THE TOTAL CHERN AND STIEFEL-WHITNEY CLASSES ARE NOT INFINITE LOOP MAPS

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1. Introduction

Let Λ be a discrete commutative ring with identity. If X is a space then

$$F(X;\Lambda) = \left\{x \in 1 + \prod_{q\geq 1} H^q(X;\Lambda)\right\}$$

has a group structure given by restricting the cohomology cup-product in $H^*(X;\Lambda)$. Also

$$F_e(X;\Lambda) = \left\{x \in 1 + \prod_{q\geq 1} H^{2q}(X;\Lambda)\right\}$$

can be made into a group by the same device. $F(X;\Lambda)$ and $F_e(X;\Lambda)$ are representable as groups by H-space structures on

$$(1.1)\qquad K(\Lambda) = \prod_{q\geq 1} K(\Lambda;q) \quad\text{and}\quad K_e(\Lambda) = \prod_{q\geq 1} K(\Lambda;2q)$$

respectively.

G. B. Segal has proved the following:

1.2. THEOREM [8]. *There are connected cohomology theories* $F^*(X;\Lambda)$ *and* $F_e^*(X;\Lambda)$ *such that* $F^0(X;\Lambda) = F(X;\Lambda)$ *and* $F_e^0(X;\Lambda) = F_e(X;\Lambda)$.

The total Stiefel-Whitney and Chern classes, w and c, respectively, are well-known natural homomorphisms. Details are to be found in [4, p. 229].

$$(1.3)\qquad w: \tilde{K}O(X) \to F(X;Z/2), \qquad c: \tilde{K}U(X) \to F_e(X;Z).$$

Both $\tilde{K}O(X)$ and $\tilde{K}U(X)$ extend to well-known connected cohomology theories $bo^*(X)$ and $bu^*(X)$, respectively. Details of connected K-theory may be found in [1].

In [8, Section 4] Segal asks: Does either w or c extend to a stable natural transformation between cohomology theories?

$$(1.4)\qquad w: bo^*(X) \to F^*(X;Z/2), \qquad c: bu^*(X) \to F_e^*(X;Z).$$

The representing spaces for $bo^0(\square)$, $bu^0(\square)$, $F(\square;Z/2)$, and $F_e(\square;Z)$ are infinite loopspaces. Segal's question may be equivalently rephrased: Does either w or c extend to a map of infinite loopspaces?

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This note proves:

1.5. THEOREM. *In* (1.3) *neither* w *nor* c *extends as in* (1.4) *to a stable natural transformation from connective K-theory to the cohomology theories constructed by Segal in* [8].

Sketch of proof. A connective cohomology theory admits a transfer homomorphism for finite covering maps. Details will be given as they are needed. A stable operation must commute with these transfers. We show that w and c fail this test for the covering $BZ/2 \to BZ/4$.

In Section 2 the ingredients of Segal's proof are recalled. In Section 3 these ingredients are related to the transfer operation in Segal's cohomology theories. There we obtain the fundamental dimension restrictions on this exotic transfer. In Section 4 we prove Theorem 1.5.

2. The infinite loopspaces $K(\Lambda)$ and $K_e(\Lambda)$

Let Λ be a discrete commutative ring. I will now briefly describe the manner in which Segal constructs infinite loopspace structures on the H-spaces $K(\Lambda)$ or $K_e(\Lambda)$ (where the multiplication is induced by cup-product) which give the connected cohomology theories of Theorem 1.2.

I will describe the proof in terms of the operads of J. P. May [6]. It is remarked in [8, Section 3] that this approach is equivalent to Segal's. However the operad approach makes the transfer more easily accessible for our purposes. I should remark that I know of no proof in the literature which shows that, when supplied the same data, the methods of [6] and [9] give rise to equivalent infinite loopspaces. However it is easy to verify that they give rise to the same transfer operations, which is all we shall use.

Let K be a graded topological group. Set $\mathscr{C}(0)$ = (point) and if $j \geq 1$ set $\mathscr{C}(j)$ equal to the topological space of continuous, graded multilinear maps from K^j to K. Then $\{\mathscr{C}(j)\}_{j\geq 0}$ is an operad. Further details may be found in [6, p. 1]. The structure maps

$$\gamma\colon \mathscr{C}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \to \mathscr{C}(j) \qquad \left(j = \sum_{s=1}^{k} j_s\right)$$

are induced by composition of maps in the following manner.

$$K^{j_1} \times \cdots \times K^{j_k} \to K \times \cdots \times K = K^k \to K.$$

Permutation of the factors of K^j makes $\mathscr{C}(j)$ into a right Σ_j-space.

The operad $\{\mathscr{C}(j)\}_{j\geq 1}$ acts naturally on K by means of the Σ_j-equivariant evaluation maps $\mathscr{C}(j) \times K^j \to K$. Operad actions are defined in [6, p. 3].

If $K = K(\Lambda)$ or $K = K_e(\Lambda)$ let $M(j) \subset \mathscr{C}(j)$ be those maps which induce the iterated cup-product on homotopy groups. Segal shows that $M(j)$ is contractible. In operad parlance $\{M(j)\}_{j\geq 0}$ is a locally contractible operad. However if K is an H-space upon which a locally contractible operad acts then K is an infinite loopspace [6, Theorem 1.3 and Proposition 3.10(ii)]. In terms of categories this result is proved in [9, Section 2]. This proves Theorem 1.2. Observe that $K_e(\Lambda)$ is a subinfinite-loopspace of $K(\Lambda)$.

