c-function which appears in the Plancherel measure for G/K [10]. In our case, this function is essentially a function of one complex variable.

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(4) $Z_\Gamma(s, \chi)$ always has certain zeroes that we call spectral zeroes. These are located at certain points s_j^+, s_j^-, $j \geq 0$, with $s_j^- = 2\rho_0 - s_j^+$. When $s_j^+ \neq 0$, the order of the zero at s_j^+ equals κn_j where κ is the integer mentioned above, and n_j is a positive integer depending on χ. These zeroes are called spectral because their location and order gives us spectral information, in the following sense: Let U be the representation of G induced from the representation T of Γ. Then certain spherical representations of G, say $\{U_j, j \geq 0\}$ will occur as summands in U. Let $\{\nu_j, j \geq 0\}$ be the parameters attached to $\{U_j\}$ in the usual way (cf. Section 1 below). We shall see that the numbers s_j^+ (or s_j^-) determine the parameters $\{\nu_j, j \geq 0\}$. Moreover, with at most one exception, the integer n_j equals the multiplicity with which U_j occurs in U.

(5) Apart from the spectral zeroes of Z_Γ, there may exist a series of "topological" zeroes or poles of Z_Γ. These exist only when $\dim(G/K)$ is even, or what is the same, when the Euler-Poincaré characteristic of M is nonzero. We can be rather precise about their location. Indeed, let $\{r_k, k \geq 0\}$ be the poles of the function $r \to c(r)^{-1}c(-r)^{-1}$ in the upper half-plane $\operatorname{Im} r \geq 0$. One sees that there is always a pole at $i\rho_0$, and we arrange matters so that $r_0 = i\rho_0$. The topological zeroes or poles of Z_Γ, when they exist, occur at the points $\rho_0 + ir_k$, $k \geq 1$. The numbers $\rho_0 + ir_k$, $k \geq 1$ are all negative integers, and for a given G, they are either *all* poles or *all* zeroes of Z_Γ. Whether we have zeroes or poles depends on the sign of the numbers id_k, $k \geq 0$, where d_k is the residue of $c(r)^{-1}c(-r)^{-1}$ at r_k. The numbers id_k are all real, nonzero, and have the same sign. If this sign is positive, then Z_Γ has poles at the points $\rho_0 + ir_k$, $k \geq 1$. Z_Γ has zeroes at $\rho_0 + ir_k$, $k \geq 1$ in the opposite case. In any case, the order of the zero or pole is always a multiple of the Euler-Poincare characteristic E of $M = \Gamma\backslash G/K$. This order is of the form $|\chi(1)e_k E|$ where e_k is an explicitly computable integer, depending on d_k, κ, etc. $e_k E$ and id_k have the same sign. Computations show that poles occur precisely when $\dim(G/K) \equiv 0 \bmod 4$.

When $G = SO_0(2n+1, 1)$, the function $r \to c(r)^{-1}c(-r)^{-1}$ is a polynomial; In these cases, $\dim(G/K) = 2n+1$ so that $E = 0$. The function Z_Γ has only spectral zeroes in this case, and its functional equation simplifies.

(5 bis) The point $s = 0$ is somewhat special in that the behavior of Z_Γ at this point has both spectral and topological aspects. Roughly speaking, the "spectral part" of Z_Γ contributes a zero of order κa_0 where a_0 is the multiplicity of the trivial representation of Γ in the representation T; the "topological" part of Z_Γ contributes a zero at $s = 0$ if $\chi(1)e_0E$ is negative, of order $|\chi(1)e_0E|$, and a pole of order $\chi(1)e_0E$ at $s = 0$ if $\chi(1)e_0E$ is positive. Here as above, e_0 is explicitly computable in terms of d_0, κ, where d_0 is the residue of $c(r)^{-1}c(-r)^{-1}$ at the pole $r_0 = i\rho_0$. The upshot is that Z_Γ has a pole (resp. zero) of order $|\kappa a_0 - \chi(1)e_0E|$ at $s = 0$, if $\kappa a_0 - \chi(1)e_0E$ is negative (resp. positive).

(6) The zeroes (poles) described above are the only zeroes (poles) of Z_Γ. When the poles do not exist, Z_Γ is an entire function, of finite order. The order can be related to the structure of (G, K). It equals $\dim(G/K)$.

(7) The spectral zeroes $\{s_j^+, s_j^-; j \geq 0\}$ lie on the line $\operatorname{Re} s = \rho_0$ except for

a finite number of indices j. Thus Z_Γ satisfies a sort of modified Riemann hypothesis. The representations U_j which correspond to the s_j^+ lying on $\operatorname{Re} s = \rho_0$ are all in the spherical principal series. Those s_j^+, s_j^- which are off the line $\operatorname{Re} s = \rho_0$ are all real, and lie in the interval $[0, 2\rho_0]$, symmetrically about ρ_0. The corresponding representations U_j are all in the spherical complementary series. One can show that for certain G and Γ, these zeroes actually occur, and that their total number can be made large as we please, by choosing G, Γ, χ properly. For a fixed G, Γ, the number of such zeroes is no bigger than a multiple of $\chi(1)$ vol $(\Gamma\backslash G)$.

(8) The logarithmic derivative of Z_Γ in the half plane $\operatorname{Re} s > 2\rho_0$ (where Z_Γ is zero-free) is related via an integral transform to a sort of theta function $\theta(t)$, $t > 0$. This theta function arises from the fundamental solution of the heat equation on M. Thus the relation between Z_Γ and θ is analogous to the relation between the classical ζ-function of Riemann and the Jacobi theta function (cf. [5]).

(9) Z_Γ has an infinite product representation in the half plane $\operatorname{Re} s > 2\rho_0$. The product runs over the conjugacy classes of primitive elements in Γ, and over a certain lattice of linear forms on a Cartan subalgebra of the Lie algebra of G. When $G = SL(2, \mathbf{R})$ the infinite product reduces to the one given by Selberg (cf. the end of Section 2 below).

(10) Z_Γ has natural properties with respect to the character χ. Thus, one has

$$Z_\Gamma(s, \chi + \chi') = Z_\Gamma(s, \chi)Z_\Gamma(s, \chi') \quad \text{and} \quad Z_\Gamma(s, \chi\chi') = \prod Z_\Gamma(s, \chi_i)^{m_i},$$

where $\chi\chi' = \sum m_i\chi_i$. We also have $Z_\Gamma(s, \chi^*) = Z_\Gamma(s, \chi)$ where χ^* is the contragredient of χ. Of course, one may phrase these properties in terms of direct sums or tensor products if one wishes.

These properties of Z_Γ are established in Section 2, after preliminaries in Section 1. Section 3 is an appendix, devoted to an auxiliary computation.

A few remarks about our results are in order. Our method uses the trace formula of Selberg in one of its simplest versions, and generalizes Selberg's method for $SL(2, \mathbf{R})$. Selberg defined his zeta function for $SL(2, \mathbf{R})$ and described its properties in [21], without any proofs. Selberg's method for $SL(2, \mathbf{R})$ was expounded by Kuga in [15], a paper based on Selberg's lectures given in the late 1950's in Princeton. In his paper, Selberg mentioned that the trace formula can be established satisfactorily for the hyperbolic spaces of higher dimension, but gave no details about it. Thus, the present paper may be regarded as an attempt to understand the situation for $\mathbf{R}$-rank one groups.

The result (7) described above implies that aribtrarily large numbers of nontempered spherical representations can occur in the representation U, for suitable G, Γ, χ. That a nontempered representation can occur in $L_2(G/\Gamma)$ has been observed by Wallach; cf. [27]. However, that representation is nonspherical. Besides, that method does not lead immediately to the assertion that arbitrarily large numbers of such representations can occur.

Unfortunately, I do not know what arithmetical significance Z_Γ might have, either locally or globally, even in the case of $SL(2)$. In particular, I do not know if these Z_Γ can be related in any way to the zeta and L-functions of Godement, Jacquet, and Langlands; cf. [9], [13].

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1. Preliminaries

Let G be a connected noncompact semisimple Lie group with finite center, K a maximal compact subgroup of G. Let $\mathfrak{g}$, $\mathfrak{k}$ be their respective Lie algebras, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with respect to the involution θ determined by $\mathfrak{k}$. For any $X \in \mathfrak{g}$, $\langle \cdot, \cdot \rangle$ denotes the Cartan-Killing form. Put $|X|^2 = -\langle X, \theta X \rangle$; then $|\cdot|$ is a norm on $\mathfrak{g}$. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. *Throughout this paper we assume that* $\dim \mathfrak{a}_\mathfrak{p} = 1$. Extend $\mathfrak{a}_\mathfrak{p}$ to a maximal abelian θ-stable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, so that $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$, with $a_\mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}$, $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{g}^C$, $\mathfrak{a}^C$ the complexifications of $\mathfrak{g}$, $\mathfrak{a}$, and let $\Phi(\mathfrak{g}^C, \mathfrak{a}^C)$ denote the set of roots of $(\mathfrak{g}^C, \mathfrak{a}^C)$. Order the dual spaces of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$ compatibly, as usual (cf. [12]), and let Φ^+ be the set of positive roots under this order. Let

$$P_+ = \{\alpha \in \Phi^+;\ \alpha \not\equiv 0 \text{ on } \mathfrak{a}_\mathfrak{p}\}, \qquad P_- = \{\alpha \in \Phi^+;\ \alpha \equiv 0 \text{ on } \mathfrak{a}_\mathfrak{p}\}.$$

Put $\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha$. For $\alpha \in \Phi^+$, let X_α be a root vector belonging to α, and put $\mathfrak{n}^C = \sum_{\alpha \in P_+} CX_\alpha$. Then if $\mathfrak{n} = \mathfrak{n}^C \cap \mathfrak{g}$, we have the Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$, $G = KA_\mathfrak{p}N$, where of course $A_\mathfrak{p} = \exp \mathfrak{a}_\mathfrak{p}$, $N = \exp \mathfrak{n}$. We will denote by W the Weyl group of $(G, A_\mathfrak{p})$.

We denote by Λ the real dual of $\mathfrak{a}_\mathfrak{p}$, by Λ^C its complexification $\Lambda + i\Lambda$. For $\lambda \in \Lambda$, we can write $\lambda = \operatorname{Re} \lambda + i \operatorname{Im} \lambda$, with $\operatorname{Re} \lambda$, $\operatorname{Im} \lambda$ in Λ.

Denote by $C^\infty(K\backslash G/K)$ the space of differentiable spherical functions (i.e., those that satisfy $f(k_1xk_2) = f(x)$, $x \in G$, $k_1, k_2 \in K$) and by $C_c^\infty(K\backslash G/K)$, those elements of $C^\infty(K\backslash G/K)$ which have compact support: The spaces $L_1(K\backslash G/K)$, $L_2(K\backslash G/K)$ have the obvious meaning. For any $\nu \in \Lambda^C$, we denote by ϕ_ν the elementary spherical function corresponding to ν (cf. [12]). Let $\Xi(x)$ denote the elementary spherical function ϕ_0, and let $\sigma(x) = |X|$, where $x = k \exp X$, $X \in \mathfrak{p}$ is the polar decomposition of $x \in G$. The Harish-Chandra-Schwartz space $\mathscr{C}(G)$ is now defined as in [11]. We have

$$\mathscr{C}(G) = \Big\{ f \in C^\infty(G);\ \sup_{x \in G} \Xi(x)^{-1}(1 + \sigma(x))^r |Df(x)| < \infty \quad \text{for all } r \geq 0, \text{ all } D \Big\} \tag{1.1}$$

where D denotes a left or right invariant differential operator on G.

We similarly define

$$(1.2)\quad \mathscr{C}_1(G) = \left\{ f \in C^\infty(G);\ \sup_{x\in G} \Xi(x)^{-2}(1 + \sigma(x))^r|Df(x)| < \infty \text{ for all } r, D \right\}.$$

Then $\mathscr{C}_1(G) \subset \mathscr{C}(G) \subset L_2(G)$, and $\mathscr{C}_1(G) \subset L_1(G)$.

The subspaces of spherical functions in $\mathscr{C}(G)$, $\mathscr{C}_1(G)$ will be denoted by $\mathscr{C}(K\backslash G/K)$ and $\mathscr{C}_1(K\backslash G/K)$ respectively.

Let Σ be the set of restrictions to $\mathfrak{a}_\mathfrak{p}$ of elements of P_+. Since $\dim \mathfrak{a}_\mathfrak{p} = 1$, one knows that we can find $\beta \in \Sigma$ such that 2β is the only other possible element in Σ. Let p be the number of roots in P_+ whose restriction to $\mathfrak{a}_\mathfrak{p}$ is β, and let q be the number of the remaining elements of P_+. We fix the element $H_0 \in \mathfrak{a}_\mathfrak{p}$ by the property $\beta(H_0) = 1$. Then one knows that

$$\langle H_0, H_0\rangle = 2p + 8q, \qquad \rho(H_0) = \tfrac{1}{2}(p + 2q),$$
$$H_\beta = (2p + 8q)^{-1}H_0 \quad\text{and}\quad \langle \rho, \rho\rangle = \tfrac{1}{4}(p + 2q)^2(2p + 8q)^{-1}.$$

Throughout this paper, we will denote by ρ_0 the number $\rho(H_0)$.

For any $h \in A_\mathfrak{p}$, we put $u(h) = \beta(\log h)$. Then $u = u(h)$ may be regarded as a parameter on the group $A_\mathfrak{p}$. By this parametrization $A_\mathfrak{p}$ can be identified with $\mathbf{R}$. Let du be the standard Lebesgue measure on $\mathbf{R}$. Via the identification of $A_\mathfrak{p}$ with $\mathbf{R}$, we get a Haar measure dh on $A_\mathfrak{p}$ which we fix from now on.

For any $\nu \in \Lambda$, we put $r = r(\nu) = \nu(H_0)$. Then r is a parameter on Λ, and maps Λ isomorphically onto $\mathbf{R}$. In these parameters, $\nu(\log h) = u(h)r(\nu)$ for $\nu \in \Lambda$, $h \in A_\mathfrak{p}$. Let dr be the Lebesgue measure on $\mathbf{R}$. Then $dr/2\pi$ is the measure on $\mathbf{R}$ dual to the measure du on $\mathbf{R}$ (in the sense of Fourier transforms). We denote by $d\nu$ the measure on Λ that we obtain from $dr/2\pi$. Then dh, $d\nu$ are dual in the sense of Fourier transforms.

Let dk be the normalized Haar measure on K. On N we fix a Haar measure normalized by the following condition: Let $\bar{n} = \theta(n^{-1})$ for $n \in N$, and for any $x \in G$, let $H(x) \in \mathfrak{a}_\mathfrak{p}$ be defined by $x = k \exp H(x)n$, $k \in K$, $n \in N$. The measure dn is to satisfy the condition $\int_N \exp(-2\rho(H(\bar{n})))\, dn = 1$. (This choice of measure on N is motivated by our need to use the Plancherel theorem on G/K repeatedly. It makes the Plancherel measure less cumbersome to write.) Having fixed the above measures on K, $A_\mathfrak{p}$, N, we fix the Haar measure dx on G given by

$$dx = \exp 2\rho(\log h)\, dk\, dh\, dn.$$

These normalizations will be adhered to throughout in what follows.

The Plancherel theorem of Harish-Chandra, for spherical functions, now takes the following form: For $f \in \mathscr{C}(K\backslash G/K)$, we have

$$(1.3)\qquad \hat{f}(\nu) = \int_G f(x)\phi_\nu(x)\, dx,$$

$$(1.4)\qquad f(x) = [W]^{-1}\int_\Lambda \hat{f}(\nu)\phi_\nu(x^{-1})c(\nu)^{-1}c(-\nu)^{-1}\, d\nu$$

where $c(\cdot)$ is the c-function of Harish-Chandra.[2] In our case $c(\cdot)$ is given by

$$c(\nu)^{-1} = \frac{\Gamma((p + q)/2)\Gamma(ir + p/2)\Gamma(ir/2 + p/4 + q/2)}{\Gamma(p + q)\Gamma(ir)\Gamma(ir/2 + p/4)} \tag{1.5}$$

where $r = r(\nu) = \nu(H_0)$, $\Gamma(\cdot)$ being the classical gamma function.

It will be convenient to write $c(r)^{-1}$ for the function of r on the right side. This double use of c will not cause any confusion. (1.5) implies that $r \mapsto c(r)^{-1}$ is a tempered function.

The Abel transform F_f of $f \in \mathscr{C}(K\backslash G/K)$ is

$$F_f(h) = \exp \rho (\log h) \int_N f(hn)\, dn, \quad h \in A_{\mathfrak{p}}. \tag{1.6}$$

It is related to $\hat{f}$ by

$$\hat{f}(\nu) = F_f^*(\nu) = \int_{A_{\mathfrak{p}}} F_f(h) \exp i\nu (\log h)\, dh. \tag{1.7}$$

Thus F_f^* is the ordinary Fourier transform of F_f. By Fourier inversion we have

$$F_f(h) = \int_\Lambda \hat{f}(\nu) \exp (- i\nu (\log h))\, d\nu, \quad h \in A_{\mathfrak{p}}. \tag{1.8}$$

For a given $f \in \mathscr{C}(K\backslash G/K)$, the function F_f induces a function on $\mathbf{R}$ via the parametrization u, and we call this function F; Its Euclidean Fourier transform will be called F^*. Thus $F(u) = F_f(h)$, where $u = u(h)$, and $F^*(r) = F_f^*(\nu) = \hat{f}(\nu)$ where $r = r(\nu)$.

