ON A CLASS OF DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE $pq + 1$

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In [3] Atkinson conjectured that a nonsolvable doubly transitive but not doubly primitive permutation group is either a normal extension of $S_z(q)$ or an automorphism group of a block design with $\lambda = 1$.

Let G be a doubly transitive but not doubly primitive permutation group of degree $pq + 1$, where q is a prime. In [2] Atkinson proved that if $p = 2, 3, 4$ then his conjecture [3] is true. More evidence supporting this conjecture appears in [13]. We will prove that the conjecture is true if p and $\frac{1}{2}(p - 1)$ are primes, $p < q$.

THEOREM. *Let G be a doubly transitive but not doubly primitive permutation group of degree $pq + 1$, where p, q, $r = \frac{1}{2}(p - 1)$ are primes and $p < q$. Then one of the following holds:*

(a) *$pq + 1 = 2^x$ for some integer x and G is sharply doubly transitive.*

(b) *$pq + 1 = 2^r$ and G is the Zassenhaus group of degree 2^r and order $2^r(2^r - 1)r$ which contains a regular normal subgroup.*

(c) *$q \equiv 1$ $(p + 1)$ and G is an automorphism group of a block design with $\lambda = 1$ and $k = p + 1$.*

Our notation for the parameters of a block design is standard; see [14]. We remark that groups in (a) and (b) are solvable and satisfy the assumptions of the theorem. Examples for (a), (b) are groups of degrees 2^{11}, 2^{23}, 2^{83}, 2^{131}, for which $2^r = 1 + pq$ and $2r + 1 = p$. (We thank Prof. P. T. Bateman for the examples.)

The incidence equations of a nontrivial block design and the Fisher's inequality implies that if $\lambda = 1$ and $v = pq + 1$ then $k = p + 1$ and $q \equiv 1$ (mod $p + 1$). Therefore all we have to prove in (c) is that G is an automorphism group of a nontrivial block design with $\lambda = 1$. Since sharply doubly transitive groups of degree $pq + 1$ are solvable all we have to prove in (a) is that G is sharply doubly transitive (see [3, 2.4]).

Notations. Let G be a doubly transitive but not doubly primitive permutation group on a set Ω. Let $\Delta_1, \Delta_2, \Delta_3, \dots,$ be a complete system of imprimitivity sets for the action of G_α on $\Omega - \{\alpha\}$, for $\alpha \in \Omega$. We call each Δ_i a G_α-block. Set $\Lambda_0 = \{\Delta_1, \Delta_2, \Delta_3, \dots\}$, and $\Lambda = \Lambda_0 - \{\Delta_1\}$. Let H be the stabilizer of Δ_1 in G_α and K the kernel of G_α in its action on Λ_0. Let A be the kernel of H on Δ_1. Let Fix (T) be the set of fixed points of the subgroup

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T. Let A_n and S_n be the alternating and the symmetric group of degree n, respectively.

Proof of the theorem. Let G be a counterexample to the theorem. Let Ω be the set on which G acts. Let $\alpha \in \Omega$ and $\beta \in \Delta_1$. Then $G_{\alpha\beta} \neq 1$ and G is not an automorphism group of a nontrivial block design on Ω with $\lambda = 1$ because of the remarks above. Let Q be a Sylow q-subgroup of G contained in G_α. By [1] we have $|Q| = q$. We now divide the proof into two parts according to the size of Λ_0. We will use [17, 11.6, 11.7] without referring to them. We note that $G_{\alpha\beta} = H_\beta$.

Case 1. $|\Lambda_0| = p$. Then $|\Delta_i| = q$ for $1 \leq i \leq p$. We use Atkinson's argument (beginning of case 3 of the proof of Theorem A of [2]), replacing 4 by p, to obtain that K is transitive and faithful on each Δ_i, $1 \leq i \leq p$, and that $A \cap K = 1$.

(1) Assume that K is solvable. Again, we use an argument of Atkinson (third paragraph of case 3 of the proof of Theorem A of [2]), replacing 4 by p, to get $K = Q$ and $K_\beta = 1$.