We will need the following observation:

2.1. LEMMA. *In the situation described above the operad action maps send*

$$M(j) \rtimes \prod_{i=1}^{j} \left(\prod_{1 \leq q \leq a_i} K(\Lambda; q) \right)$$

into $\prod_{1 \leq q \leq a} K(\Lambda; q)$ *where* $a = \sum_{i=1}^{j} a_i$.

Proof. $M(j)$ consists of graded multilinear maps. Such maps preserve total degree.

This simple observation restricts the dimension of the transfer of a Stiefel-Whitney or Chern class (see Lemma 3.2) in Segal's cohomology theories. This fact is used in Section 4 to show that w and c do not commute with transfers.

3. Transfers

Now let us recall the transfer operations associated with Segal's infinite loopspaces $K(\Lambda)$. Further details concerning the transfer may be found in [5].

Let $Y \to Y/(Z/n) = X$ be an n-fold cyclic covering. There is a map $\phi: X \to M(n) \rtimes_{\Sigma_n} Y^n$ [5, Section 3]. Let $f: Y \to K$ be a homotopy class then the transfer, $tr_K(f)$, is represented by the composition

$$X \xrightarrow{\phi} M(n) \rtimes_{\Sigma n} Y^n \xrightarrow{1 \rtimes_{\Sigma n} f^n} M(n) \rtimes_{\Sigma n} K^n \xrightarrow{d_n} K. \tag{3.1}$$

Here d_n is the Dyer-Lashof map which is induced by one of the structure maps of the action of $\{M(j)\}_{j \geq 0}$ on K.

In the above situation we have the following result:

3.2. LEMMA. *Let* $f \in \prod_{1 \leq q \leq a} H^q(Y; \Lambda)$. *Then*

$$tr_K(f) \in \prod_{1 \leq q \leq an} H^q(X; \Lambda).$$

Proof. In (3.1) the map $1 \times_{\Sigma_n} f^n$ maps into $M(n) \times_{\Sigma_n} (\prod_{1 \leq q \leq a} K(\Lambda; q))^n$ which, by Lemma 2.1, maps into $\prod_{1 \leq q \leq an} K(\Lambda; q)$.

4. Proof of Theorem 1.5

Suppose that $w: BO \to \prod_{q \geq 1} K(Z/2; q) = K$ is an infinite loop map. From [3; 6] we know that the following diagram is Σ_n-homotopy commutative

$$\begin{array}{ccc} E\Sigma_n \times BO^n & \xrightarrow{\bar{d}_n} & BO \\ \downarrow{\scriptstyle \mu \times w^n} & & \downarrow{\scriptstyle w} \\ M(n) \times K^n & \xrightarrow[d_n]{} & K \end{array} \tag{4.1}$$

In (4.1), μ is an equivariant map and d_n, $\bar{d}_n$ are the appropriate Dyer-Lashof maps, i.e., the operad action maps.

Let $Y \to Y/(Z/n) = X$ be an n-fold cyclic covering. Then (4.1) immediately implies

$$w(tr_{bo}(f)) = tr_K(w(f)) \in \prod_{q \geq 0} H^q(X; Z/2). \tag{4.2}$$

Here tr_{bo} is the K-theory transfer which is defined by the procedure of Section 3 with BO and $\bar{d}_n$ replacing K and d_n. In [7, Chapter VIII, Section 1] it is shown that this transfer agrees with that defined by means of the direct image construction [2].

For the covering $BZ/2 \to BZ/4$, (4.2) does not hold. Consider $\xi - 1 \in KO(BZ/2)$ where ξ is the Hopf line bundle. Then

$$H^*(BZ/2; Z/2) = P(w_1(\xi)) \quad \text{and} \quad w(\xi - 1) = 1 + w_1(\xi).$$

Therefore, by Lemma 3.2,

$$tr_K(w(\xi - 1)) \in \prod_{1 \leq q \leq 2} H^q(BZ/4; Z/2). \tag{4.3}$$

Let ν be the 2-plane bundle over $BZ/4$ associated to the $Z/4$-representation in which the generator rotates the plane through $\pi/2$. Let ξ' be the pullback of ξ via the nontrivial map $BZ/4 \to BZ/2$. Then

$$H^*(BZ/4; Z/2) = E(w_1(\xi')) \otimes P(w_2(\nu)).$$

From the transfer in representation rings we easily see that

$$tr_{bo}(\xi - 1) = (\nu - 2) - (\xi' - 1)$$

whence

$$w(tr_{bo}(\xi - 1)) = \frac{1 + w_2(\nu)}{1 + w_1(\xi')} = 1 + w_1(\xi') + w_2(\nu) + w_1(\xi')w_2(\nu).$$

This does not lie in $\prod_{1 \leq q \leq 2} H^q(BZ/4; Z/2)$ which contradicts (4.2) and (4.3).

A similar calculation shows that

$$tr_K(c(\theta - 1)) \neq c(tr_{bu}(\theta - 1)) \in \prod_{q \geq 1} H^{2q}(BZ/4; Z)$$

where θ is the complexification of $\xi \in KO(BZ/2)$.

4.4. *Remark.* One can make sense of the total Stiefel-Whitney and Chern class for other coefficient rings. For example one might reduce modulo n or localise the Chern class. In these cases, too, these total classes will fail to commute with transfer. The expressions $tr_K(w(F - n))$ and $tr_K(c(F - n))$ will always have low dimensions comparable with the dimension of the vector bundle, F. However,

$$w(tr_{bo}(F - n)) = \frac{w(tr_{bo}(F))}{w(tr_{bo}(n))} \quad \text{and} \quad c(tr_{bu}(F - n)) = \frac{c(tr_{bu}(F))}{c(tr_{bu}(n))}$$

can be arranged to have high dimensional nonzero components.

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