The formulas above now become

$$\hat{f}(\nu) = F^*(r) = \int_{-\infty}^{\infty} F(u) \exp iru\, du, \quad r = r(\nu), \tag{1.9}$$

$$F_f(h) = F(u) = (1/2\pi) \int_{-\infty}^{\infty} F^*(r) \exp (- iru)\, dr, \quad u = u(h). \tag{1.10}$$

The inversion formula (1.4) gives, for $x = 1$,

$$\begin{aligned} f(1) &= ([W]^{-1}/2\pi) \int_{-\infty}^{\infty} F^*(r)c(r)^{-1}c(-r)^{-1}\, dr \\ &= (1/4\pi) \int_{-\infty}^{\infty} F^*(r)c(r)^{-1}c(-r)^{-1}\, dr, \end{aligned} \tag{1.11}$$

which will be used incessantly below.

Now let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. We assume that Γ has no elements of finite order. Then every element $\gamma \in \Gamma$ is conjugate in

[2] $[W]$ stands for the order of the Weyl group W.

G to an element of the Cartan subgroup A = centralizer of $\mathfrak{a}$ in G. $A = A_{\mathfrak{t}}A_{\mathfrak{p}}$; choose an element $h(\gamma)$ of A to which γ is conjugate, and let $h(\gamma) = h_{\mathfrak{t}}(\gamma)h_{\mathfrak{p}}(\gamma)$. We then define $u_\gamma = \beta(\log h_{\mathfrak{p}}(\gamma))$. Thus $u_\gamma = u(h_{\mathfrak{p}}(\gamma))$. Though u_γ will depend on the choice of $h(\gamma)$, its absolute value $|u_\gamma|$ depends only on γ. ($|u_\gamma|$ is essentially the length of the shortest geodesic in the free homotopy class associated to γ on the manifold $\Gamma\backslash G/K$; cf. [7].)

An element $\gamma \in \Gamma$, $\gamma \neq 1$ is called primitive if it cannot be expressed as δ^n, for some $n > 1$, $\delta \in \Gamma$. It can be proved [7] that every $\gamma \neq 1$ is equal to a positive power of a unique primitive element δ. The integer $j(\gamma)$ is defined by $\gamma = \delta^{j(\gamma)}$.

Our chief tool is Selberg's trace formula. Let T be a finite dimensional unitary representation of Γ, with character χ. Denote by U the unitary representation of G induced by T. U is a discrete direct sum of irreducible unitary representations of G, occurring with finite multiplicities. Let $\{U_j, j \geq 0\}$ be the spherical representations that occur in U, and let $n_j(\chi)$ be their multiplicities. For technical reasons, we always let U_0 be the trivial representation of G. Its multiplicity $n_0(\chi)$ is equal to a_0, where a_0 is the multiplicity of the trivial representation of Γ in T. Thus $n_0(\chi)$ may be zero. We shall nevertheless include U_0 in the collection $\{U_j\}$. Each U_j is completely determined by its elementary spherical function, say ϕ_{ν_j}, with $\nu_j \in \Lambda^{\mathbf{C}}$.[3] Since U_j is unitary, ϕ_{ν_j} is positive definite, and one knows, cf. [6], that $\langle \nu_j, \nu_j\rangle + \langle \rho, \rho\rangle \geq 0$. From this it follows that ν_j is either purely real, i.e., $\nu_j \in \Lambda$ or purely imaginary, i.e., $\nu_j \in i\Lambda$. We choose and fix ν_j so that when it is real, we have $\nu_j(H_0) \geq 0$, and when it is purely imaginary, we have $i\nu_j(H_0) < 0$. Since U_0 is the trivial representation, we have that $\nu_0 = i\rho$.

The notion of an admissible function f is defined as usual, cf. [8], and one has the trace formula

$$\begin{aligned} \sum_{j\geq 0} n_j(\chi)\hat{f}(\nu_j) \\ (1.12)\qquad &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \chi(\gamma) f(x^{-1}\gamma x)\, d\dot{x} \\ &= \chi(1)\operatorname{vol}(\Gamma\backslash G) f(1) + \sum_{\gamma\in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) F_f(h_{\mathfrak{p}}(\gamma)). \end{aligned}$$

which was derived in [7]. Here vol $(\Gamma\backslash G)$ stands for the volume of $\Gamma\backslash G$ in the invariant measure $d\dot{x}$ which arises on $\Gamma\backslash G$ when we equip Γ with counting measure, and $C(h)$ is a positive function depending only on the structure of G. The number $C(h(\gamma))F_f(h_{\mathfrak{p}}(\gamma))$ depends only on the G-conjugacy class of γ. C_Γ is a set of representatives in Γ for the Γ-Conjugacy class of elements of Γ.[4] $C(h(\gamma))$ is given by

$$(1.13)\qquad C(h(\gamma)) = \varepsilon_R^A(h(\gamma))\xi_\rho(h_{\mathfrak{p}}(\gamma)) \prod_{\alpha\in P_+} (1 - \xi_\alpha(h(\gamma))^{-1})^{-1}$$

[3] We then say that ν_j occurs in the spectrum.

[4] Since Γ has no nontrivial elements of finite order, it follows that no nontrivial element of Γ can be conjugate to its own inverse. Hence we can choose C_Γ in such a way that it is stable under $\gamma \to \gamma^{-1}$.

Here, for any α, ξ_α stands for the character of $A = A_{\mathfrak{h}}A_{\mathfrak{p}}$ defined by $\xi_\alpha(h) = \exp \alpha (\log h)$, and $\varepsilon_R^A(h)$ is, for $h \in A$, equal to the sign of $\prod_{\alpha \in \Phi_R^+} (1 - \xi_\alpha(h)^{-1})$, Φ_R^+ being the set of *real* roots of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{a}^{\mathbf{C}})$, i.e., those that are real on $\mathfrak{a}$. As seen in [7], $C(h)$ is a positive function on A.

The actual value of $C(h(\gamma))$ plays no role in the sequel.

One concludes from this formula, as in [7], that the numbers

$$\{|u_\gamma|;\, \gamma \in C_\Gamma - \{1\}\}$$

are bounded away from zero.

Because of [8], every $f \in \mathscr{C}_1(K\backslash G/K)$ is admissible, and can be used in the trace formula. Both sides then converge absolutely.

We shall write $r_j^+(\chi) = \nu_j(H_0)$, and $r_j^-(\chi) = -\nu_j(H_0)$ and put $s_j^+(\chi) = \rho_0 + ir_j^+$, $s_j^-(\chi) = \rho_0 + ir_j^-$, for $j \geq 0$. Though all these quantities depend on χ, when there is no risk of confusion we shall omit explicit mention of χ, and write n_j, s_j^+, s_j^+, etc. Now

$$\langle \nu_j, \nu_j \rangle + \langle \rho, \rho \rangle = |H_0|^{-2}((r_j^+)^2 + \rho_0^2)$$

as is easily seen, so that $\langle \nu_j, \nu_j \rangle + \langle \rho, \rho \rangle \geq 0$ implies $(r_j^+)^2 + \rho_0^2 \geq 0$. Thus $(r_j^+)^2$ is real and lies in $[-\rho_0^2, \infty)$. Note that either (i) $\operatorname{Re} s_j^+ = \rho_0$ or (ii) $\operatorname{Im} s_j^+ = 0$ and s_j^+ lies in the interval $[0, \rho_0)$. (Note that $s_0^+ = 0$.)

Clearly the numbers s_j^+ (or s_j^-) determine the numbers $\nu_j(H_0)$, and hence the linear forms ν_j. These in turn determine the spherical representation U_j. We know that when ν_j is real, i.e., $\nu_j \in \Lambda$, U_j is in the spherical principal series. This corresponds to $\operatorname{Re} s_j^+ = \rho_0$. On the other hand when $\nu_j \in i\Lambda$ and $\nu_j \neq 0$, U_j is in the spherical complementary series. This corresponds to s_j^+ in the interval $[0, \rho_0)$. In terms of the parameters r, u, and the functions F, F^* defined above, (1.9), (1.10), the trace formula takes the form

$$\begin{aligned} &\sum_{j \geq 0} n_j F^*(r_j^+) \\ &\quad = \chi(1) \operatorname{vol}(\Gamma\backslash G) f(1) + \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) F(u_\gamma) \end{aligned} \tag{1.14}$$

which will be the form most often used below.

It will be useful to have a simple condition on F or F^* which will imply that $f \in \mathscr{C}_1(K\backslash G/K)$. This can be easily done, by using the results of [24] (actually since rank $(G/K) = 1$, one could proceed directly as well). Suppose $F^*(z)$ satisfies (i) $F^*(z) = F^*(-z)$, (ii) for some $\varepsilon > 0$, F^* is holomorphic in the strip $\{z \in \mathbf{C}, |\operatorname{Im} z| \leq \rho_0 + \varepsilon\}$ and (iii) F^* is a rapidly decreasing function of $\operatorname{Re} z$ when z is on the boundary of this strip; then one can show that there exists $f \in \mathscr{C}_1(K\backslash G/K)$ such that $\hat{f}(\nu) = F^*(r)$; $r = r(\nu) = \nu(H_0)$. These conditions are easily translated in terms of $F(u)$. If $F(u)$ is C^∞, $F(-u) = F(u)$ and for some $\varepsilon > 0$,

$$\sup_u (\exp(\rho_0 + \varepsilon)|u|)|F(u)| < \infty,$$

then F^* will satisfy the above conditions.

The following propositions will be used below.

PROPOSITION 1.1 (Cf. [8]). *There exists an integer $d > 0$ such that*

$$\sum_{j \geq 0} n_j(1 + \langle \nu_j, \nu_j \rangle + \langle \rho, \rho \rangle)^{-d} < \infty.$$

In particular, this implies the convergence of

$$\sum_{j \geq 0} n_j(1 + r_j^+(\chi)^2 + \rho_0^2)^{-d}.$$

It follows that the numbers $r_j^+(\chi)^2$ do not have a finite point of accumulation. Thus $r_j^+(\chi)^2$ can lie in $[-\rho_0^2, 0)$ for only finitely many indices j. We shall assume the indices j to be chosen so, that $r_j^+(\chi)^2$ is an increasing sequence.

PROPOSITION 1.2 (Cf. [4], [6]). *For any $x \geq 0$ let $N(x)$ be defined by*

$$N(x) = \sum_{\{j;\, (r_j^+)^2 \leq x\}} n_j. \tag{1.15}$$

Then as $x \to \infty$ we have $N(x) \sim C_G \operatorname{vol}(\Gamma\backslash G)x^n$, where $n = \dim G/K$, and C_G is a constant depending only on G.

Note that $\langle \nu_j, \nu_j \rangle + \langle \rho, \rho \rangle$ is just the negative of the eigenvalue of the casimir operator of G operating on ϕ_{ν_j}. Thus the numbers $-(\rho_0^2 + r_j^+(\chi)^2)$ can be interpreted as the eigenvalues of the Laplace Beltrami operator of G/K (in a suitable metric). If ∇ is this operator, one can interpret the numbers $n_j(\chi)$ as the multiplicity of the eigenvalue $-(\rho_0^2 + r_j^+(\chi)^2)$ when ∇ operates on the smooth sections of the vector bundle E_χ whose base is $\Gamma\backslash G/K$ and the fibers are $\mathbf{C}^m$, where $m =$ degree (T), and Γ operates on the fibers via the representation T in the usual way.

Finally, we note that when T is the trivial one-dimensional representation of Γ, which (as well as its character) we denote by 1, then the trivial representation of G occurs in U with multiplicity one. The trivial representation of G corresponds to the element $i\rho \in i\Lambda$. Thus in this case the first term on the left side of (1.14) is precisely $F^*(i\rho_0)$, which will be used below.

2. The zeta function

With T, χ fixed as in Section 1 above, we shall define $Z_\Gamma(s, \chi)$ by writing down its logarithmic derivative with respect to s. This logarithmic derivative, which will be called $\Psi_\Gamma(s, \chi)$, will be written down in the form of a series convergent in $\operatorname{Re} s > 2\rho_0$. The series comes from the application of the trace formula to a suitable admissible function. We will first define Ψ_Γ and study it.

Let $\varepsilon_0 > 0$ be a fixed real number and let g be a real-valued function in $C^\infty(\mathbf{R})$ such that: (i) g is even, (ii) g vanishes in some neighborhood of zero, (iii) g is constant, equal to c, say, in $\{x \in \mathbf{R};\, |x| \geq \varepsilon_0\}$ and (iv) $0 \leq g \leq c$. Such functions surely exist. The constant c will be chosen conveniently later on.

Now let s be a complex variable and define

$$g(s, u) = g(|u|) \exp((\rho_0 - s)|u|), \quad u \in \mathbf{R}. \tag{2.1}$$

Then, for fixed s, $g(s, u)$ is an even smooth function of u and $g(s, u) = c \exp(\rho_0 - s)|u|$ if $|u| \geq \varepsilon_0$.

For $h_\mathfrak{p} \in A_\mathfrak{p}$, let $u(h_\mathfrak{p}) = \beta(\log h_\mathfrak{p})$. We regard $u = u(h_\mathfrak{p})$ as a parameter on $A_\mathfrak{p}$, and thus functions on $\mathbf{R}$ can be regarded as functions on $A_\mathfrak{p}$. Let F_s be defined on $A_\mathfrak{p}$ by $F_s(h_\mathfrak{p}) = g(s, u(h_\mathfrak{p}))$. Then F_s is a C^∞, W-invariant function on $A_\mathfrak{p}$, and outside a compact set, we have $F_s(h_\mathfrak{p}) = \exp(\rho_0 - s)|\beta(\log h_\mathfrak{p})|$. It follows from the remarks in Section 1, (cf. [24]) that F_s is the Abel transform of a function f_s in $\mathscr{C}_1(K\backslash G/K)$, provided that $\operatorname{Re} s > 2\rho_0$. Thus

$$F_s(h) = F_{f_s}(h) = g(s, u(h)) \quad \text{with } f_s \in \mathscr{C}_1(K\backslash G/K) \tag{2.2}$$

Since f_s is admissible, we have the trace formula

$$\begin{aligned} &\sum_{j \geq 0} n_j \hat{f}_s(v_j) \\ &= \chi(1) f_s(1) \operatorname{vol}(\Gamma\backslash G) + \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) F_{f_s}(h_\mathfrak{p}(\gamma)) \\ &= \chi(1) f_s(1) \operatorname{vol}(\Gamma\backslash G) + \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) g(s, u_\gamma), \end{aligned} \tag{2.3}$$

where both sides converge absolutely for $\operatorname{Re} s > 2\rho_0$.

The numbers $\{|u_\gamma|; \gamma \in C_\Gamma - \{1\}\}$ are bounded away from zero; cf. Section 1. If we choose and fix ε_0 so small that it is smaller than the smallest of these numbers, we have $g(s, u_\gamma) = c \exp(\rho_0 - s)|u_\gamma|$. Hence we get the following proposition.

PROPOSITION 2.1. *Let T be a finite-dimensional unitary representation of Γ, with character χ. Then the series*

$$\Psi_\Gamma(s, \chi, g) = g(\varepsilon_0) \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(\rho_0 - s)|u_\gamma| \tag{2.4}$$

converges absolutely, uniformly with respect to χ for each s in the half-plane $\operatorname{Re} s > 2\rho_0$. *The convergence is uniform with respect to s in each half-plane* $\operatorname{Re} s \geq 2\rho_0 + \varepsilon$, *where $\varepsilon > 0$.*

The uniformity statement with respect to χ comes from observing that $|\chi(\gamma)| \leq \chi(1) = \text{degree}(T)$, and that $C(h(\gamma)) > 0$ for every γ. Thus the series (2.4) is dominated by a multiple of $\Psi_\Gamma(\operatorname{Re} s, \mathbb{1}, g)$ where $\mathbb{1}$ is the trivial character of Γ.

Observe that, because of (2.3) we have

$$\Psi_\Gamma(s, \chi, g) = \sum_{j \geq 0} n_j \hat{f}_s(v_j) - \chi(1) f_s(1) \operatorname{vol}(\Gamma\backslash G);$$

We will show next that each term on the right has a meromorphic continuation to the whole plane. This gives us a meromorphic continuation of $\Psi_\Gamma(s, \chi, g)$.

For any complex number r, let $H(r) = \int_0^\infty g'(u) \exp iru\, du$. Because of the properties of g, g' is in $C_c^\infty(\mathbf{R})$, and $g'(u) = 0$ if $|u| \geq \varepsilon_0$. Hence an application of the Paley-Wiener theorem gives us the following lemma.