It follows that $G_{\alpha\beta}$ is a subgroup of S_{p-1}, the symmetric group on $p - 1$ points. Since $Q \lhd H$ and $Q \cap A = 1$ we conclude that H/A is a Frobenius group of degree q so that $G_{\alpha\beta}/A$ is cyclic of order dividing $q - 1$.

If G_α/Q is solvable, it is a Frobenius group on Λ_0 so that $G_{\alpha\beta}$ is semiregular on Λ. Thus $G_{\alpha\beta}$ is cyclic and $|G_{\alpha\beta}|$ divides $2r$. Since A fixes at least $q + 1$ points we have $A \neq G_{\alpha\beta}$ (by Lemma 1 of [2]). Hence if $A \neq 1$, A is a normal Sylow subgroup of $G_{\alpha\beta}$ contradicting Lemma 1 of [2]. It follows that $A = 1$ and $G_{\alpha\beta}$ is semiregular and faithful on both $\Delta_1 - \{\beta\}$ and Λ. Hence $G_{\alpha\beta\gamma} = 1$ for $\gamma \in \Omega - \{\alpha, \beta\}$. Hence G is a Zassenhaus group and since $p \neq q$ we use [5], [4], [16] to conclude that G contains a regular normal subgroup. Also $1 + pq = 2^a$ where $a = |G_{\alpha\beta}|$ and a is a prime. Since $1 + pq \neq 4$ we have $a = r$. This is a contradiction because G is a counterexample.

Therefore G_α/Q is nonsolvable. We claim that $|G_{\alpha\beta}: A| = 2$. We have G_α doubly transitive on Λ_0.

By [6] we get that either $p \leq 11$ and $G_\alpha/Q \simeq PSL(2, p)$ or G_α/Q is triply transitive on Λ_0. If $p = 11$ then $H/Q \simeq G_{\alpha\beta} \simeq A_5$ and since $A \neq G_{\alpha\beta}$ we have $A = 1$. This is impossible because $G_{\alpha\beta}/A$ is cyclic. If $p = 7$ then $G_{\alpha\beta} \simeq S_4$ and since $G_{\alpha\beta}/A$ is cyclic we conclude that $|G_{\alpha\beta}: A| = 2$. If $p = 5$ then $G_{\alpha\beta} \simeq A_4$ and A is a Sylow 2-subgroup of $G_{\alpha\beta}$. Since A fixes more than two points, Lemma 1 of [2] gives a contradiction.

If G_α/Q is triply transitive then $H/Q \simeq G_{\alpha\beta}$ is doubly transitive on Λ. It follows that $A \neq 1$ because otherwise $G_{\alpha\beta}$ would be regular on Λ. Since $A \lhd G_{\alpha\beta}$ we get that A is transitive on Λ (see [17, 9.9]). In particular r divides $|A|$. Let $g \in G$ such that $A^g \subseteq G_{\alpha\beta}$. If A^g fixes no point of $\Omega - \Delta_1 - \{\alpha\}$ then $\text{Fix}\,(A^g) = \Delta_1 \cup \{\alpha\}$ because $|\text{Fix}\,(A^g)| = |\text{Fix}\,(A)|$. Thus $A^g = A$ and A is weakly closed in $G_{\alpha\beta}$, contradicting Lemma 1 of [2]. Thus, there is a

$$\theta \in (\Omega - \Delta_1 - \{\alpha\}) \cap \text{Fix}\,(A^g).$$

Let i be such that $\theta \in \Delta_i$ and let $H_i = \{h \in H \mid \Delta_i h = \Delta_i\}$. It follows that A^g fixes Δ_i. Hence $A^g \subseteq H_i$ so $r \mid |H_i|$. Since H is transitive on Λ we have $|H: H_i| = 2r$ so that $r^2 \mid |G_\alpha: Q|$. Now [1] implies that G_α/Q contains A_p so that $G_{\alpha\beta}$ contains A_{p-1}. If $p = 5$ then either $G_{\alpha\beta}$ is A_4 or S_4 and we finish as above. If $p > 5$ then $|G_{\alpha\beta}: A| \leq 2$ and since $A \neq G_{\alpha\beta}$ we conclude that the index is 2.