LEMMA 2.3. *H is an entire function of r. Moreover, for each integer $n \geq 1$, we can find $C_n > 0$ such that we have the estimates*

$$(2.5)\qquad \begin{aligned} |H(r)| &\leq C_n|r|^{-n} && \text{if } \operatorname{Im} r \geq 0 \\ &\leq C_n|r|^{-n} \exp(\varepsilon_0 |\operatorname{Im} r|) && \text{if } \operatorname{Im} r < 0. \end{aligned}$$

Let us also observe:

LEMMA 2.4. *Let f_s be as above in* (2.2), *with* $\operatorname{Re} s > 2\rho_0$ *and let $v \in \Lambda$. Then wherever $\hat{f}_s(v)$ exists, we have*

$$(2.6)\qquad \hat{f}_s(v) = \frac{H(is - i\rho_0 + v(H_0))}{s - \rho_0 - iv(H_0)} + \frac{H(is - i\rho_0 - v(H_0))}{s - \rho_0 + iv(H_0)}.$$

Proof.

$$\begin{aligned} \hat{f}_s(v) &= \int_{A_\mathfrak{p}} F_{f_s}(h) \exp iv\,(\log h)\, dh \\ &= \int_{A_\mathfrak{p}} g(s, u(h)) \exp iv(H_0)u(h)\, dh \\ &= \int_{-\infty}^{\infty} g(s, u) \exp iv(H_0)u\, du \\ &= \int_{-\infty}^{\infty} g(|u|) \exp((\rho_0 - s)|u| + iv(H_0)u)\, du \\ &= \int_{-\infty}^{0} g(-u) \exp(s - \rho_0 + iv(H_0))u\, du \\ &\quad + \int_0^\infty g(u) \exp(\rho_0 - s + iv(H_0))u\, du \\ &= \int_0^\infty g(u) \exp(-s + \rho_0 - iv(H_0))u\, du \\ &\quad + \int_0^\infty g(u) \exp(-s + \rho_0 + iv(H_0))u\, du. \end{aligned}$$

Integrate by parts and remember that $g(0) = 0$, and that $\operatorname{Re} s > 2\rho_0$. The lemma follows. ∎

We now define $A(s) = \sum_{j \geq 0} n_j \hat{f}_s(v_j)$ for $\operatorname{Re} s > 2\rho_0$.

PROPOSITION 2.5. *The function $A(s)$ has a meromorphic continuation to the whole complex plane. The poles of A occur at the points s_j^+ and s_j^-, $j \geq 0$, where*

$s_j^+ = \rho_0 + i\nu_j(H_0)$ and $s_j^- = \rho_0 - i\nu_j(H_0)$. These poles are all simple; the residues at s_j^+ and s_j^- both equal $n_j H(0)$, $j = 0, 1, 2, \ldots$, if $s_j^+ \neq s_j^-$. (It is understood that the poles at s_0^+ and s_0^- are present only if $n_0 > 0$.) Finally, if $s_j^+ = s_j^-$ for some j, the residue of $A(s)$ at s_j^+ is $2n_j H(0)$.

Proof. We have

$$A(s) = \sum_{j \geq 0} n_j \left\{ \frac{H(is - i\rho_0 + \nu_j(H_0))}{s - \rho_0 - i\nu_j(H_0)} + \frac{H(is - i\rho_0 - \nu_j(H_0))}{s - \rho_0 + i\nu_j(H_0)} \right\}$$
$$= \sum_{j \geq 0} n_j \left\{ \frac{H(i(s - s_j^+))}{s - s_j^+} + \frac{H(i(s - s_j^-))}{s - s_j^-} \right\}$$

in the half plane $\operatorname{Re} s > 2\rho_0$. Each term on the right is meromorphic in s, and thanks to the estimate of Lemma 2.3 and Proposition 1.2, the series converges absolutely, uniformly for s running over a compact set disjoint from $\{s_j^{\pm}\}_{j \geq 0}$. ∎

We will now consider the term $\chi(1) f_s(1) \operatorname{vol}(\Gamma \backslash G)$, and show that it is meromorphic in s.

By the Plancherel formula for G/K, we have

$$f_s(1) = [W]^{-1} \int_{\Lambda} \hat{f}_s(\nu) c(\nu)^{-1} c(-\nu)^{-1} \, d\nu. \tag{2.7}$$

Now $\hat{f}_s(\nu)$ is given by (2.6), and $c(\nu)$ is as in (1.5). If we recall the normalizations of the dh, $d\nu$ in Section 1, in terms of the parameters $u = u(h)$, $r = r(\nu)$ introduced there, we can write this as

$$f_s(1) = \frac{[W]^{-1}}{2\pi} \int_{-\infty}^{\infty} h(r, s) c(r)^{-1} c(-r)^{-1} \, dr \tag{2.8}$$

where

$$h(r, s) = \frac{H(i(s - \rho_0 - ir))}{s - \rho_0 - ir} + \frac{H(i(s - \rho_0 + ir))}{s - \rho_0 + ir} \tag{2.9}$$

and

$$c(r)^{-1} = \frac{\Gamma((p + q)/2)\Gamma(ir + p/2)\Gamma(ir/2 + p/4 + q/2)}{\Gamma(p + q)\Gamma(ir)\Gamma(ir/2 + p/4)} \tag{2.10}$$

and Γ is the classical Γ-function.

Observe that the substitution $r \to -r$ interchanges the two terms on the right side of (2.9). Moreover $[W] = 2$ in our case. It follows that

$$f_s(1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H(i(s - \rho_0 - ir))}{s - \rho_0 - ir} c(r)^{-1} c(-r)^{-1} \, dr. \tag{2.11}$$

We now shift the integration into the complex plane by using a rectangular contour with vertices at $-R$, $+R$, $R + iR$, $-R + iR$; The function $r \to$

$c(r)^{-1}c(-r)^{-1}$ is meromorphic in the upper half plane, and can only have simple poles. Let r_k, $k \geq 0 \ldots$ be the poles, if any, and let d_k be the residue of $c(r)^{-1}c(-r)^{-1}$ at the pole r_k. Using the residue theorem one finds

$$f_s(1) = i \sum_{\{k;\, \operatorname{Im} r_k \leq R\}} \frac{H(is - i\rho_0 + r_k)}{s - \rho_0 - ir_k} \cdot d_k + I_+ + I_- + J$$

where $I_+ I_-$ are the contributions from the vertical sides of our contour and J is the integral coming from the top side. Now, because $\operatorname{Im} r \geq 0$, and $\operatorname{Re} s > 2\rho_0$, we have $\operatorname{Im}(is - i\rho_0 + r) \geq 0$, so that the estimate

$$|H(is - i\rho_0 + r)| \leq C_n |s - \rho_0 + r|^{-n}$$

of Lemma 2.3 is available.

On the other hand, $|c(r)^{-1}c(-r)^{-1}| \leq C_1 |r|^d$ as remarked in Section 1. Since the integer n is at our disposal, we see easily from these estimates that I_+, I_-, J all tend to zero as $R \to \infty$. It follows that

$$(2.12) \qquad f_s(1) = i \sum_{k \geq 0} \frac{H(is - i\rho_0 + r_k)}{s - \rho_0 - ir_k} \cdot d_k, \quad \operatorname{Re} s > 2\rho_0.$$

Of course, if $c(r)^{-1}c(-r)^{-1}$ has no poles in the upper half plane then this sum is to be interpreted as zero.

The poles r_k of $c(r)^{-1}c(-r)^{-1}$ and the residues d_k at these poles can be calculated for all the groups G of split rank one. We omit the tedious calculation, and summarize the results in Table I at the end of this paper.

Note that when $G = SO_0(n, 1)$ with n odd, the function $c(r)^{-1}c(-r)^{-1}$ is a polynomial in r, and has no poles. Thus in that case, $f_s(1) = 0$. (This accords with the known facts. Indeed, in this case, the inverse of the Abel transform $f \to F_f$ is given by a differential operator, and since F_{f_s} vanishes near the identity, due to the properties of g, one can deduce that $f_s(1) = 0$.)

In all the other cases $c(r)^{-1}c(-r)^{-1}$ has simple poles, and those in the upper half plane are tabulated in Table I. *One notes that* r_k *is purely imaginary,* $|r_k| = O(k)$, *and* $|d_k| = O(k^a)$ *where a is a positive integer, depending only on G, and that* $\rho_0 + ir_k$ *equals either* $-k$ *or* $-2k$, $k \geq 0$.

We now claim that the series on the right side of (2.12) converges absolutely, uniformly with respect to s varying over a compact subset U of the complex plane, provided that U is disjoint form the points $\{\rho_0 + ir_k;\, k \geq 0\}$. Indeed, for s in U, we see that $\operatorname{Im}(is - i\rho_0 + r_k) > 0$ for large enough k. For such k, the estimate

$$|H(is - i\rho_0 + r_k)| \leq C_n |is - i\rho_0 + r_k|^{-n}$$

of Lemma 2.3 is available; Since s is confined to U which misses $\rho_0 + ir_k$, we have

$$|H(is - i\rho_0 + r_k)| \leq C_1(n) |r_k|^{-n}$$

for large k, with C_1 independent of k. Using the estimates on r_k, d_k given by Table I, we conclude by choosing n large that the series on the right side does indeed converge as claimed.

It follows that the series defines a meromorphic function of s with simple poles at the points $\rho_0 + ir_k$, $k \geq 0$, and the residue of this function at the pole $\rho_0 + ir_k$ is equal to $iH(0)\, d_k$. We summarize these observations.

PROPOSITION 2.6. *For* Re $s > 2\rho_0$, *we have*

$$\chi(1) f_s(1) \operatorname{vol}(\Gamma\backslash G) = i\chi(1) \operatorname{vol}(\Gamma\backslash G) \sum_{k \geq 0} \frac{H(is - i\rho_0 + r_k)}{s - \rho_0 - ir_k}\, d_k \tag{2.13}$$

where $\{r_k;\, k \geq 0\}$ *are the poles of the function* $(c(r)c(-r))^{-1}$ *in the upper half plane and* d_k *is the residue of that function at* r_k. (*The series is to be interpreted at zero when the set* $\{r_k\}$ *is empty.*) *The series converges absolutely, uniformly for s in any compact set disjoint from the numbers* $\{\rho_0 + ir_k\}$, *and defines a meromorphic function of s in the whole complex plane, thus giving us a meromorphic continuation of the left side of* (2.13). *This function has simple poles at the points* $\rho_0 + ir_k$, $k \geq 0$, *and has the residue* $i\chi(1) \operatorname{vol}(\Gamma\backslash G) H(0)\, d_k$ *at the pole* $\rho_0 + ir_k$.

Note that $\rho_0 + ir_k$ is a nonpositive integer in all cases, either equal to $-k$ or $-2k$. Also, since d_k is purely imaginary, the residue $i\chi(1) \operatorname{vol}(\Gamma\backslash G) H(0)\, d_k$ is real.

By the Gauss-Bonnet theorem applied to $\Gamma\backslash G/K$, we can relate $\operatorname{vol}(\Gamma\backslash G) = \operatorname{vol}(\Gamma\backslash G/K)$ to the Euler-Poincaré characteristic E of the manifold $\Gamma\backslash G/K$. As is seen in Section 3, for our normalization of Haar measure, it turns out that $\operatorname{vol}(\Gamma\backslash G)$ is a *rational* multiple of E. Also, Table I shows that $i\, d_k$ is a rational number, whose denominator depends *only* on (G, K), and not on k. It follows that $i \operatorname{vol}(\Gamma\backslash G)\, d_k = e_k E/\kappa$, where κ is a positive integer depending on the pair (G, K) alone, and e_k is an integer. Note that $e_k E$ and $i\, d_k$ have the same sign.

Recall that, in defining $\Psi_\Gamma(s, \chi, g)$ we had used a constant c, with $g(x) = c$ when $x \geq \varepsilon_0$. We now choose c to be equal to the integer κ. The corresponding $\Psi_\Gamma(s, \chi, g)$ will be denoted simply by $\Psi_\Gamma(s, \chi)$. Note that then, $H(0) = \int_0^\infty g'(u)\, du = g(\varepsilon_0) - g(0) = \kappa$. Thus, taking into account Propositions 2.5 and 2.6, and the definition of $\Psi_\Gamma(s, \chi)$, we get the following proposition. Bear in mind that the sets $\{s_j^+, j \geq 0\}$ and $\{\rho_0 + ir_k, k \geq 0\}$ have just the point 0 in common, because $0 = s_0^+ = \rho_0 + ir_0$. Also recall that a_0 is the multiplicity of the trivial representation of Γ in the representation T.

PROPOSITION 2.7. *For* Re $s > 2\rho_0$, *define*

$$\Psi_\Gamma(s, \chi) = \kappa \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma) |u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(\rho_0 - s)|u_\gamma|. \tag{2.14}$$

This series converges absolutely, uniformly in any half plane Re $s \geq 2\rho_0 + \delta$, *so that* Ψ_Γ *is holomorphic in* Re $s > 2\rho_0$. $\Psi_\Gamma(s, \chi)$ *has meromorphic continuation* *to the whole complex plane, via the relation* $\Psi_\Gamma(s, \chi) = A(s) - \chi(1)f_s(1)\,\mathrm{vol}\,(\Gamma\backslash G)$. *The poles of* Ψ_Γ *are all simple, and are as follows:*

$$\begin{array}{llll} & \textit{Pole} & \textit{Residue} & \\ & s_j^+ = \rho_0 + iv_j(H_0) & \kappa n_j & j \geq 1 \\ (2.15) & s_j^- = \rho_0 - iv_j(H_0) & \kappa n_j & j \geq 1 \\ & \rho_0 + ir_k & -\chi(1)e_k E & k \geq 1. \\ & 0 & \kappa a_0 - \chi(1)e_0 E & \end{array}$$

If $0 \in \Lambda$ *occurs in the spectrum, i.e., if for some* j, *we have* $v_j = 0$, *then for that* j, $s_j^+ = s_j^- = \rho_0$, *and the residue at this pole is* $2\kappa n_j$.

Because the function $\Psi_\Gamma(s, \chi)$ has only *simple* poles with *integer* residues, we can find a *meromorphic* function $Z_\Gamma(s, \chi)$ such that

$$\frac{d}{ds}\log Z_\Gamma(s, \chi) = \Psi_\Gamma(s, \chi).$$

Z_Γ will be defined up to a multiplicative constant, which we will now fix. As we have seen, if for some j, $v_j = 0$, then Z_Γ will have a zero at ρ_0 of order $2\kappa n_j$. We will denote this even integer $2\kappa n_j$ by m_0. Of course, $m_0 = 0$ if all the v_j are nonzero, i.e., if $0 \in \Lambda$ is not in the spectrum. We now normalize Z_Γ by requiring that $(s - \rho_0)^{-m_0}Z_\Gamma(s, \chi) \to 1$ as $s \to \rho_0$. This determines Z_Γ completely. We shall call this *the* zeta function attached to the data (G, K, Γ, χ). It is obvious that the points s_j^+, s_j^-, with $j \geq 1$ are zeroes of Z_Γ of order κn_j respectively, and the points $\rho_0 + ir_k$, $k \geq 1$ are either zeroes or poles of Z_Γ according to whether $-\chi(1)e_k E$ is positive or negative, the order of the zero or pole being $|\chi(1)e_k E|$. Since $\chi(1) > 0$, and $e_k E$ and $i\,d_k$ have the same sign, the sign of $-\chi(1)e_k E$ can be read off from Table I at the end of this paper. That table shows that the sign of $i\,d_k$ is independent of k, and depends on (G, K) alone. When the sign is positive, $\rho_0 + ir_k$, $k \geq 1$, are *all* poles of Z_Γ. They are *all* zeroes of Z_Γ in the opposite case. By Table I, poles occur precisely when $\dim\,(G/K) \equiv 0 \bmod 4$.

The point $s = 0$, which is common to the sets $\{s_j^+, j \geq 0\}$ and $\{\rho_0 + ir_k, k \geq 0\}$ is somewhat special as is made clear in the above proposition. It will be a zero of Z_Γ if $\kappa a_0 - \chi(1)e_0 E$ is positive, and a pole if $\kappa a_0 - \chi(1)e_0 E$ is negative. In each case the order of the zero or pole will be $|\kappa a_0 - \chi(1)e_0 E|$.

At this point, we have proved that Z_Γ has properties (1), (2), (4), (5), and (5 bis) described in Section 0. It is also clear that we have proved the first statement of (6).

Our next task is to show that Z_Γ enjoys the functional equation claimed in (3) of Section 0. For brevity, we write $\Phi(t) = \kappa\,\mathrm{vol}\,(\Gamma\backslash G)\chi(1)c(it)^{-1}c(-it)^{-1}$. Then Φ is meromorphic and its poles are simple.

We shall first show that the logarithmic derivative Ψ_Γ of Z_Γ has a functional equation.

PROPOSITION 2.8. *We have*

$$\Psi_\Gamma(s, \chi) + \Psi_\Gamma(2\rho_0 - s, \chi) + \Phi(s - \rho_0) \equiv 0, \quad s \in \mathbf{C}. \tag{2.16}$$

Proof. The three terms are all meromorphic with simple poles. The poles of Ψ_Γ are at s_j^+, s_j^- and at $\rho_0 + ir_k$. Since $s_j^- = 2\rho_0 - s_j^+$, it follows that $\Psi_\Gamma(s, \chi) + \Psi_\Gamma(2\rho_0 - s, \chi)$ has only simple poles at

$$\{\rho_0 + ir_k, \rho_0 - ir_k, k \geq 0\},$$

with residues $-\chi(1)e_k E$, $\chi(1)e_k E$ respectively. On the other hand the poles of $\Phi(s - \rho_0)$ are at $s = \rho_0 + ir_k$ and $s_0 = \rho_0 - ir_k$, and the residues are $\chi(1)e_k E$ and $-\chi(1)e_k E$ respectively. It follows that $\Psi_\Gamma(s, \chi) + \Psi(2\rho_0 - s, \chi) + \Phi(s - \rho_0)$ is an entire function. Call this function $Q(s)$. We will show by applying the trace formula that $Q(s) \equiv 0$.

Let $r = i(\rho_0 - s)$ so that $s = \rho_0 + ir$. It will be convenient to use the variable r instead of s. The above functions Ψ, Q can be regarded as functions of r. We denote them by ψ, q when so regarded. Thus $\psi_\Gamma(r, \chi) = \Psi(\rho_0 + ir, \chi)$, $q(r) = Q(\rho_0 + ir)$. Also during the rest of this proof we shall write $\psi(r)$ instead of $\psi_\Gamma(r, \chi)$.

Let r_j^+, r_j^- be defined by $s_j^+ = \rho_0 + ir_j^+$, and $s_j^- = \rho_0 + ir_j^-$. Then $r_j^+ = v_j(H_0)$ and $r_j^- = -v_j(H_0)$. We shall show that

$$\psi(r) + \psi(-r) + \Phi(ir) \equiv 0, \quad r \in \mathbf{C}.$$

Note that $\Phi(ir) = \kappa\chi(1) \operatorname{vol}(\Gamma\backslash G)c(r)^{-1}c(-r)^{-1}$.

Fix $\varepsilon > 0$, and let F^* be a function on $\mathbf{C}$ which satisfies: (i) $F^*(z) = F^*(-z)$, (ii) F^* is holomorphic in $\{z; |\operatorname{Im} z| \leq \rho_0 + \varepsilon\}$, and (iii) F^* is rapidly decreasing in $\operatorname{Re} z$ on the boundaries of this strip. Then, as we saw in Section 1, we can find $f \in \mathscr{C}_1(K\backslash G/K)$ such that $\hat{f}(v) = F^*(r)$, $r = r(v)$. By (1.9), (1.10) we have

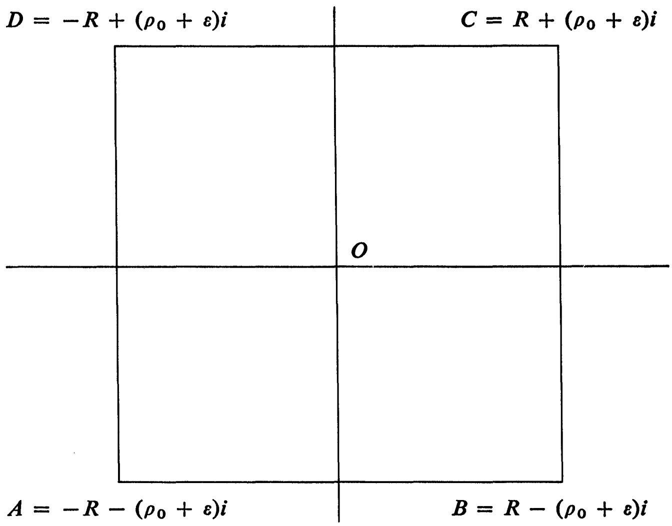
$$\begin{aligned} F_f(h) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(r) \exp(-iru)\, dr = F(u), \quad u = u(h), \\ \hat{f}(v) &= \int_{-\infty}^{\infty} F(u) \exp iru\, dr = F^*(r), \qquad r = r(v). \end{aligned} \tag{2.17}$$

Note that $\hat{f}(v_j) = F^*(v_j(H_0)) = F^*(r_j^+)$. Using all this in the trace formula (2.3), we get

$$\begin{aligned} \sum_{j \geq 0} n_j F^*(r_j^+) &= \chi(1) \operatorname{vol}(\Gamma\backslash G) f(1) \\ &\quad + \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \cdot F(u_\gamma) \\ &= \chi(1) \operatorname{vol}(\Gamma\backslash G) \frac{1}{4\pi}\int_{-\infty}^{\infty} F^*(r)c(r)^{-1}c(-r)^{-1}\, dr \\ &\quad + \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) F(u_\gamma). \end{aligned} \tag{2.18}$$

where we used the Plancherel theorem for G/K to express $f(1)$ as an integral.

Now let Ω_R be the rectangular contour in the complex r-plane with vertices at $A = -R - (\rho_0 + \varepsilon)i$, $B = R - (\rho_0 + \varepsilon)i$, $C = R + (\rho_0 + \varepsilon)i$ and $D = -R + (\rho_0 + \varepsilon)i$, as in the figure below.



$F^*(r)$ is holomorphic inside Ω_R. The poles of $\psi(r)$, and the residues at these poles are described in Proposition 2.7. By Cauchy's Residue Theorem,

$$\tag{2.19} \begin{aligned} &\int_{\Omega_R} F^*(r)\psi(r)\,dr \\ &\quad = 2\pi i\left\{ i\chi(1)e_0EF^*(i\rho_0) + \sum_{\{j;\,|r_j|\leq R\}} (F^*(r_j^+) + F^*(r_j^-))(-in_j\kappa)\right\} \end{aligned}$$

But $r_j^+ = -r_j^-$ and $F^*(r) = F^*(-r)$. So

$$\tag{2.20} \lim_{R\to\infty}\int_{\Omega_R} F^*(r)\psi(r)\,dr = -2\pi\left\{\chi(1)e_0EF^*(i\rho_0) - \sum_{j\geq 0} 2\kappa n_jF^*(r_j^+)\right\}.$$

Now,

$$\tag{2.21} \begin{aligned} &\int_{\Omega_R} F^*(r)\psi(r)\,dr \\ &\quad = \int_A^B F^*(r)\psi(r)\,dr + \int_C^D F^*(r)\psi(r)\,dr + I_+ + I_- \end{aligned}$$

where I_+, I_- are the integrals along the vertical sides of Ω_R. Now,

$$\int_C^D F^*(r)\psi(r)\,dr = -\int_A^B F^*(-r)\psi(-r)\,dr,$$

and since $F^*(r) = F^*(-r)$, we get

$$(2.22)\quad \int_{\Omega_R} F^*(r)\psi(r)\,dr = \int_A^B F^*(r)(\psi(r) - \psi(-r))\,dr + I_+ + I_-.$$

Since $q(r) = \psi(r) + \psi(-r) + \Phi(ir)$, we get

$$(2.23)\quad \begin{aligned} \int_{\Omega_R} F^*(r)\psi(r)\,dr &= 2\int_A^B F^*(r)\psi(r)\,dr + \int_A^B F^*(r)\Phi(ir)\,dr \\ &\quad - \int_A^B F^*(r)q(r)\,dr + I_+ + I_-. \end{aligned}$$

Now let $R \to \infty$. Using the properties of F^*, ψ, one sees that $I_+, I_- \to 0$ as $R \to \infty$, and

$$(2.24)\quad \begin{aligned} &\lim_{R\to\infty} \int_{\Omega_R} F^*(r)\psi(r)\,dr \\ &= \int_{-\infty}^{\infty} F^*(x - i(\rho_0 + \varepsilon))\{2\psi(x - i(\rho_0 + \varepsilon)) \\ &\qquad + \Phi(i(x - (\rho_0 + \varepsilon)i)) - q(x - (\rho_0 + \varepsilon)i)\}\,dx \end{aligned}$$

Now $\Psi_\Gamma(s)$ has the representation (2.14) when $\operatorname{Re} s > 2\rho_0$. So,

$$(2.25)\quad \begin{aligned} &\psi(x - i(\rho_0 + \varepsilon)) \\ &\qquad = \kappa \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp -i(x - (\rho_0 + \varepsilon)i)|u_\gamma|. \end{aligned}$$

Since F^* is holomorphic in the strip $\{x + iy;\ -\rho_0 - \varepsilon \le y \le 0\}$, we can shift integration to get

$$(2.26)\quad \begin{aligned} &2\int_{-\infty}^{\infty} F^*(x - i(\rho_0 + \varepsilon))\psi(x - i(\rho_0 + \varepsilon))\,dx \\ &\qquad = 2\kappa \sum_\gamma \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \int_{-\infty}^{\infty} (\exp - ix|u_\gamma|)F^*(x)\,dx \\ &\qquad = 2\kappa \sum_\gamma \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \int_{-\infty}^{\infty} (\exp - ixu_\gamma)F^*(x)\,dx \\ &\qquad = 4\pi\kappa \sum_\gamma \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) F(u_\gamma) \end{aligned}$$

where we used $F^*(x) = F^*(-x)$ for the last equality but one, and (2.17) for the last step.

Next, consider $\int_{-\infty}^{\infty} F^*(x - i(\rho_0 + \varepsilon))\Phi(i(x - i(\rho_0 + \varepsilon)))\,dx$. Here $F^*(z)$ is holomorphic in $\{z = x + iy,\ -\rho_0 - \varepsilon \le y \le 0\}$ while $\Phi(iz)$ is meromorphic there with possibly a pole at $z = -\rho_0$, with residue $i\chi(1)e_0E$. Using the Residue theorem, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} F^*(x - i(\rho_0 + \varepsilon))\Phi(i(x - (i\rho_0 + \varepsilon)i))\,dx \\
&\qquad = \int_{-\infty}^{\infty} F^*(x)\Phi(ix)\,dx + 2\pi i(i\chi(1)e_0E)F^*(i\rho_0) \\
&\qquad = \kappa \operatorname{vol}(\Gamma\backslash G)\chi(1)\int_{-\infty}^{\infty} F^*(x)c(x)^{-1}c(-x)^{-1}\,dx \\
&\qquad\quad - 2\pi\chi(1)e_0EF^*(i\rho_0) \\
&\qquad = 4\pi\kappa \operatorname{vol}(\Gamma\backslash G)\chi(1)f(1) - 2\pi\chi(1)e_0EF^*(i\rho_0).
\end{aligned} \tag{2.27}$$

Finally, since F^* and q are holomorphic in the strip

$$\{x + iy;\ -\rho_0 - \varepsilon \leq y \leq 0\},$$

we have

$$\int_{-\infty}^{\infty} F^*(x - i(\rho_0 + \varepsilon))q(x - i(\rho_0 + \varepsilon))\,dx = \int_{-\infty}^{\infty} F^*(x)q(x)\,dx. \tag{2.28}$$

Now (2.20) together with (2.24), (2.26), (2.27), (2.28) leads to

$$\begin{aligned}
&4\pi\kappa \sum_{j\geq 0} n_jF^*(r_j^+) - 2\pi\chi(1)e_0EF^*(i\rho_0) \\
&\qquad = 4\pi\kappa \sum_{\gamma\in C_\Gamma-\{1\}} \chi(\gamma)|u_\gamma|j(\gamma)^{-1}C(h(\gamma))\cdot F(u_\gamma) \\
&\qquad\quad + 4\pi\kappa \operatorname{vol}(\Gamma\backslash G)\chi(1)f(1) - 2\pi\chi(1)e_0EF^*(i\rho_0) \\
&\qquad\quad - \int_{-\infty}^{\infty} F^*(x)q(x)\,dx.
\end{aligned} \tag{2.29}$$

Recalling (2.18), we have from this

$$\int_{-\infty}^{\infty} F^*(x)q(x)\,dx = 0. \tag{2.30}$$

Now F^* can be varied over a large class of functions; For example any function of the form $P(x)\exp(-\alpha x^2)$ where $\alpha > 0$ and $P(x)$ is an even polynomial in x will fulfill the assumptions on F^*. Since $q(x)$ is an even function of x, one deduces from (2.30) that $q(x) = 0$, $x \in \mathbf{R}$. But q is entire, hence $q \equiv 0$. It follows that $Q(s) \equiv 0$ and Proposition 2.8 is proved. ■

We are now in a position to get the functional equation for Z_Γ.

THEOREM 2.9. *We have*

$$Z_\Gamma(2\rho_0 - s, \chi) = Z_\Gamma(s, \chi)\exp\int_0^{s-\rho_0} \Phi(t)\,dt \tag{2.31}$$

where $\Phi(t) = \kappa \operatorname{vol}(\Gamma\backslash G)\chi(1)c(it)^{-1}c(-it)^{-1}$.

Proof. Let us note first of all that the expression $\exp \int_0^{s-\rho_0} \Phi(t)\,dt$ is well defined, when $\int_0^{s-\rho_0} \Phi(t)\,dt$ is interpreted as a contour integral. Indeed, $\Phi(t)$ is meromorphic and its residues at the poles are always integral. It follows that two different contours from 0 to $s - \rho_0$ will lead to values for $\int_0^{s-\rho_0} \Phi(t)\,dt$ that differ by an integral multiple of $2\pi i$. Hence $\exp \int_0^{s-\rho_0} \Phi(t)\,dt$ is well defined.

Since

$$\Psi_\Gamma(s, \chi) = \frac{d}{ds}(\log Z_\Gamma(s, \chi)),$$

it is evident that Proposition 2.8 leads by integration to

$$Z_\Gamma(2\rho_0 - s, \chi) = \alpha Z_\Gamma(s, \chi) \exp \int_0^{s-\rho_0} \Phi(t)\,dt. \tag{2.32}$$

where α is a nonzero constant. We claim that $\alpha = 1$.

Indeed, let m_0 be the multiplicity of the zero of Z_Γ at ρ_0 ($m_0 = 0$ if ρ_0 is not a zero), and recall that m_0 is even (cf. Proposition 2.5). Hence $(s - \rho_0)^{m_0} = (\rho_0 - s)^{m_0}$. Thus from (2.32) we get

$$(\rho_0 - s)^{-m_0} Z_\Gamma(2\rho_0 - s, \chi) = \alpha (s - \rho_0)^{-m_0} Z_\Gamma(s, \chi) \exp \int_0^{s-\rho_0} \Phi(t)\,dt. \tag{2.33}$$

Now suppose $Z_\Gamma(s, \chi) = (s - \rho_0)^{m_0} F(s)$ in a neighborhood of ρ_0; then $F(\rho_0) = 1$, so $(s - \rho_0)^{-m_0} Z_\Gamma(s, \chi) \to 1$ as $s \to \rho_0$. On the other hand

$$Z_\Gamma(2\rho_0 - s, \chi) = (\rho_0 - s)^{m_0} F(2\rho_0 - s)$$

in a neighborhood of ρ_0, so $(\rho_0 - s)^{-m_0} Z_\Gamma(2\rho_0 - s, \chi) \to 1$ as $s \to \rho_0$. Thus letting $s \to \rho_0$ in (2.33), we see that $\alpha = 1$. Theorem 2.9 is completely proved. ∎

For $G = SL(2, \mathbf{R})$, this functional equation was observed by Selberg [21, p. 75]. In that case, $c(r)^{-1} c(-r)^{-1} = r \tanh \pi r$ and $\Phi(t) = -\chi(1) \operatorname{vol}(\Gamma \backslash G) t \tan \pi t$. Thus the above equation reduces to the one given by Selberg, if we remember that for $G = SL(2, \mathbf{R})$ we have $\kappa = 1$ (cf. Section 3).

When $G = SO_0(2n + 1, 1)$, Φ is a polynomial, and so is $\int_0^{s-\rho_0} \Phi(t)\,dt$. Thus the functional equation is simpler in that case. In all other cases, $\Phi(t)$ will equal a polynomial times $t \tan \pi t$, so that $\int_0^{s-\rho_0} \Phi(t)\,dt$ is not an elementary function. The simplicity of the formulas in the case $SO_0(2n + 1, 1)$ is due to the structural fact that there is then just one conjugacy class of Cartan subgroups in G.

Since $s_j^+ = \rho_0 + i\nu_j(H_0)$, it is clear that s_j^+ determines $\nu_j(H_0)$; Moreover, since rank $(G/K) = 1$, this in turn determines the linear form ν_j. Thus a knowledge of the zeroes s_j^+ and their multiplicities κn_j is equivalent to the knowledge of the parameters ν_j, and the multiplicities n_j, since κ depends on (G, K) alone. As we have remarked in Section 1, the parameters v_j correspond to spherical representations U_j occurring in U, and the integers n_j to their multiplicities. Now if ϕ_{ν_j} is the elementary spherical function of the representation U_j, then

ϕ_{ν_j} is positive definite and the eigenvalue of the Laplacian operating on ϕ_{ν_j} is real and nonpositive. In our parametrization, this means that

$$-\langle \nu_j, \nu_j \rangle - \langle \rho, \rho \rangle \leq 0.$$

Now $\langle \nu_j, \nu_j \rangle = \nu_j(H_0)^2 \langle \beta, \beta \rangle$ and $\langle \rho, \rho \rangle = \rho(H_0)^2 \langle \beta, \beta \rangle = \rho_0^2 \langle \beta, \beta \rangle$. So we obtain $\nu_j(H_0)^2 + \rho_0^2 \geq 0$. It follows that $\nu_j(H_0)$ is either real or purely imaginary. In the latter case we must have $|\nu_j(H_0)| \leq \rho_0$. Because of Proposition 1.1, such ν_j can only be finite in number. Now it is well known, cf. [25], that the ν_j which are real correspond to representations U_j of the spherical principal series, and the purely imaginary ν_j correspond to representations in the spherical complementary series of G. Since $s_j^+ = \rho_0 + i\nu_j(H_0)$, we find that the zeroes s_j^+ lie on the line $\operatorname{Re} s = \rho_0$ except for the finitely many indices j for which $\nu_j(H_0)$ is purely imaginary and $\neq 0$. For these j, s_j^+ is real, and lies in $[0, \rho_0)$. s_j^- lies in $(\rho_0, 2\rho_0]$, and is symmetrically opposed to s_j^+ around ρ_0. Thus except for these zeroes, finite in number, the spectral zeroes of Z_Γ satisfy the modified Riemann hypothesis $\operatorname{Re} s_j^\pm = \rho_0$ (cf. [21]).