Since $G_{\alpha\beta}$ does not fix a third point [2, Lemma 1] we conclude that the $G_{\alpha\beta}$-orbits on $\Delta_1 - \{\beta\}$ are of size 2. Let $\{\alpha_i, \beta_i\}$, $1 \leq i \leq \frac{1}{2}(q-1)$, be these orbits. Since Fix $(G_{\alpha\beta}) = \{\alpha, \beta\}$ we have that $N_G(G_{\alpha\beta}) = G_{\{\alpha,\beta\}}$. But $G_{\alpha\beta} \subset G_{\{\alpha_i,\beta_i\}}$ for all i so that $|G_{\{\alpha_i,\beta_i\}}: G_{\alpha\beta}| = 2$ and so $G_{\{\alpha_i,\beta_i\}} = N_G(G_{\alpha\beta}) = G_{\{\alpha,\beta\}}$ for all i. This implies that $\Delta_1 - \{\beta\}$ is $G_{\{\alpha,\beta\}}$-invariant contradicting Lemma 2 of [2]. This contradiction implies that K is nonsolvable.

(2) We can assume that K is nonsolvable. Since $A \cap K = 1$ we get

$$K \simeq AK/A \subseteq H/A$$

so that H/A is nonsolvable and consequently H is doubly transitive on Δ_1. Thus $\Gamma_1 = \Delta_1 - \{\beta\}$ is a $G_{\alpha\beta}$-orbit. Now Γ_1 is not $G_{\{\alpha,\beta\}}$-invariant by Lemma 2 of [2] so that there is another $G_{\alpha\beta}$-orbit, Γ_2, of size $q-1$ which is not $G_{\{\alpha,\beta\}}$-invariant. Lemma 3 of [2] yields yet another $G_{\alpha\beta}$-orbit Γ_3, of size $q-1$ which is $G_{\{\alpha,\beta\}}$-invariant. It follows that $\Gamma_1 \neq \Gamma_2$, $\Gamma_1 \neq \Gamma_3$, $\Gamma_2 \neq \Gamma_3$. By an argument appearing in the second paragraph of case 3 of [2], we get that Γ_3 and Γ_2 cannot intersect the same Δ_i and that none of them is contained in a G_α-block. Also K_β has at most two orbits on any Δ_i such that $\Gamma_j \cap \Delta_i \neq \emptyset$, $j = 2, 3$.

Let t be the number of Δ_i's such that $\Delta_i \cap \Gamma_2 \neq \emptyset$. Then $t \neq 1$, $t \neq p-1$. The set of these t Δ_i's is a $G_{\alpha\beta}$-orbit on Λ and since $H = KG_{\alpha\beta}$, the set is an H-orbit on Λ. As $t \neq p-1$, H/K is not transitive on Λ and therefore G_α/K is a solvable Frobenius group on Λ_0. Thus $H/K \simeq G_{\alpha\beta}/K_\beta$ is semiregular on Λ, its order is t and $t \mid 2r$. It follows that $t = 2, r$.

Let $\mathscr{A}_i = \{\Delta_j \mid \Delta_j \cap \Gamma_i \neq \emptyset\}$ for $i = 2, 3$. Then $\mathscr{A}_i$ is a $G_{\alpha\beta}$-orbit on Λ of size t. Let $\Delta \in \mathscr{A}_i$ for some i. Since both $\Delta \cap \Gamma_i$ and $\Delta - \Gamma_i$ are K_β-invariant, they have to be the two K_β-orbits on Δ (K_β cannot be transitive on Δ as $|K_\beta|_q = 1$). Since $K_\beta \lhd G_{\alpha\beta}$, Γ_i is a union of K_β-orbits of equal sizes. From $\Gamma_i = \bigcup_{\Delta \in \mathscr{A}_i} (\Delta \cap \Gamma_i)$ we conclude that $|\Delta \cap \Gamma_i| = (q-1)/t$. Thus K_β has two orbits of sizes m and $q - m$ on $\Delta \in \mathscr{A}_i$, where $m = (q-1)/t$.

By [11, B1], we get $|\text{Fix}\,(K_\beta)| = 2$ so that $N_{G_\alpha}(K_\beta) = G_{\alpha\beta}$ ($K_\beta \neq 1$ as K is doubly transitive on each Δ_i). Since G_α/K is solvable, $|G_\alpha: K| = tp$ and we can choose $h \in G_\alpha - K$ such that $|hK| = p$. Let $C_1 = \langle h \rangle$ and let $C = \{h^i \mid 1 \leq i \leq p\}$. Using the bar notation in $\bar{G}_\alpha = G_\alpha/K$ we have that $\bar{C} = \bar{C}_1$. Also $\bar{C}$ is a regular normal subgroup of $\bar{G}_\alpha$ and therefore $\bar{C} \cap \bar{H} = 1$. Since $N_{G_\alpha}(K_\beta) \subseteq H$ we get $C \cap N_{G_\alpha}(K_\beta) = 1$. It follows that the set $I = \{(K_\beta)^a \mid a \in C\}$ contains exactly p different subgroups of index q in K. Assume that $a, b \in C$ and $(K_\beta)^a$ and $(K_\beta)^b$ are conjugate in K. Then $K_{\beta a h_1} = K_{\beta b}$ for some $h_1 \in K$ and since conjugates of K_β in G_α have exactly one fixed point, each, in $\Omega - \{\alpha\}$

we conclude that $\beta a h_1 = \beta b$. Since $h_1 \in K$, $\{\beta a, \beta b\} \subseteq \Delta_i$ for some $1 \leq i \leq p$ and therefore $\Delta_1 a = \Delta_1 b$. Since C is regular on Λ_0, this implies that $a = b$.

Hence no element of I is conjugate in K to another element of I. A lemma of Ito [7, Lemma 1] implies that for each pair $\langle E, F\rangle$, $\{E, F\} \subseteq I$, there is a symmetric block design on the cosets of E in K. In this design $k = |E: E \cap F|$ and $v = q$.

Let a_1, a_2 belong to C, $a_1 \neq a_2$, such that

$$\beta a_1 \in \Delta_{i_1}, \quad \beta a_2 \in \Lambda_{i_2} \quad \text{and} \quad \{\Delta_{i_1}, \Delta_{i_2}\} \subseteq \mathscr{A}_3 \cup \mathscr{A}_2.$$

This is possible because C is transitive and regular on Λ_0 and $|\mathscr{A}_3 \cup \mathscr{A}_2| = 2t \geq 4$. Let k_i, λ_i be the parameters of the mentioned design for $\langle K_\beta, (K_\beta)^{a_i}\rangle$, for $i = 1, 2$ and let k_3, λ_3 be the parameters for $\langle (K_\beta)^{a_1}, (K_\beta)^{a_2}\rangle$. Since K_β has two orbits of sizes m and $q - m$ on Δ_{i_j}, $j = 1, 2$, we get

$$k_i = |K_\beta\colon K_{\beta, \beta a_i}| = |(\beta a_i)^{K_\beta}| = \text{either } m \text{ or } q - m.$$

The equations of a symmetric design with $v = q$ implies that

$$\lambda_i = m(m-1)/(q-1) \text{ or } (q-m)(q-m-1)/(q-1), \text{ respectively.}$$

It follows that $k_i - \lambda_i = m(q - m)/(q - 1)$ for $i = 1, 2$. In particular $k_1 - \lambda_1 = k_2 - \lambda_2$. Another lemma of Ito [7, equation (12)] implies that for some natural number a,

$$(k_1 - \lambda_1)(k_2 - \lambda_2)(k_3 - \lambda_3) = a^2.$$

This implies that $k_3 - \lambda_3$ is a square, contradicting Lemma 5 of [7]. This completes the proof of Case I.

Case II. $|\Lambda_0| = q$. Then $|\Delta_i| = p$ for $1 \leq i \leq q$. We break the proof into two parts.