If T, T' are finite dimensional unitary representations of Γ, with characters χ, χ', their direct sum $T \oplus T'$ will have character $\chi + \chi'$. Now the induced representations U_χ, $U_{\chi'}$ and $U_{\chi+\chi'}$ will satisfy $U_{\chi+\chi'} \cong U_\chi \oplus U_{\chi'}$. It follows that $n_j(\chi + \chi') = n_j(\chi) + n_j(\chi')$. Thus if $m_0(\chi)$, $m_0(\chi')$ and $m_0(\chi + \chi')$ are the multiplicities with which the spherical representation corresponding to $0 \in \Lambda$ occurs in U_χ, $U_{\chi'} U_{\chi+\chi'}$ respectively, we see that $m_0(\chi + \chi') = m_0(\chi) + m_0(\chi')$. Now consider $\Psi_\Gamma(s, \chi + \chi')$. It is obviously linear in the variable χ. Hence

$$\Psi_\Gamma(s, \chi + \chi') = \Psi_\Gamma(s, \chi) + \Psi_\Gamma(s, \chi'),$$

which readily leads, via the above relation for m_0, to

$$Z_\Gamma(s, \chi + \chi') = Z_\Gamma(s, \chi) Z_\Gamma(s, \chi').$$

Similarly decomposing the tensor product $T \otimes T'$ into irreducibles T_i with characters χ_i, we get

$$\chi\chi' = \sum_i m_i \chi_i \quad \text{and} \quad Z_\Gamma(s, \chi\chi') = \prod_i Z_\Gamma(s, \chi_i)^{m_i}.$$

Finally, observe that the contragredient χ^* of χ obeys $\chi^*(\gamma) = \chi(\gamma^{-1})$. In the expression for $\Psi_\Gamma(s, \chi)$, the factors $|u_\gamma|$, $j(\gamma)$, $C(h(\gamma))$ are all invariant under $\gamma \to \gamma^{-1}$, as is the factor $\exp(s - \rho_0)|u_\gamma|$. Since Γ is torsion free, no nontrivial element of Γ can be conjugate in Γ to its own inverse. Thus $C_\Gamma - \{1\}$ can be chosen to be stable under $\gamma \to \gamma^{-1}$. We see from the above observations that $\Psi_\Gamma(s, \chi^*) = \Psi_\Gamma(s, \chi)$. On the other hand, it is obvious that $m_0(\chi^*) = m_0(\chi)$. It follows that $Z_\Gamma(s, \chi^*) = Z_\Gamma(s, \chi)$.

At this point, we have established all the statements made in Section 0, except (8), (9), and the latter halves of (6) and (7), concerning the occurrence of zeroes in $[0, 2\rho_0]$ and the order of Z_Γ when it is entire. We shall now proceed to these matters.

In order to prove (8), we consider, for $t > 0$, the function

$$v \to \exp(-(r(v)^2 + \rho_0^2)t),$$

where $r(v) = v(H_0)$, $v \in \Lambda$. It is seen that this function is the Fourier transform of a function in $\mathscr{C}_1(K\backslash G/K)$ which we will call h_t. h_t is essentially the fundamental solution g_t of the heat equation on G/K discussed in [6]. In fact, if ω is the Casimir operator of G, so that g_t is the fundamental solution of $\omega u = \partial u/\partial t$, then one checks easily that h_t is the spherical fundamental solution of $(1/c^2)\omega u = \partial u/\partial t$, where $c^2 = |H_0|^2 = 2p + 8q$. So we have $h_t = g_{tc^2}$. Using this h_t in the trace formula we have

$$\hat{h}_t(v) = \exp(-(r(v)^2 + \rho_0^2)t), \tag{2.34}$$

$$\begin{aligned} F_{h_t}(h) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-(r^2 + \rho_0^2)t) \exp iru\, dr, \\ &= (4\pi t)^{-1/2} \exp(-(\rho_0^2 t + u^2/4t)), \quad u = u(h). \end{aligned} \tag{2.35}$$

so that the trace formula is

$$\begin{aligned} &\sum_{j\geq 0} n_j \exp(-(\rho_0^2 + r(v_j)^2)t) \\ &\quad = \chi(1)\operatorname{vol}(\Gamma\backslash G)h_t(1) \\ &\qquad + (4\pi t)^{-1/2} \sum_{\gamma\in C_\Gamma-\{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(-(\rho_0^2 t + u_\gamma^2/4t)). \end{aligned} \tag{2.36}$$

Now define the theta function θ by

$$\begin{aligned} \theta(t) &= \sum_{j\geq 0} n_j \exp(-(\rho_0^2 + r(v_j)^2)t) - \chi(1)\operatorname{vol}(\Gamma\backslash G)h_t(1) \\ &= \sum_{\gamma\in C_\Gamma-\{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma))(4\pi t)^{-1/2} \exp(-(\rho_0^2 t + u_\gamma^2/4t)). \end{aligned} \tag{2.37}$$

Now multiply by $\exp(-s(s - 2\rho_0)t)$ and integrate term by term with respect to t between $[0, \infty)$. The procedure can be justified easily since $\operatorname{Re} s > 2\rho_0$, and we obtain for s real and $\operatorname{Re} s > 2\rho_0$,

$$\begin{aligned} &\int_0^\infty \exp(-s(s - 2\rho_0)t)\theta(t)\, dt \\ &\qquad = \sum_\gamma \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \\ &\qquad\quad \times \int_0^\infty (4\pi t)^{-1/2} \exp(-((s - \rho_0)^2 t) + u_\gamma^2/4t)\, dt \\ &\qquad = \sum_\gamma \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma))(2(s - \rho_0))^{-1} \exp(-(s - \rho_0)|u_\gamma|) \\ &\qquad = 2(s - \rho_0)^{-1}\kappa^{-1}\Psi_\Gamma(s, \chi) \end{aligned} \tag{2.38}$$

where we used the formula

$$\int_0^\infty (4\pi t)^{-1/2} \exp(-(x^2 t + y^2/4t)) = (2x)^{-1} \exp(-xy),$$

valid for $x > 0, y > 0$.

(2.38) holds for complex s such that $\operatorname{Re} s > 2\rho_0$ by analytic continuation, and we get

$$\frac{d}{ds}(\log Z_\Gamma(s, \chi)) = \kappa 2(s - \rho_0) \int_0^\infty \theta(t) \exp(-s(s - 2\rho_0)t)\, dt \tag{2.39}$$

which is the relation claimed in (8) of Section 0.

This relation is clearly analogous to the classical relation between the logarithmic derivative of the Riemann ζ-function and the Jacobi theta function via the Mellin transform. See [5, p. 67] for instance. In the classical case, this relation together with the functional properties of the theta function given by Poisson summation forms the basis of a proof of the functional equation for ζ. In the present case, (2.39) can be made the basis of a proof of the functional equation for Z_Γ. For $G = SL(2, \mathbf{R})$, a sketch of the proof is given in [16]. For other G the proof is much more cumbersome. It involves the explicit inversion of the Abel transform $f \to F_f$; We do not discuss it here.

The numbers $r(v_j) = v_j(H_0)$ are what we called r_j^+ above. As we have seen, a finite number of these are purely imaginary. Suppose that r_j^+ is purely imaginary and nonzero for $j = 0, 1, \ldots, N$. Then $s_j^+ = \rho_0 + ir_j^+$ lies in $[0, \rho_0)$. Let $d' = \sum_{j=0}^N n_j$. Then the number of exceptional zeroes of $Z_\Gamma(s, \chi)$ that lie in $[0, 2\rho_0]$ *off* the line $\operatorname{Re} s = \rho_0$ is $2d'$. We intend to get an estimate for this number. The result will be that for a given Γ, $2d' \leq C(\Gamma)\chi(1) \operatorname{vol}(\Gamma\backslash G)$ where $C(\Gamma)$ is a positive number depending only on Γ and not on χ.

This estimate is obtained by looking at the trace formula applied to h_t. Clearly, we have $0 \leq \rho_0^2 + (r_j^+)^2 \leq \rho_0^2$ for $0 \leq j \leq N$. Hence

$$\begin{aligned} d' \exp(-\rho_0^2 t) &\leq \sum_{j=0}^N n_j \exp(-(\rho_0^2 + (r_j^+)^2)t) \\ &\leq \sum_{j=0}^\infty n_j \exp(-(\rho_0^2 + (r_j^+)^2)t). \end{aligned} \tag{2.40}$$

Now the right side of this, by the trace formula (1.12) equals

$$\int_{\Gamma\backslash G} \left(\sum_{\gamma \in \Gamma} \chi(\gamma) h_t(x^{-1}\gamma x) \right) d\dot{x}.$$

This latter expression is of course positive, and is dominated by

$$\chi(1) \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma} h_t(x^{-1}\gamma x)\, d\dot{x},$$

because h_t is positive. It follows that

$$
\begin{aligned}
d' &\le \exp(\rho_0^2 t)\chi(1) \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} h_t(x^{-1}\gamma x)\, d\dot{x} \\
&= \chi(1) \int_{\Gamma\backslash G} H_t(\dot{x}, \dot{x})\, d\dot{x}
\end{aligned}
\tag{2.41}
$$

where $H_t(\dot{x}, \dot{y}) = \exp \rho_0^2 t \sum_{\gamma\in\Gamma} h_t(x^{-1}\gamma y)$.

Since h_t is admissible, $H_t(\dot{x}, \dot{y})$ is continuous on $\Gamma\backslash G \times \Gamma\backslash G$. Let $C_t = \sup_{\dot{x}\in\Gamma\backslash G} H_t(\dot{x}, \dot{x})$. Then

$$
d' \le C_t\chi(1) \operatorname{vol}(\Gamma\backslash G). \tag{2.42}
$$

The left side is independent of t, and our claim follows by choosing $C(\Gamma) = 2C_1$. Of course, this may not be the best possible value for $C(\Gamma)$.

We now turn to the question of existence of these exceptional zeroes of $Z_\Gamma(s, \chi)$. If T is the trivial one-dimensional representation of Γ, then the trivial representation of G occurs with multiplicity one in U, so that $Z_\Gamma(s, \chi)$ certainly has a zero at $s = 0$, or order 1. More generally, if T is the trivial (reducible) representation of Γ on an m-dimensional space, which we shall call $\mathfrak{1}_m$, then $Z_\Gamma(s, \chi)$ will have a zero of order m at $s = 0$. We shall give a condition on the dual space of Γ which will ensure that we can find nontrivial χ for which $Z_\Gamma(s, \chi)$ will turn out to have zeroes in the interval $(0, \rho_0)$. Thus in these cases, nontrivial spherical complementary series representations will in fact occur in U.

For a fixed integer $m \ge 1$, let $\hat{\Gamma}_m$ be the set of equivalence classes of (not necessarily irreducible) unitary representations of Γ, of degree m. Each element of $\hat{\Gamma}_m$ is determined by its character. We topologize $\hat{\Gamma}_m$ by the topology of uniform convergence of the characters on compact (i.e., finite) subsets of Γ. The trivial representation of degree m is denoted by $\mathfrak{1}_m$.

PROPOSITION 2.10. *Suppose that in the topology of $\hat{\Gamma}_m$ described above, the point $\mathfrak{1}_m$ is not isolated. Then there exists a nontrivial representation T of degree m, with character χ, such that $Z_\Gamma(s, \chi)$ has at least one zero in the open interval $(0, \rho_0)$. Moreover, the number of zeroes of $Z_\Gamma(s, \chi)$ in $(0, \rho_0)$ is then at least $m - a_0$, where a_0 is the multiplicity with which trivial representation $\mathfrak{1}_1$ occurs in T. In particular, if T is irreducible, $Z_\Gamma(s, \chi)$ will have at least m zeroes in $(0, \rho_0)$. Finally, if η is any number such that $0 < \eta < \rho_0$ it is possible to choose T in such a way that the interval $(0, \eta)$ contains at least $m - a_0$ zeroes.*

Proof. The proof uses the trace formula applied to h_t, written as (2.36) above. We can also write it in the form

$$
\begin{aligned}
&(4\pi t)^{1/2} \sum_{j\ge 0} n_j(\chi) \exp(-r_j^+(\chi)^2 t) \\
&\qquad = \chi(1) \operatorname{vol}(\Gamma\backslash G)(t/4\pi)^{1/2} \int_{-\infty}^{\infty} \exp(-r^2 t) c(r)^{-1} c(-r)^{-1}\, dr \\
&\qquad\quad + \sum_{\gamma\in C_\Gamma - \{1\}} \chi(\gamma)|u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(-u_\gamma^2/4t)
\end{aligned}
\tag{2.43}
$$

where we have written $n_j(\chi)$, $r_j^+(\chi)$ to emphasize their dependence on χ, and have used the Plancherel theorem to express $h_t(1)$ as an integral.

The first term on the right will be denoted by $J_1(\chi, t)$ and the second by $J_2(\chi, t)$. As to the left side, the numbers $r_j^+(\chi)^2$ always fall in the interval $[-\rho_0^2, \infty)$. Let $I_1 = [-\rho_0^2, 0)$, $I_2 = [0, \infty)$. We split up sums on the index j (such as the sum on the left side of (2.43)) into two sums over $\{j; r_j^+(\chi)^2 \in I_1\}$ and $\{j; r_j^+(\chi)^2 \in I_2\}$ and denote them by $\sum_1$ and $\sum_2$ respectively.

We shall also tacitly assume that the indices j are so ordered that $r_j^+(\chi)^2$ increases with j.

Now put

$$F_1(\chi, t) = (4\pi t)^{1/2} \sum_1 n_j(\chi) \exp(-r_j^+(\chi)^2 t)$$

and

$$F_2(\chi, t) = (4\pi t)^{1/2} \sum_2 n_j(\chi) \exp(-r_j^+(\chi)^2 t).$$

Note that $F_1 \geq 0$, $F_2 \geq 0$. Then (2.43) takes the form

$$F_1(\chi, t) + F_2(\chi, t) = J_1(\chi, t) + J_2(\chi, t). \tag{2.44}$$

Remember that $\mathbb{1}$ denotes the trivial character of Γ. We shall be interested in studying the number $N(\chi) = \sum_1 n_j(\chi)$. Clearly, $N(\chi)$ is precisely the number of zeroes (counting multiplicities) of $Z_\Gamma(s, \chi)$ in $[0, \rho_0)$.

In what follows, we shall find it convenient to denote wholesale by $\varepsilon(t)$ any function of t which approaches zero as $t \to \infty$, not necessarily the same function in each case.

We assume $t \geq 1$; It is easy to see that

$$F_2(\mathbb{1}, 1) \leq J_1(\mathbb{1}, 1) + J_2(\mathbb{1}, 1), \tag{2.45}$$

$$\begin{aligned} F_2(\chi, 1) &\leq J_1(\chi, 1) + J_2(\chi, 1) \\ &\leq \chi(1)J_1(\mathbb{1}, 1) + \chi(1)J_2(\mathbb{1}, 1), \end{aligned} \tag{2.46}$$

where we used the fact that $J_2(\chi, 1)$ is real, and $|\chi(\gamma)| \leq \chi(1)$.

It follows that we can find $M_1 > 0$ such that

$$F_2(\chi, t) \leq \chi(1)M_1(4\pi t)^{1/2} \quad \text{for all } \chi \text{ and } t \geq 1. \tag{2.47}$$

Next, when $\chi = \mathbb{1}$, the trivial representation of G occurs in U. So $r_0^+(\mathbb{1}) = i\rho_0$, and the multiplicity $n_0(\mathbb{1}) = 1$. It follows that

$$F_1(\mathbb{1}, t) = (1 + \varepsilon(t))(4\pi t)^{1/2} \exp \rho_0^2 t. \tag{2.48}$$

Now, using the above, we conclude

$$F_2(\chi, t) = \varepsilon(t)(4\pi t)^{1/2} \exp \rho_0^2 t \quad \text{for every } \chi, \tag{2.49}$$

$$J_1(\chi, t) = \varepsilon(t)(4\pi t)^{1/2} \exp \rho_0^2 t \quad \text{for every } \chi. \tag{2.50}$$

From (2.48)–(2.50), and the trace formula (2.44) with $\chi = \mathbb{1}$, we conclude

$$J_2(\mathbb{1}, t) = (1 + \varepsilon(t))(4\pi t)^{1/2} \exp \rho_0^2 t. \tag{2.51}$$

Let $\varepsilon > 0$ be a given number, and choose t_0 so large that all the functions $\varepsilon(t)$ appearing above are smaller than ε in absolute value. From now on we only consider $t \geq t_0$.

Now consider $(4\pi t)^{-1/2} \exp(-\rho_0^2 t) J_2(\mathbf{1}, t)$. This is the infinite sum

$$(4\pi t)^{-1/2} \exp(-\rho_0^2 t) \sum_{\gamma \in C_\Gamma - \{1\}} |u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(-u_\gamma^2/4t).$$

We know from (2.51) that for $t \geq t_0$ it lies between $1 - \varepsilon$ and $1 + \varepsilon$. Fix such a t, and let F_t be a finite subset of $C_\Gamma - \{1\}$ so large, that

$$(4\pi t)^{-1/2} \exp(-\rho_0^2 t) \sum_{\gamma \notin F_t} |u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(-u_\gamma^2/4t)$$

is less than ε. Then we find that

$$\sum_{\gamma \notin F_t} |u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(-u_\gamma^2/4t) \tag{2.52}$$

lies between $(1 - 2\varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t$ and $(1 + 2\varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t$. We may assume that F_t is stable under $\gamma \to \gamma^{-1}$.

$$J_2(\chi, t, F_t) = \sum_{\gamma \in F_t} \chi(\gamma) |u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(-u_\gamma^2/4t), \tag{2.53}$$

$$J_2(\chi, t, F_t^c) = J_2(\chi, t) - J_2(\chi, t, F_t). \tag{2.54}$$

Then, $J_2(\chi, t, F_t)$ is real, and since $|\chi(\gamma)| \leq \chi(1)$, we find for any χ, that

$$\begin{aligned} J_2(\chi, t, F_t^c) &\leq \chi(1) J_2(\mathbf{1}, t, F_t^c) \\ &\leq \chi(1)\varepsilon(4\pi t)^{1/2} \exp \rho_0^2 t. \end{aligned} \tag{2.55}$$

Now by our hypothesis on $\hat{\Gamma}_m$, we can find a representation $T \in \hat{\Gamma}_m$ such that its character χ satisfies $|\chi(\gamma) - m| \leq \varepsilon$ for any $\gamma \in F_t$. It follows, since $\chi(1) = m$, that

$$\begin{aligned} &|J_2(\chi, t, F_t) - m J_2(\mathbf{1}, t, F_t)| \\ &\qquad \leq \sum_{\gamma \in F_t} |\chi(\gamma) - m| \, |u_\gamma| j(\gamma)^{-1} C(h(\gamma)) \exp(-u_\gamma^2/4t) \\ &\qquad \leq \varepsilon J_2(\mathbf{1}, t, F_t) \\ &\qquad \leq \varepsilon(1 + 2\varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t \end{aligned} \tag{2.56}$$

by (2.52). Hence

$$\begin{aligned} (m - \varepsilon)(1 - 2\varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t &\leq J_2(\chi, t, F_t) \\ &\leq (m + \varepsilon)(1 + 2\varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t \end{aligned} \tag{2.57}$$

Now (2.57), (2.55), and the estimate on $J_1(\chi, t)$ implies that the right side of the trace formula (2.44) satisfies

$$\begin{aligned} &((m - \varepsilon)(1 - 2\varepsilon) - m\varepsilon - \varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t \\ &\qquad \leq J_1(\chi, t) + J_2(\chi, t) \\ &\qquad \leq ((m + \varepsilon)(1 + 2\varepsilon) + m\varepsilon + \varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t. \end{aligned} \tag{2.58}$$

It follows that the left side $F_1(\chi, t) + F_2(\chi, t)$ must satisfy this inequality also. Moreover, since $F_2(\chi, t)$ satisfies (2.49), we conclude finally that

$$F_1(\chi, t) \geq ((m - \varepsilon)(1 - 2\varepsilon) - m\varepsilon - 2\varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t. \tag{2.59}$$

Clearly, if ε is chosen small enough, the right side is positive. It follows that $F_1(\chi, t)$ is not zero. Now

$$F_1(\chi, t) = (4\pi t)^{1/2} \sum_1 n_j(\chi) \exp(-r_j^+(\chi)^2 t). \tag{2.60}$$

It is easy to see that the trivial representation of G occurs in U as often as χ contains the trivial one-dimensional representation of Γ. Let a_0 be this multiplicity. Then clearly, the term corresponding to $j = 0$ in the above sum is $a_0(4\pi t)^{1/2} \exp \rho_0^2 t$. Thus, recalling the definition of $N(\chi)$ we find

$$\begin{aligned} N(\chi)(4\pi t)^{1/2} \exp \rho_0^2 t &\geq (4\pi t)^{1/2} \sum_1 n_j(\chi) \exp(-r_j^+(\chi)^2 t) \\ &= F_1(\chi, t) \\ &\geq ((m - \varepsilon)(1 - 2\varepsilon) - m\varepsilon - 2\varepsilon)(4\pi t)^{1/2} \exp \rho_0^2 t \end{aligned} \tag{2.61}$$

which implies

$$N(\chi) \geq (m - \varepsilon)(1 - 2\varepsilon) - m\varepsilon - 2\varepsilon. \tag{2.62}$$

Now since ε is arbitrary, we find that

$$N(\chi) \geq m. \tag{2.63}$$

Of the $N(\chi)$ zeroes of $Z_\Gamma(s, \chi)$ in $[0, \rho_0)$ exactly a_0 correspond to $s = 0$. It follows that the number of zeroes of $Z_\Gamma(s, \chi)$ in $(0, \rho_0)$ is at least $m - a_0$. Since χ is nontrivial, $a_0 \neq m$. Finally, if χ is irreducible, we must have $a_0 = 0$. Thus all the assertions of our proposition are proved, except the last one. For the last one, we can repeat the argument used above almost verbatim, with $I_1 = [-\rho_0^2, -\eta^2]$ and $I_2 = (\eta^2, \infty)$. ∎

For a given m, it does not seem easy to give structural conditions on Γ which will imply that $\hat{\Gamma}_m$ satisfies the hypothesis of the above proposition, except when $m = 1$. In the case $m = 1$, if $\Gamma/[\Gamma, \Gamma]$ is infinite, then $\hat{\Gamma}_1$ will satisfy the hypothesis of Proposition 2.10. For, then $\Gamma/[\Gamma, \Gamma]$ is a finitely generated infinite abelian group, so its rank is ≥ 1. It follows that $\hat{\Gamma}_1$ contains a torus, so that the trivial representation $\mathbb{1}$ of Γ is not isolated.

In the case of $G = SL(2, \mathbf{R})$, $\Gamma/[\Gamma, \Gamma]$ is infinite. Indeed

$$\Gamma/[\Gamma, \Gamma] \cong H_1(\Gamma\backslash G/K) = \mathbf{Z}^{2g}$$

where g is the genus of $\Gamma\backslash G/K$. Since $g \geq 2$, Proposition 2.10 applies.

Actually, whenever $\Gamma/[\Gamma, \Gamma]$ is infinite, one can do more: namely, given any integer $N > 0$, we can find a subgroup Γ_0 of finite index in Γ such that the zeta function $Z_{\Gamma_0}(s, \mathbb{1})$ has at least N zeroes in $(0, \rho_0)$. The proof of this assertion for $G = SL(2, \mathbf{R})$ is contained in a recent paper of B. Randol [18]. An examination of his proof shows that once one has established the existence of a nontrivial character $\chi \in \hat{\Gamma}_1$ such that $Z_\Gamma(s, \chi)$ has a zero in $[0, \rho_0)$, the rest of his proof does not depend on any property of $SL(2, \mathbf{R})$ at all. Imitating it, we get our assertion without difficulty. We omit the proof.

It should be observed that for $G = SL(2, \mathbf{R})$, a result analogous to the above was mentioned without proof by Selberg. See the footnote on page 74 of [21].

In view of the discussion preceding Proposition 2.10, the above observations mean that when $\Gamma/[\Gamma, \Gamma]$ is infinite and $\Gamma\backslash G$ is compact, given any $N > 0$, we can choose a subgroup Γ_0 of finite index in Γ such that the representation of G on $L_2(\Gamma_0\backslash G)$ contains at least N subrepresentations of the spherical complementary series. These representations are not tempered. The question of the existence of such nontempered representations has attracted some attention recently. That nontempered representations can occur was observed by Wallach [27]. He observed, using the results of Hotta-Wallach [28] and Johnson-Wallach [29], that if $G = SO_0(n, 1)$ with $n \geq 4$, and if Γ is a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ is compact and $\Gamma/[\Gamma, \Gamma]$ is infinite then there exists a nontempered representation π_1 of G whose multiplicity in $L_2(\Gamma\backslash G)$ equals the rank of the free summand of the abelian group $\Gamma/[\Gamma, \Gamma]$ (which also equals, of course, the first Betti number of $\Gamma\backslash G/K$). Moreover, he also observed that via the result of Vinberg [25], such Γ do indeed exist if $n = 4$ or 5. Our result above differs from this in two ways. First, the representation π_1 mentioned here is nonspherical. Second, it is not known whether π_1 can occur with arbitrarily large multiplicity, i.e., it is not known whether Γ can be chosen with rank $\Gamma/[\Gamma, \Gamma]$ arbitrarily large.

A natural question that arises is whether a given group G has discrete subgroups Γ such that $\Gamma\backslash G$ is compact and $\Gamma/[\Gamma, \Gamma]$ is infinite. Because of the results of Kazhdan [14], such groups cannot exist if rank $(G/K) > 1$. Moreover, these results of Kazhdan, combined with the results of Kostant [30], imply that if $G = Sp(n, 1)$, $n \geq 2$, or if $G = F_{4(-20)}$, then such Γ cannot exist. Thus, there remain the cases $G = SO_0(n, 1)$, $n \geq 2$, and $G = SU(n, 1)$, $n \geq 2$. That such subgroups Γ exist for $G = SO_0(n, 1)$ with $2 \leq n \leq 5$ was observed by Vinberg [25]. Recently, J. Millson has shown that such Γ exist (and can even be assumed arithmetic) when $G = SO_0(n, 1)$ with n arbitrary [31]. (I gather that this result has also been independently obtained by W. Thurston.) It is not known whether discrete subgroups Γ with these properties exist for the groups $SU(n, 1)$.

We finally come to the assertion concerning the order of Z_Γ when it is an entire function. Here we shall be content to give a sketch of the proof of the fact that Z_Γ has finite order, and that the order can be related to the structure of (G, K). The reader may consult [23] where a similar argument is given in detail for a different zeta function.

Let $\varepsilon > 0$ be fixed. In the half plane $\operatorname{Re} s > 2\rho_0 + \varepsilon$, $Z_\Gamma(s, \chi)$ is clearly bounded; Now the function $\int_0^{s-\rho_0} \Phi(t)\, dt$ which occurs in the functional equation is surely a tempered function of s. More precisely, in absolute value, it is less than or equal to $A|s|^d$ for some constant A and integer d. It follows from the functional equation for Z_Γ that $|Z_\Gamma(s, \chi)| \leq \exp A|s|^d$ for s in the half plane $\operatorname{Re} s < -\varepsilon$. We shall show that in the strip $-\varepsilon \leq \operatorname{Re} s \leq 2\rho_0 + \varepsilon$, $Z_\Gamma(s, \chi)$ does not grow too fast as $\operatorname{Im} s \to \infty$. Then an application of the Phragmén-Lindelöf theorem will show that $|Z_\Gamma(s, \chi)| \leq C \exp A|s|^d$ for all s, implying that the order of Z_Γ is finite and $\leq d$.

To achieve this we recall that the logarithmic derivative $\Psi_\Gamma(s, \chi)$ equals $A(s) - \chi(1) \operatorname{vol}(\Gamma\backslash G) f_s(1)$ where

$$A(s) = \sum_{j \geq 0} n_j \left\{ \frac{H(i(s - s_j^+))}{s - s_j^+} + \frac{H(i(s - s_j^-))}{s - s_j^-} \right\} \tag{2.64}$$

and

$$f_s(1) = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r, s) c(r)^{-1} c(-r)^{-1} \, dr \tag{2.65}$$

with

$$h(r, s) = \frac{H(i(s - \rho_0 - ir))}{s - \rho_0 - ir} + \frac{H(i(s - \rho_0 + ir))}{s - \rho_0 + ir} \tag{2.66}$$

Using the asymptotic estimate of Proposition 1.2, one can prove the following lemma:

LEMMA 2.11. *For all sufficiently large $x > 0$, we can find two numbers x_1 and x_2 and a constant $A' > 0$ such that*

(i) $0 < x_1 < x < x_2 < 2x$, *and*

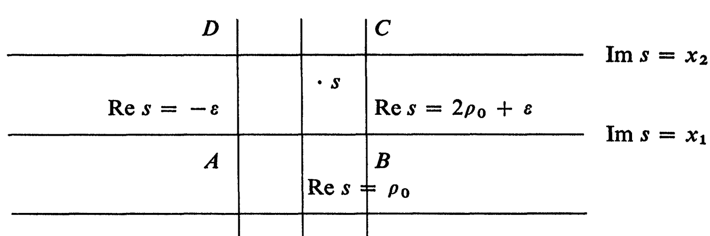
(ii) *for all $j \geq 0$, we have*

$$|x_1 - r_j^\pm| \geq \frac{A'}{x^{n-1}} \quad \textit{and} \quad |x_2 - r_j^\pm| \geq \frac{A'}{x^{n-1}}$$

where $n = \dim G/K$.

The proof of this lemma is a simple application of the pigeonhole principle. One cuts up the interval $(0, x)$ (resp. $(0, 2x)$) into pieces of length about A/x^{n-1} where A is small and then using the asymptotic estimate, one concludes that one of these intervals must be free of any $r_j^\pm$; Thus one finds an interval of length A/x^{n-1} in $(0, x)$ which does not contain any $r_j^\pm$. The midpoint of such an interval will serve as x_1. x_2 is produced similarly. Fix such an x, x_1, x_2, etc.

One can now estimate $A(s)$ where s lies on the boundary of the rectangle $ABCD$



by using the expression (2.64). It turns out that for s on the boundary, one has for some A_1, A_2,

(2.67) $\qquad |A(s)| \le A_1 x^N + A_2$ where N is a large integer.

On the other hand one shows easily that

(2.68) $\qquad |f_s(1)| \le A_1 x^N + A_2$ for s on the boundary $ABCD$.

It follows that

(2.69) $\qquad |\Psi_\Gamma(s, \chi)| \le 2A_1 x^N + 2A_2$ for s on the boundary.

By integration, one finds

(2.70) $\qquad |Z_\Gamma(s, \chi)| \le \exp(C_1 x^{N+1} + D_1)$ for s on the boundary.

However, Z_Γ is holomorphic in the interior of $ABCD$, so by the maximum modulus principle, we get $|Z_\Gamma(s, \chi)| \le \exp(C_1 x^{N+1} + D_1)$ for all s inside. In particular for any s with $\operatorname{Im} s = x$, inside $ABCD$, we get $|Z_\Gamma(s, \chi)| \le \exp(C_1 |\operatorname{Im} s|^{N+1} + D_1)$ and it follows that

(2.71) $\qquad |Z_\Gamma(s, \chi)| \le \exp(C_1 |s|^{N+1} + D_1), \quad -\varepsilon \le \operatorname{Re} s \le \rho_0 + \varepsilon$

for $|\operatorname{Im} s|$ sufficiently large. This verifies the hypotheses of Phragmén-Lindelöf and we conclude that when Z_Γ is entire, it has finite order which is less than or equal to the integer d. It is easy to see that d equals exactly the dimension of G/K. This is because

$$|c(r)^{-1} c(-r)^{-1}| \le C_1(1 + |r|)^{n-1}$$

where $n = \dim G/K$, as an examination of c will show.

On the other hand, Proposition 1.2 leads without difficulty to the following: The series $\sum_{r_j^+ \neq 0} n_j / |r_j^+|^k$ converges if $k > n = \dim G/K$, and diverges if $k \le n$. It follows that the exponent of convergence of the zeroes of the entire function Z_Γ is at least n. By a well-known theorem, this implies that the order of $Z_\Gamma \ge n$. Together with what we showed above, this implies that the order of $Z_\Gamma = n = \dim G/K$.

It only remains to prove the assertion (9) of Section 0.

Recall that an element $\gamma \in \Gamma$, $\gamma \neq 1$ is called primitive if it cannot be written in the form δ^j where $\delta \in \Gamma$ and $j > 1$. It was shown in [7] that each $\gamma \in \Gamma$, $\gamma \neq 1$ can be expressed uniquely in the form $\delta^{j(\gamma)}$ where δ is primitive and $j(\gamma)$ is a positive integer. If Prim_Γ is a complete set of representatives for the conjugacy classes of primitive elements in Γ, we can put

$$C_\Gamma = \bigcup_{\delta \in \mathrm{Prim}_\Gamma} \{\delta^j; j \ge 1\}.$$

The infinite product representation will involve a product over Prim_Γ, just as in [21].

Enumerate the roots in P_+ as $\alpha_1, \ldots, \alpha_t$, and let L be the set of linear functions on $\mathfrak{a}$ which are of the form $\sum_{i=1}^t m_i\alpha_i$, with m_i nonnegative and integral. For $\lambda \in L$, define m_λ to be the number of distinct ordered t-tuples $(m_1, \ldots, m_t)$ such that $\lambda = \sum_{i=1}^t m_i\alpha_i$, and let ξ_λ be the character of the Cartan subgroup A which corresponds to λ.

For any $\gamma \in \Gamma$, $\gamma \neq 1$, we have chosen an element $h(\gamma) \in A$ which is conjugate to γ; $h(\gamma) = h_{\mathfrak{p}}(\gamma)h_{\mathfrak{k}}(\gamma)$. We now further demand that $h(\gamma)$ be chosen so that $h_{\mathfrak{p}}(\gamma)$ lies in $A_+ = \exp \mathfrak{a}_{\mathfrak{p}}^+$, where $\mathfrak{a}_{\mathfrak{p}}^+$ is the positive Weyl chamber in $\mathfrak{a}_{\mathfrak{p}}$. With this understood, the product for Z_Γ is given by (in Re $s > 2\rho_0$)

$$(2.72)\quad Z_\Gamma(s, \chi) = C \prod_{\delta \in \mathrm{Prim}_\Gamma} \prod_{\lambda \in L} (\det (I - T(\delta)\xi_\lambda(h(\delta))^{-1} \exp (-su_\delta)))^{m_\lambda \kappa}$$

where C is a constant $\neq 0$, $u_\delta = \beta (\log h_{\mathfrak{p}}(\delta))$, and κ is as defined above. I is identity matrix, and T is the representation of Γ with character χ. det means determinant.