(1) Assume that G_α/K is not solvable. Then H is transitive on Λ. If all $G_{\alpha\beta}$-orbits on Λ are of size more than $p - 1$, then all $G_{\alpha\beta}$-orbits on $\Omega - \{\alpha\} - \Delta_1$ are of size bigger than $p - 1$. Thus all the orbits of $G_{\alpha\beta}$ of size at most $p - 1$ are in $\Delta_1 \cup \{\alpha\}$. Since Fix $(G_{\alpha\beta}) = \{\alpha, \beta\}$ by Lemma 1 of [2], we conclude that $\Delta_1 - \{\beta\}$ is $G_{\{\alpha,\beta\}}$-invariant. This is impossible because of Lemma 2 of [2]. Hence there is at least one $G_{\alpha\beta}$-orbit on Λ of size less or equal to $p - 1$. In particular $G_{\alpha\beta}$ is not transitive on Λ. Since H is transitive on Δ_1, $|H\colon G_{\alpha\beta}| = p$. If $K \neq 1$ we get $H = G_{\alpha\beta}K$ and since H is transitive on Λ, so is $G_{\alpha\beta}$. Since this is impossible we get $K = 1$.

Let P be a Sylow-p-subgroup of G contained in H. It follows that $H = PG_{\alpha\beta}$. If P fixes some $\Delta_i \in \Lambda$ then $(\Delta_i)^H = (\Delta_i)^{G_{\alpha\beta}} \subset_{\neq} \Lambda$, contradicting the transitivity of H on Λ. Thus P fixes no point of Λ and consequently $p \mid q - 1$. Since $|H\colon G_{\alpha\beta}| = p$, all $G_{\alpha\beta}$-orbits on Λ are of size at least $(q - 1)/p$. By the remark at the beginning of this case we have that $(q - 1)/p \leq p - 1$. Let $q - 1 = pm$, then $m \leq 2r$.

If $G_\alpha \simeq A_q$ or S_q in its action on Λ_0 then it is 3-transitive on Λ_0 so that H is doubly transitive on Λ. Since $A \subseteq G_{\alpha\beta}$, A is not transitive on Λ so that

$A = 1$ (see 9.9, [17]). But $\frac{1}{2}(q-1)!$ divides $|H|$ and $|H|$ divides $p!$. This is impossible. Thus $G_\alpha \not\simeq A_q$ or S_q on Λ_0.

Suppose $(m, r) \neq 1$. Since m is even and $1 < m \leq 2r$ we have that either $m = r = 2$ or $m = 2r$. If $m = r = 2$ then $p = 5$ and $q = 11$. Since G_α is a nonsolvable transitive permutation group of degree 11 and $G \not\simeq A_{11}$ or S_{11}, we get that either $G_\alpha \simeq PSL(2, 11)$ or M_{11} (see [9]). If $G_\alpha \simeq PSL(2, 11)$ then $H \simeq A_5$ so that $A = 1$ and $G_{\alpha\beta} \simeq A_4$. Now [18] gives a contradiction. If $G_\alpha \simeq M_{11}$ then H is triply transitive on Λ. As $A \lhd H$, A is either 1 or transitive on Λ. Since $A \subseteq G_{\alpha\beta}$ and $G_{\alpha\beta}$ is not transitive on Λ we have $A = 1$. Hence H is a transitive permutation group of degree 5 and order $8 \cdot 9 \cdot 10$. This is impossible. We conclude that $m = 2r$ and $q - 1 = p(p-1)$. Since $p \neq 3$, $r = 3$, or $r \equiv 2\ (3)$. But $r \equiv 2\ (3)$ implies $q \equiv 0\ (3)$ which is impossible. Thus $r = 3$, $p = 7$, and $q = 43$. By [9, Section 5], $G_\alpha \simeq A_q$ or S_q on Λ_0 which is impossible. Hence $(m, r) = 1$.