When $G = PSL(2, \mathbf{R})$, P_+ consists of a single element β, L consists of $\{k\beta, k \geq 0\}$, and $m_\lambda = 1$ for each $\lambda \in L$. Moreover, $h(\delta) = h_{\mathfrak{p}}(\delta)$ in this case, so that $\xi_\lambda(h(\delta))$ is equal to $\exp k\beta(h_{\mathfrak{p}}(\delta))$. Remembering that $\kappa = 1$ in this case, we recover the product formula of Selberg [21] for Z_Γ as a special case of (2.72) up to the constant factor C. The factor C will be commented upon below. It is due to a difference in the normalization of Z_Γ.

The proof of (2.72) proceeds from the formula (2.14) for the logarithmic derivative of Z_Γ, valid for Re $s > 2\rho_0$. In that formula, recall that $C(h(\gamma))$ was given by (1.13). Because of our special choice of $h(\gamma)$, we see that $\varepsilon_R^A(h(\gamma)) = 1$, and $u_\gamma \geq 0$, and we find that

$$C(h(\gamma)) = \xi_\rho(h_{\mathfrak{p}}(\gamma))^{-1} \prod_{\alpha \in P_+} (1 - \xi_\alpha(h(\gamma))^{-1})^{-1}.$$

Thus (2.14) can be written, as in [7],

$$\begin{aligned} &\frac{d}{ds} \log Z_\Gamma(s, \chi) \\ (2.73)\qquad &= \kappa \sum_{\delta \in \mathrm{Prim}_\Gamma} \sum_{j \geq 1} \left\{\chi(\delta^j)u_\delta \prod_{\alpha \in P_+} (1 - \xi_\alpha(h(\delta))^{-j})^{-1} \exp (-sju_\delta)\right\} \end{aligned}$$

Now expand $(1 - \xi_\alpha(h(\delta))^{-j})^{-1}$ as a power series, (which converges because $\xi_\alpha(h_{\mathfrak{p}}(\delta))^{-1} < 1$ by our choice of $h(\delta)$), $\sum_{m \geq 0} \xi_\alpha(h(\delta))^{-jm}$, and multiply together these series for the various $\alpha \in P_+$. We find that the product

$$\prod_{\alpha \in P_+} (1 - \xi_\alpha(h(\delta))^{-j})^{-1}$$

equals the sum $\sum_{\lambda \in L} m_\lambda \xi_\lambda(h(\delta))^{-j}$. Hence (2.73) becomes, with a rearrangement,

$$(2.74)\quad \frac{d}{ds} \log Z_\Gamma(s, \chi) = \kappa \sum_{\delta \in \mathrm{Prim}_\Gamma} \sum_{\lambda \in L} \sum_{j \geq 1} u_\delta m_\lambda \chi(\delta^j)\xi_\lambda(h(\delta))^{-j} \exp (-sju_\delta).$$

Now let $\varepsilon_1(\delta), \varepsilon_2(\delta), \ldots, \varepsilon_d(\delta)$ be the eigenvalues of $T(\delta)$. Then $\chi(\delta^j)$ equals $\sum_{i=1}^d (\varepsilon_i(\delta))^j$. Then

$$
\begin{aligned}
\frac{d}{ds} \log Z_\Gamma(s, \chi) &= \kappa \sum_{i=1}^{d} \sum_{\delta} \sum_{\lambda \in L} m_\lambda u_\delta \sum_{j \geq 1} \varepsilon_i(\delta)^j \xi_\lambda(h(\delta))^{-j} \exp(-sju_\delta) \\
&= \kappa \sum_{i=1}^{d} \sum_{\delta} \sum_{\lambda \in L} m_\lambda u_\delta \frac{\varepsilon_i(\delta)\xi_\lambda(h(\delta))^{-1} \exp(-su_\delta)}{1 - \varepsilon_i(\delta)\xi_\lambda(h(\delta))^{-1} \exp(-su_\delta)}.
\end{aligned} \tag{2.75}
$$

Integrating this logarithmic derivative, we find

$$
\begin{aligned}
Z_\Gamma(s, \chi) &= C \prod_{i=1}^{d} \prod_{\delta \in \mathrm{Prim}_\Gamma} \prod_{\lambda \in L} (1 - \varepsilon_i(\delta)\xi_\lambda(h(\delta))^{-1} e^{-su_\delta})^{m_\lambda \kappa} \\
&= C \prod_{\delta \in \mathrm{Prim}_\Gamma} \prod_{\lambda \in L} (\det(I - T(\delta)\xi_\lambda(h(\delta))^{-1} e^{-su_\delta}))^{m_\lambda \kappa}
\end{aligned} \tag{2.76}
$$

where $C \neq 0$. These manipulations are valid if $\operatorname{Re} s > 2\rho_0$.

In [21], Selberg *defines* Z_Γ by giving this product representation, and choosing $C = 1$. We have on the other hand normalized our Z_Γ by stipulating that $\lim_{s \to \rho_0} (s - \rho_0)^{-m_0} Z_\Gamma(s, \chi) = 1$; cf. the remarks following Proposition 2.7 above. We could, of course, renormalize Z_Γ so that $C = 1$ in (2.76) without losing any property of Z_Γ. We have now proved all the assertions of Section 0.

We conclude with a remark about the assumption that Γ is torsion free. If this assumption is dropped, most of the above assertions can still be made in a somewhat modified form. First, apart from the terms on the right side of the trace formula, there would be in addition a finite number of terms corresponding to conjugacy classes of elements of Γ that are of finite order. The contribution of such an element to the trace formula (2.3) can be computed by using the results of [19]. It turns out that if γ is an element of finite order in Γ, then the integral $\int_{\Gamma_\gamma \backslash G} f_s(x^{-1}\gamma x)\, d\dot{x}$ which equals

$$
\operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f_s(x^{-1}\gamma x)\, d\dot{x},
$$

can be expressed in terms of $\hat{f}_s(v)$ as an integral on the parameter v. If we call this term $I(f_s, \gamma)$, one can show that $I(f_s, \gamma)$ is meromorphic in s, and has poles in the upper half plane precisely at the points r_k, $k \geq 0$. Thus there is no problem in continuing $\Psi_\Gamma(s, \chi, g)$ analytically in this case to a meromorphic function.

If now one attempts to construct Z_Γ as before, one must relate the volume $\operatorname{vol}(\Gamma \backslash G)$ to the generalized Euler number of $\Gamma \backslash G/K$ in the sense of Satake [20]. Note that $\Gamma \backslash G/K$ is no longer a manifold, but it is a V-manifold in Satake's sense, and the singular points of $\Gamma \backslash G/K$ correspond exactly to the elliptic conjugacy classes of Γ. Therefore one expects that if one did the computations called for by [20] in our case, one would see that

$$
\left\{ \chi(1) \operatorname{vol}(\Gamma \backslash G) f_s(1) - \sum_{\gamma \text{ elliptic}} I(f_s, \gamma) \right\}
$$

would have simple poles with integer residues at r_k. There would then be no impediment to getting an analogous theory even in the case where Γ is not assumed torsion free. We have not, however, carried out this suggestion.

3. Appendix: An auxiliary computation

This section is devoted to a computation which will show the existence of the integer κ that we referred to in Section 2.

If a, b are any real numbers such that $a = rb$ with r a rational number, we shall write $a \sim b$. A similar convention will be used for functions, forms, etc. We wish to establish that in our normalization of the measures, we have $\operatorname{vol}(\Gamma\backslash G) \sim E$, where E is the Euler-Poincaré characteristic of the manifold $M = \Gamma\backslash G/K$. Of course, we assume throughout that $\dim M$ is even, equal to $2m$ say.

We denote by $\langle\cdot, \cdot\rangle_\theta$ the form $-\langle\cdot, \theta\cdot\rangle$, where $\langle\cdot, \cdot\rangle$ is the Cartan-Killing form of $\mathfrak{g}$. Then $\langle\cdot, \cdot\rangle_\theta$ is a positive definite form on $\mathfrak{g} \times \mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is an orthogonal[5] decomposition of $\mathfrak{g}$. Let $\mathfrak{s}$ be the orthogonal complement of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{p}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{s}$. Since we also have $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$ direct sum, we see that $\dim \mathfrak{s} = \dim \mathfrak{n}$. Call this integer t. Let $\mathfrak{n}_\beta$ be the subspace of $\mathfrak{n}$ on which $ad(H)$ acts via the scalar $\beta(H)$, and $\mathfrak{n}_{2\beta}$ the subspace on which $ad(H)$ acts by $2\beta(H)$, $H \in \mathfrak{a}_\mathfrak{p}$; cf. Section 1. Then $\dim \mathfrak{n}_\beta = p$, $\dim \mathfrak{n}_{2\beta} = q$, $\mathfrak{n} = \mathfrak{n}_\beta + \mathfrak{n}_{2\beta}$ is an orthogonal direct sum and $t = p + q$.

An element of $\mathfrak{g}$ will be viewed as a left invariant vector field on G. Elements of the dual of $\mathfrak{g}$ can then be viewed as invariant 1-forms. Let $\mathfrak{b}$ be any subspace of $\mathfrak{g}$ with orthocomplement $\mathfrak{b}^\perp$. Let $B_1, \ldots, B_b$ be a basis of $\mathfrak{b}$ and $B_1^*, \ldots, B_b^*$ be elements in $\mathfrak{g}^*$ defined by $B_i^*(B_j) = \delta_{ij}$, $B_i^* \equiv 0$ on $\mathfrak{b}^\perp$. Then we can define the form $\omega_\mathfrak{b}$ by

$$\omega_\mathfrak{b} = (\operatorname{Det}\langle B_i, B_j\rangle_\theta)^{1/2} B_1^* \wedge B_2^* \wedge \ldots \wedge B_b^*. \tag{3.1}$$

The form $\omega_\mathfrak{b}$ depends only on $\mathfrak{b}$, and not on the choice of the basis $B_1, \ldots, B_b$. When we have $\mathfrak{g} = \mathfrak{b} + \mathfrak{b}^\perp$, clearly,

$$\omega_\mathfrak{g} = \omega_\mathfrak{b} \wedge \omega_{\mathfrak{b}^\perp}, \tag{3.2}$$

suitable ordering of bases being tacitly understood.

These considerations imply in particular that

$$\omega_\mathfrak{g} = \omega_\mathfrak{k} \wedge \omega_{\mathfrak{a}_\mathfrak{p}} \wedge \omega_\mathfrak{s}, \tag{3.3}$$

$$\omega_\mathfrak{p} = \omega_{\mathfrak{a}_\mathfrak{p}} \wedge \omega_\mathfrak{s}. \tag{3.4}$$

The form $\omega_\mathfrak{g}$ gives us a G-invariant volume form on the group G. The form $\omega_\mathfrak{k}$, gives by restriction to K, an invariant volume element on K. The normalized volume element of K is then $\omega_\mathfrak{k}(K)^{-1}\omega_\mathfrak{k}$. The form $\omega_\mathfrak{p}$ can be thought of as a volume element on G/K, invariant under G.

[5] Throughout this section, orthogonality is understood with respect to the form $\langle\cdot, \cdot\rangle_\theta$.

Now let v be the invariant form on G which corresponds to our choice of the Haar measure. Recall that we have normalized the Haar measure dn on N by the requirement that $\int_N \exp(-2\rho(H(\bar{n})))\, dn = 1$, and that our parametrization on $A_\mathfrak{p}$ is via the parameter $u = u(h) = \beta(\log h)$, $h \in A_\mathfrak{p}$. Taking this into account, we get (cf. [12, chapter X, p. 373]),

$$v = c^{-1} C_N^{-1} \omega_\mathfrak{k}(K)^{-1} \omega_\mathfrak{k} \wedge \omega_{\mathfrak{a}_\mathfrak{p}} \wedge \omega_\mathfrak{n} \tag{3.5}$$

where $c = (\langle H_0, H_0 \rangle_\theta)^{1/2} = (2p + 8q)^{1/2}$, and

$$C_N = \int_N e^{-2\rho(H(\bar{n}))} \omega_\mathfrak{n}. \tag{3.6}$$

The integral C_N can be computed by the technique of Godement-Schiffman and Gindikin-Karpelevic, as quoted in [26, vol. II, p. 323].

If we introduce orthonormal coordinates $\xi_1, \ldots, \xi_p$ in $\mathfrak{n}_\beta$ and $\eta_1, \ldots, \eta_q$ in $\mathfrak{n}_{2\beta}$, we get

$$C_N = \int_{R^p \times R^q} ((1 + |\xi|^2/2c^2)^2 + 2|\eta|^2/c^2)^{-1/2(p+2q)}\, d\xi\, d\eta \tag{3.7}$$

where $|\xi|^2 = \sum \xi_i^2$, $|\eta|^2 = \sum \eta_i^2$, etc. c is, of course, $(2p + 8q)^{1/2}$ as above. This integral can be evaluated by standard methods which we omit. The result is

$$\begin{aligned} C_N &= c^{p+q} 2^{(p-q)/2} \pi^{(p+q+1)/2} 2^{-(p+q-1)} (((p + q - 1)/2)!)^{-1} \\ &= c^{p+q} 2^{(p-q)/2} \pi^m 2^{-2m+2} ((m - 1)!)^{-1}. \end{aligned} \tag{3.8}$$

where $m = (p + q + 1)/2$. Note that $m = \frac{1}{2} \dim G/K$. Thus we see that

$$C_N \sim c^{p+q} 2^{(p+q)/2} \pi^m \tag{3.9}$$

so

$$c^{-1} C_N^{-1} \sim c^{-p-q-1} 2^{-(p-q)/2} \pi^{-m} \sim 2^{-(p-q)/2} \pi^{-m} \tag{3.10}$$

since $c = (2p + 8q)^{1/2}$, and so $c^{p+q+1} = c^{2m} \sim 1$.

Now the form $\omega_\mathfrak{k} \wedge \omega_{\mathfrak{a}_\mathfrak{p}} \wedge \omega_\mathfrak{n}$ is certainly an invariant ($\dim G$)-form on G, and as such, it must be a constant multiple of the form $\omega_\mathfrak{g}$. We will now compute the constant that relates these two by choosing suitable bases.

Let σ be the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Then σ operates on P_+ in a natural way; we denote by α^σ the image of $\alpha \in P_+$ under σ (cf. [12, p. 222]). A root $\alpha \in P_+$ is real if and only if $\alpha = \alpha^\sigma$. It follows that we can find a subset P_+^0 of P_+ with the property that $P_+ = P_+^r \cup P_+^0 \cup (P_+^0)^\sigma$ where P_+^r is the set of real roots in P_+, and the union is disjoint. Now let $E_\alpha \in \mathfrak{g}^C$ be a root vector corresponding to α. We can choose E_α, $\alpha \in \Phi^+(\mathfrak{g}^C, \mathfrak{a}^C)$ in such a way that

$$[E_\alpha, E_{-\alpha}] = 2H_\alpha/\langle \alpha, \alpha \rangle, \qquad \langle E_\alpha, E_{-\alpha} \rangle = 2\langle \alpha, \alpha \rangle^{-1}. \tag{3.11}$$

This can always be done; cf. [12].

Now consider the vectors

$$\{E_\alpha, \alpha \in P_+^r\}, \{(E_\alpha + \sigma E_\alpha)/2, \alpha \in P_+^0\} \quad \text{and} \quad \{(E_\alpha - \sigma E_\alpha)/2i, \alpha \in P_+^0\}.$$

It is easily seen that these all lie in $\mathfrak{n}$; A computation, using standard facts about σ shows that these vectors are orthogonal with respect to $\langle \cdot, \cdot \rangle_\theta$. Thus they form an orthogonal basis of $\mathfrak{n}$. Normalizing these vectors, we get an orthonormal basis $N_1, \ldots, N_t$ of $\mathfrak{n}$.