Let R be a Sylow r-subgroup of H. If H/A is nonsolvable, R fixes one point on Δ_1 and has two orbits of size r on the rest of the points. This is also the case when H/A is solvable unless $|H: A| = p$ or $2p$. In this case $|G_{\alpha\beta}: A| \leq 2$. This is impossible because Lemma 1 of [2] implies that $|G_{\alpha\beta}: A| = 2$ and we can get a contradiction as in the end of (1) of Case I. We conclude that every Sylow r-subgroup of H has three orbits on Δ_1, their sizes are: $1, r, r$. Since $|G_{\alpha\beta}: A| \leq 2$ is impossible we also get $1 \neq |G_{\alpha\beta}|_r = |H|_r$ so that $G_{\alpha\beta}$ is either transitive or has two orbits of size r on $\Delta_1 - \{\beta\}$. Let $g \in G_{\{\alpha,\beta\}} - G_{\alpha\beta}$. Since $\Delta_1 - \{\beta\}$ is not $G_{\{\alpha,\beta\}}$-invariant (by Lemma 2 of [2]), there is a $G_{\alpha\beta}$-orbit, Γ_0, on $\Delta_1 - \{\beta\}$ such that $\Gamma_0 g \not\subseteq \Delta_1$. Let $\Gamma = \Gamma_0 g$, then $|\Gamma| = r$ or $2r$.

Let $H_i = \{h \in H \mid \Delta_i h = \Delta_i\}$, $2 \leq i \leq q - 1$. Since $|H: H_i| = q - 1 = pm$ and $(m, r) = 1$ we get $|H_i|_r = |H|_r$ for $2 \leq i \leq q - 1$. If no H_i fixes a point of Δ_1 then each H_i must have two orbits of sizes r and $r + 1$ on Δ_1. Thus, for $i > 1$,

$$|\Delta_i^{G_{\alpha\beta}}| = |G_{\alpha\beta}: G_{\alpha\beta} \cap H_i| = |G_{\alpha\beta}: (H_i)_\beta| = \left|\frac{H_i: (H_i)_\beta}{H: G_{\alpha\beta}}\right| \cdot |H: H_i| = \theta m,$$

where $\theta = r$ or $r + 1$. If $m > 2$ then all $G_{\alpha\beta}$-orbits on Λ are of size more than $2r$ which is impossible. Hence $m = 2$. By a theorem of Neumann [10], $G_\alpha \simeq A_q$ or S_q on Λ unless $q = 23, 11$. Since $G_\alpha \not\simeq A_q$ or S_q we conclude that $q = 23, 11$ and by [9, Section 5], we have $G_\alpha \simeq M_{23}$ on Λ_0 or $G_\alpha \simeq PSL(2, 11)$. If $G_\alpha \simeq M_{23}$ then H is 3-transitive on Λ so that A is either 1 or transitive on Λ. Since $A \subseteq G_{\alpha\beta}$ we have $A = 1$. But $q = 23$ implies $p = 11$ and since $|H| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, [9] gives a contradiction. If $q = 11$, $p = 5$, and $H \simeq A_5$. Thus $A = 1$ and $G_{\alpha\beta} \simeq A_4$. A contradiction follows from [18].

Therefore at least one H_i fixes a point of Δ_1. Since all H_i's are conjugate in H, each H_i has a fixed point on Δ_1 and is either transitive or has two orbits of size r on the rest of the points in Δ_1. Thus for $i \geq 2$,

$$|\Delta_i^{G_{\alpha\beta}}| = |G_{\alpha\beta}: (H_i)_\beta| = \theta m,$$

where $\theta = 1, r, 2r$. As above $m > 2$ so that all $G_{\alpha\beta}$-orbits on Λ which have size at most $p - 1$, are of size $m < 2r$. Let $\mathscr{A} = \{\Delta \in \Lambda \mid \Delta \cap \Gamma \neq \emptyset\}$. Then $\mathscr{A}$ is a $G_{\alpha\beta}$-orbit on Λ. But $\mathscr{A}$ is also a complete system of imprimitivity sets for the action of $G_{\alpha\beta}$ on Γ. Thus $|\mathscr{A}|$ divides Γ. Hence $|\mathscr{A}| = 1, 2, r, 2r$. In particular $|\mathscr{A}| \leq p - 1$ so that $|\mathscr{A}| = m$. But $m \neq 1, 2, r, 2r$. This is a contradiction.

(2) We assume now that G_α/K is solvable. It follows that $G_{\alpha\beta}/K_\beta$ is a semiregular group on Λ, as H/K is a Frobenius complement on it. Since H is transitive on Δ_1 and $K \lhd H$, K is 1/2-transitive on Δ_1. If K fixes Δ_1 pointwise it fixes all Ω so that $K = 1$. If K is transitive on Δ_1, it is transitive on each Δ_i. Hence, either $K = 1$ or K is transitive on each Δ_i. As in (1), $|G_{\alpha\beta}\colon A| > 2$ so that $|G_{\alpha\beta}\colon A|_r \neq 1$ and we can define Γ and $\mathscr{A}$ as in (1). Since $G_{\alpha\beta}/K_\beta$ is semiregular and $\mathscr{A}$ is a $G_{\alpha\beta}$-orbit on Λ we get that $|\mathscr{A}| = |G_{\alpha\beta}\colon K_\beta|$. Let $t = |\mathscr{A}|$, then $|G_\alpha\colon K| = tq$ and $t \mid 2r$.

If $K_\beta = 1$ then $G_{\alpha\beta}$ is semiregular and faithful on Λ and $|G_{\alpha\beta}| = 1, 2, r, 2r$. It follows that $G_{\alpha\beta\gamma} = 1$ for $\gamma \in \Omega - \Delta_1 - \{\alpha\}$. Since $A \lhd G_{\alpha\beta}$ and $A \neq G_{\alpha\beta}$ (by Lemma 1 of [2]) we have that either $A = 1$ or A is a Sylow subgroup of $G_{\alpha\beta}$. By Lemma 1 of [2], $A = 1$. Then $|H| = tp$ so that H is solvable. Thus $G_{\alpha\beta}$ is also semiregular on Δ_1. Hence $G_{\alpha\beta\gamma} = 1$ for $\gamma \in \Omega - \{\alpha, \beta\}$ and G is a Zassenhaus group. By the characterization of these groups we have that $1 + pq = 2^t$ and t is a prime. Since $1 + pq \neq 4$ we get $t = r$. Also G contains a regular normal subgroup. This contradicts the fact that G is a counterexample.

Hence $K_\beta \neq 1$. By [11, B1], we get Fix $(K_\beta) = \{\alpha, \beta\}$. Also K is transitive on each Δ_i. If K acts nonfaithfully on the Δ_i's then: $PSL(n, s) \leq G \leq P\Gamma L(n, s)$ (see either [13A(a)] or [12, Proposition 4] and [11]). Then G is an automorphism group of a block design with $\lambda = 1$ (see [11]). Hence K is faithful on each Δ_i. If K is solvable, there is exactly one class of subgroups of index p in K so that K_β fixes one point in each Δ_i contradicting $|\text{Fix}\ (K_\beta)| = 2$.

Therefore K is nonsolvable. Since $|\text{Fix}\ (K_\gamma)| = 2$ for $\gamma \in \Delta_i$, $i > 1$, K_γ fixes no point of Δ_1 so that K_β is not conjugate to K_γ in K. By a paper of Ito [8], K is not triply transitive on Δ_1 and by another paper of Ito [6], $p \leq 11$ and $K \simeq PSL(2, p)$. Let $g \in G_\alpha$ such that $Q = \langle g \rangle$. Then $g \notin K$, $|g| = q > p$, and g normalizes K. Since $PSL(2, p)$ does not admit an automorphism of order $q > p$ if $p = 5, 7, 11$ we conclude that g and therefore Q centralizes K. Now $G_\alpha = HQ = G_{\alpha\beta}KQ = G_{\alpha\beta}QK$. Let $h \in G_\alpha$ be such that $\beta h = \gamma \in \Delta_i$, $i > 1$. Then $h = h_1h_2h_3$ where $h_1 \in G_{\alpha\beta}$, $h_2 \in Q$, and $h_3 \in K$. Then $(K_\beta)^h = K_\gamma$. But $K_\beta \lhd G_{\alpha\beta}$ and Q centralizes K_β. Thus $(K_\beta)^{h_3} = K_\gamma$. This is a contradiction because K_β and K_γ are not conjugate in K, as $|\text{Fix}\ (K_\gamma)| = 2$. This contradiction completes the proof of the theorem.

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