Next, consider $N_i - \theta N_i$. Clearly this belongs to $\mathfrak{p}$. Moreover, both $\mathfrak{n}$ and $\theta\mathfrak{n}$ are orthogonal to $\mathfrak{a}_\mathfrak{p}$ under $\langle \cdot, \cdot \rangle_\theta$. Hence $N_i - \theta N_i \in \mathfrak{s}$. In fact, one can show that these elements are mutually orthogonal in $\mathfrak{s}$. The norm of $N_i - \theta N_i$ is easily computed. It turns out to equal $\sqrt{2}$. Thus if

$$Y_i = \frac{1}{\sqrt{2}}(N_i - \theta N_i),$$

the vectors $Y_1, \ldots, Y_t$ are an orthonormal basis of $\mathfrak{s}$. Similarly, if

$$X_i = \frac{1}{\sqrt{2}}(N_i + \theta N_i),$$

we get an orthonormal set of vectors in $\mathfrak{k}$, which we extend to an orthonormal basis of $\mathfrak{k}$, call it $X_1, \ldots, X_d$. Note that

$$N_i = \frac{1}{\sqrt{2}}(X_i + Y_i), \quad i = 1, \ldots, t.$$

Let H_1 be an element of norm one in $\mathfrak{a}_\mathfrak{p}$. Then the set

$$\{X_1, X_2, \ldots, X_d; H_1; N_1, N_2, \ldots, N_t\}$$

is a basis of $\mathfrak{g}$. We can use it to compute $\omega_\mathfrak{g}$, following (3.1). Since

$$\langle X_i, N_j \rangle = \frac{1}{\sqrt{2}}\delta_{ij}, \quad 1 \leq i, j \leq t,$$

we obtain

$$\begin{aligned} \omega_\mathfrak{g} &= (\sqrt{2})^{-t} X_1^* \wedge \cdots \wedge X_d^* \wedge H_1^* \wedge N_1^* \wedge \cdots \wedge N_t^* \\ &= (\sqrt{2})^{-t} \omega_\mathfrak{k} \wedge \omega_{\mathfrak{a}_\mathfrak{p}} \wedge \omega_\mathfrak{n}. \end{aligned} \tag{3.12}$$

Combining (3.12), (3.5), (3.9) we see, since $t = p + q$, that

$$\begin{aligned} v &= c^{-1} C_N^{-1} \omega_\mathfrak{k}(K)^{-1} 2^{(p+q)/2} \omega_\mathfrak{g} \\ &\sim \omega_\mathfrak{k}(K)^{-1} 2^{-(p-q)/2} 2^{(p+q)/2} \pi^{-m} \omega_\mathfrak{g} \\ &\sim \pi^{-m} \omega_\mathfrak{k}(K)^{-1} \omega_\mathfrak{g} \\ &\sim \pi^{-m} \omega_\mathfrak{k}(K)^{-1} \omega_\mathfrak{k} \wedge \omega_\mathfrak{p} \end{aligned} \tag{3.13}$$

It follows that

$$\text{vol}\,(\Gamma\backslash G) \sim \pi^{-m}\omega_{\mathfrak{p}}(\Gamma\backslash G/K). \tag{3.14}$$

where $\omega_{\mathfrak{p}}(\Gamma\backslash G/K)$ is the volume of $\Gamma\backslash G/K$ with respect to the volume form $\omega_{\mathfrak{p}}$, which we repeat, was obtained from the Cartan-Killing form.

It remains to check that the Euler-Poincaré characteristic E satisfies

$$E \sim \pi^{-m}\omega_{\mathfrak{p}}(\Gamma\backslash G/K). \tag{3.15}$$

To check this last point, one may either use the results of Ono [17, Section 3] or proceed directly via the Gauss-Bonnet theorem. (In using Ono's results, it must be borne in mind that $\mathfrak{k}'$, the derived algebra of $\mathfrak{k}$, has two Euclidean structures, namely one that is obtained by restricting to $\mathfrak{k}$ the Euclidean structure on $\mathfrak{g}$ given by $\langle\cdot, \cdot\rangle_\theta$, and the other obtained by the Cartan-Killing form of $\mathfrak{k}'$. Ono uses the latter in his computation of the volume of a compact semisimple group while we have used the former.) In either case, it seems that one is forced to use the classification. If we proceed via Ono's results, we have to use the Dynkin diagrams of the various groups.

We shall indicate here how one can verify (3.15) directly via the Gauss-Bonnet theorem, as given in [1] or [22] for example.

For any compact oriented Riemannian manifold M of even dimension $2m$, the Gauss-Bonnet theorem tells us that the Euler-Poincaré characteristic $E(M)$ is given by

$$E(M) = \pi^{-(m+1)/2}\Gamma\left(\frac{m+1}{2}\right)\int_M R\,d\omega \tag{3.16}$$

where $d\omega$ is the Riemannian volume form of M and R is a function on M defined locally by

$$R = (2^m \det g(2m)!)^{-1} \times \sum_{i,j} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} \cdots R_{i_{2m-1} i_{2m} j_{2m-1} j_{2m}} \varepsilon(i)\varepsilon(j)$$

Here $g = (g_{kl})$ is the Riemannian metric tensor, and R_{rstu} are the components of the Riemannian curvature tensor. The sum is over all possible permutations $i = (i_1, i_2, \ldots, i_{2m})$ and $j = (j_1, \ldots, j_{2m})$ of $(1, 2, \ldots, 2m)$, and $\varepsilon(i)$, $\varepsilon(j)$ are the signatures of those permutations. Of course, the components g_{ij}, R_{ijkl} are computed via the usual basis of the tangent space at any point viz $\{\partial/\partial x_i\}$.

In our case, M is $\Gamma\backslash G/K$, and since the covering manifold G/K is homogeneous, and the metric on it is G-invariant, it follows that R is a constant, so we need to compute it just at the coset K in G/K.

We shall show, by choosing a suitable basis of $\mathfrak{p}$, that the number R is a rational number. This will imply (3.15), via (3.14). In fact, let $Y_1, \ldots, Y_{2m}$ be a mutually orthogonal basis of $\mathfrak{p}$. Then the components of the Riemannian curvature, are given by

$$R(Y_i, Y_j)Y_k = \sum_l R_{ijkl} Y_l,$$

where $R(\cdot, \cdot)$ is the curvature tensor of G/K, viewed as a map of $\mathfrak{p} \times \mathfrak{p}$ to End $\mathfrak{p}$. As is well known, $R(Y_i, Y_j)Y_k = -[[Y_i Y_j], Y_k]$; cf. [12]. Thus we find $R_{ijkl} = -\langle [[Y_i, Y_j], Y_k], Y_l \rangle_\theta / \langle Y_l, Y_l \rangle_\theta$. Since $\theta Y_l = -Y_l$, this is seen to equal

$$\langle [Y_i, Y_j], [Y_k, Y_l] \rangle / \langle Y_l, Y_l \rangle.$$

Thus our assertion about the rationality of R will follow if we can find a $\langle \cdot, \cdot \rangle$-orthogonal basis of $\mathfrak{p}$, say $Y_1, \ldots, Y_{2m}$, such that (i) R_{ijkl} are all rational and (ii) $\det g = \det \langle Y_i, Y_j \rangle$ is rational.

In fact one can do a little better. One can find an orthogonal basis of $\mathfrak{p}$ such that (i) $\langle [Y_i, Y_j], [Y_k, Y_l] \rangle$ are all rational and (ii) $\langle Y_l, Y_l \rangle$ are all rational.

This is done as follows. Let σ be the involution of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}$, as above, and let τ be the involution of $\mathfrak{g}^{\mathbf{C}}$ with respect to the compact real form $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Then $\sigma\tau = \tau\sigma = \theta$, $\theta\sigma = \sigma\theta = \tau$, and $\theta\tau = \tau\theta = \sigma$. For any root α, we let H_α be the element of $\mathfrak{a}^{\mathbf{C}}$ such that $\langle H, H_\alpha \rangle = \alpha(H)$ for all $H \in \mathfrak{a}^{\mathbf{C}}$. By using the classification one can show that for each root α, we can choose a root vector E_α in $\mathfrak{g}^{\mathbf{C}}_\alpha$ so that the following properties hold: (i) $\langle E_\alpha, E_{-\alpha} \rangle$ is rational. (ii) If α, β are roots such that $\alpha + \beta$ is a root, then $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ where $N_{\alpha,\beta}$ is rational (and even integral). (iii) $\tau E_\alpha = E_{\alpha^\tau}$ where α^τ is the image of the root α under τ. (iv) $\sigma E_\alpha = c_\alpha E_{\alpha^\sigma}$, where c_α is ± 1 or $\pm\sqrt{-1}$ for $\alpha \in P_+$. The standard Chevalley bases of the complexifications of $\mathfrak{so}(n, 1)$, $\mathfrak{su}(n, 1)$, and $\mathfrak{sp}(n, 1)$ have these properties. In the case of $\mathfrak{f}_{4(-20)}$ one has to use the explicit multiplication table given by Cartan [2, p. 343] for the complex Lie algebra $\mathfrak{f}_4^{\mathbf{C}}$, together with the description of the involutions τ, σ given in [2, p. 352] and [2, p. 351] respectively. (The vectors $X_{\alpha\beta\gamma\delta}$ in Cartan's notation serve as our E_α, $\alpha \in P_+$.)

Having obtained such root vectors, we now consider the elements

$$\{E_\alpha;\, \alpha \in P^r_+\}, \qquad \{E_\alpha + \sigma E_\alpha,\, (E_\alpha - \sigma E_\alpha)/i;\, \alpha \in P^0_+\}.$$

As mentioned above, these elements form an orthogonal basis of $\mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle_\theta$. In our case P^r_+ has exactly one element. Let H_α be the element of $\mathfrak{a}^{\mathbf{C}}$ corresponding to it. Then $H_\alpha \in \mathfrak{a}_\mathfrak{p}$, and H_α, together with the above elements gives us an orthogonal basis of $\mathfrak{a}_\mathfrak{p} + \mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle_\theta$. Call this basis $Z_1, \ldots, Z_{2m}$. Then $\{Z_i - \theta Z_i;\, i = 1, \ldots, 2m\}$ is seen to be an orthogonal basis of $\mathfrak{p}$. We call this basis $Y_1, \ldots, Y_{2m}$. Because of the properties of the chosen vectors E_α, this basis is easily seen to have the following properties: (i) $\langle [Y_i, Y_j], [Y_k, Y_l] \rangle$ are all rational (and, of course, known a priori to be real) and (ii) $\langle Y_l, Y_l \rangle$ are all rational. This completes the proof of (3.15).

The existence of the root vectors E_α with the above properties can be obtained from a general result of Chevalley [3] (slightly refined to give the additional property (iv)). However, this does not in principle avoid the classification, because Chevalley's paper uses the classification at one point. Thus in our case it seems simpler to proceed directly. In Table II of the appendix we have listed the basis $Y_1, \ldots, Y_d$ for the classical groups under consideration, using their matrix realizations. The case of $F_{4(-20)}$ must, however, be handled abstractly as described above.

Having established (3.15), the existence of the integer κ follows, as we have remarked in Section 2, upon observing that the numbers $i\, d_k$ are all rational, with denominators that depend only on (G, K) and not on the particular pole r_k. By explicit (and sometimes excessive) computation, one could actually determine κ for $SO_0(2n, 1)$, $SU(n, 1)$, $Sp(n, 1)$. For $SO_0(2n, 1)$ the value of κ is easy to compute. One finds that $\kappa = (2n - 1)^n$. This explains why this integer never explicitly appears in Selberg's paper. There, Selberg is dealing with $SL(2, \mathbf{R})$ or $SO_0(2, 1)$ essentially, and $n = 1$, so that $\kappa = 1$ in that case.

Finally, a word about the use of classification that has been made in this section, and which one should wish to avoid. While we may have appeared to use the classification in discussing the poles and residues of $r \to c(r)^{-1}c(-r)^{-1}$ in Section 2, a moment's thought reveals that this is merely a matter of convenience of description, i.e., we could have described Table I in terms of the integers p, q if we had wished to do so. Thus the use of classification made to arrive at Table I is inessential. However, for the computations of this section, resulting in (3.15), the use of classification is not merely a matter of convenience, i.e., I have not been able to avoid it.

Remark. In arriving at (3.11) above, we computed in effect the constant factor which relates two different normalizations of Haar measure on G. In fact, let $d_+ p$ be the Riemannian measure on G/K arising from the matric on G/K given by the Cartan-Killing form, and let $d_+ g$ be the Haar measure on G such that $d_+ g = dk\, d_+ p$. On the other hand, let $d_+ a$, $d_+ n$ be the Haar measures on $A_{\mathfrak{p}}$, N obtained from the euclidean structures on $\mathfrak{a}_p$, $\mathfrak{n}$ determined by $\langle \cdot, \cdot \rangle_\theta$, and let dg be the Haar measure on G such that

$$dg = \exp 2\rho\,(\log a)\, dk\, d_+ a\, d_+ n.$$

Then our argument shows actually that $d_+ g = (\sqrt{2})^{-t}\, dg$, where $t = \dim \mathfrak{n}$. The argument does not depend on the fact that rank $(G/K) = 1$. Thus we have computed in effect the relation between the normalizations of Haar measure given by the Iwasawa decomposition and the polar decomposition of G. The exact constant relating these does not seem to have been written down explicitly in the literature except in the special cases $SL(2, \mathbf{R})$ and $SO_0(n, 1)$.

The constant C_N can also be computed in general when rank $(G/K) > 1$. Indeed, if Σ is the set of restrictions to $\mathfrak{a}_p$ of the roots in P_+, and if Σ_0 is the subset $\{\alpha \in \Sigma, \alpha/n \notin \Sigma$ for any integer $n > 1\}$, then corresponding to each $\alpha \in \Sigma_0$, one gets a symmetric space S^α of rank one, and if C_{N_α} is the constant corresponding to this symmetric space, one can show that $C_N = \prod_{\alpha \in \Sigma_0} C_{N_\alpha}$, as is clear from the method of Grindikin-Karpelevič; cf. [26, vol. II, Chapter 9].

TABLE I

$$c(r)^{-1} = \frac{\Gamma((p+q)/2)}{\Gamma(p+q)} \times \frac{\Gamma(ir+p/2)}{\Gamma(ir)} \times \frac{\Gamma(ir/2+p/4+q/2)}{\Gamma(ir/2+p/4)}$$

$$\rho_0 = \tfrac{1}{2}(p+2q)$$

G	p	q	r_k	$i\,d_k$
$SO_0(2n+1,1)$, $n \geq 1$	$2n$	0	Void	Void
$SO_0(2n,1)$, $n \geq 1$	$2n-1$	0	$i(\rho_0+k)$, $k \geq 0$	$\frac{(-1)^n(2n+2k-1)}{2^{4n-3}}\binom{2n+k-2}{n-1}\binom{n+k-1}{n-1}$
$SU(n,1)$, $n \geq 2$	$2(n-1)$	1	$i(\rho_0+2k)$, $k \geq 0$	$\frac{(-1)^n(n+2k)}{2^{2n-2}}\binom{n+k-1}{n-1}\binom{n+k-1}{n-1}$
$Sp(n,1)$, $n \geq 2$	$4(n-1)$	3	$i(\rho_0+2k)$, $k \geq 0$	$\frac{2n+2k+1}{2^{4n}}\binom{2n+k}{2n-1}\binom{2n+k-1}{2n-1}$
$F_{4(-20)}$	8	7	$i(\rho_0+2k)$, $k \geq 0$	$\frac{2k+11}{2^{20}}\binom{k+10}{7}\binom{k+7}{7}$

In the cases SU, Sp, and F_4, if we write $p = 2m$ and $q = 2l - 1$, then the poles r_k are at $i(\rho_0 + 2k) = i(m + 2l + 2k - 1)$, $k \geq 0$, with residue d_k, where

$$i\,d_k = (-1)^{m+l}\frac{m+2l+2k-1}{2^{2m+4l-4}} \times \binom{m+2l+k-2}{m+l-1}\binom{m+l+k-1}{m+l-1}$$

TABLE II

As usual E_{ij} denotes a matrix with 1 in the ijth place and zeroes elsewhere.

(i) $G = SO_0(2n, 1)$. The Cartan-Killing form is

$$\langle x, y\rangle = (2n-1)\,\mathrm{Trace}\,(xy).$$

Here

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & x_{12} \\ {}^t x_{12} & 0 \end{pmatrix};\ x_{12} \text{ any real } 2n \times 1 \text{ matrix}\right\}.$$

The dimension of $\mathfrak{p}$ is $2n$. The basis $\{Y_1, \ldots, Y_{2n}\}$ is given by

$$Y_j = E_{j,\,2n+1} + E_{2n+1,\,j}, \quad 1 \leq j \leq 2n.$$

(ii) $G = SU(n, 1)$. The Cartan-Killing form is

$$\langle x, y\rangle = 2(n+1)\,\mathrm{Trace}\,xy.$$

Here

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix};\ Z \text{ any complex } n \times 1 \text{ matrix and } Z^* = {}^t\bar{Z}\right\}.$$

The dimension of $\mathfrak{p}$ is $2n$. The basis $\{Y_1, \ldots, Y_{2n}\}$ consists of the matrices S_j and T_j where

$$S_j = E_{j,n+1} + E_{n+1,j}, \quad 1 \leq j \leq n,$$
$$T_j = (E_{j,n+1} - E_{n+1,j})/i, \quad 1 \leq j \leq n.$$

(iii) $G = Sp(n, 1)$. The Cartan-Killing form is

$$\langle x, y\rangle = 2(n+2) \text{ Trace } (xy).$$

Here

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z_{12} & 0 & Z_{14} \\ Z_{12}^* & 0 & {}^tZ_{14} & 0 \\ 0 & Z_{14} & 0 & -Z_{12} \\ Z_{14}^* & 0 & {}^tZ_{12} & 0 \end{pmatrix} ; \ Z_{12}, Z_{14} \text{ any complex } n \times 1 \text{ matrices} \right\}$$

The dimension of $\mathfrak{p}$ is $4n$. The basis $\{Y_1, \ldots, Y_{4n}\}$ consists of the matrices S_j, T_j, U_j, V_j, where

$$\begin{aligned} S_j &= E_{j,n+1} + E_{n+1,j} - E_{n+1+j,2n+2} - E_{2n+2,n+1+j}, & 1 \le j \le n, \\ T_j &= (1/i)(E_{j,n+1} - E_{n+1,j} + E_{n+1+j,2n+2} - E_{2n+2,n+1+j}), & 1 \le j \le n, \\ U_j &= E_{j,2n+2} + E_{n+1,n+1+j} + E_{n+1+j,n+1} + E_{2n+2,j}, & 1 \le j \le n, \\ V_j &= (1/i)(E_{j,2n+2} + E_{n+1,n+1+j} - E_{n+1+j,n+1} - E_{2n+2,j}), & 1 \le j \le n. \end{aligned}$$

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