ILLINOIS JOURNAL OF MATHEMATICS
Volume 21, Number 4, December 1977

RECURSION IN THE EXTENDED SUPERJUMP

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The history of the investigation into recursion in the superjump is a long and complicated one, with many people contributing pieces of information. The final word on the type three object defined first by R. O. Gandy [3],

$$S(F, \alpha. e) \simeq \begin{cases} 0 & \text{if } \{e\}(\alpha, F) \text{ converges} \\ 1 & \text{otherwise,} \end{cases}$$

was had by Leo Harrington [4], [5] after Peter Aczel and Peter Hinman obtained partial results. The results obtained were:

(A) The first ordinal not recursive in the superjump, ω_1^S, equals ρ_0, the first recursively Mahlo ordinal.

(B) 1-sc $S = L\rho_0 \cap 2^\omega$ where $L\rho_0$ is the collection of sets constructible before ρ_0.

The basic interest in the superjump stems from the fact that, unlike the *normal* type three objects, which involve ineluctibly uncountable computations, the superjump applied to a type two object can be viewed as a countable computation. This can be seen more clearly by replacing the superjump by the equivalent

$$\mathscr{E}(F) \simeq \begin{cases} 0 & \text{if } \exists\alpha\varepsilon \text{ 1-sc } F[F(\alpha) = 0] \\ 1 & \text{otherwise.} \end{cases}$$

Then, of course, we see that the value of $\mathscr{E}$ applied to F only depends on 1-sc F. The fact that S (and $\mathscr{E}$) are strictly weaker than ${}^3E$ makes it impossible to apply the techniques of Shoenfield and Sacks without alteration, and it is the reason that the analysis of recursion based S has been so difficult.

After result (B) above, the situation remained unsatisfactory, because of the fact that 1-env $S = \Pi_2^1$. As has been noted by Harrington, and others, this fact arises because some computations from the superjump may diverge for "the wrong reasons." For example, if a λ-term which defines a partial type two object $\dot{F}$ arises and is taken as an argument for S, the computation will diverge because $\dot{F}$ is not total, even though $\dot{F}$ may be defined at all "relevant" objects, for example, on 1-sc $\dot{F}$.

Received March 15, 1976.

[1] This paper is taken from the author's doctoral dissertation, Cornell University, August 1974.

Harrington, in [5] has eliminated this pathology by introducing the notion of "reindexing," a procedure by which the 1-env S is "shrunk" by casting out all computation which diverge for "avoidable" reasons.

The result of these manipulations of the notion of 1-envelope is to obtain a collection which will serve as an appropriate substitute for that notion, and which is a Spector class Γ, whose self dual class Δ is the 1-section of S.

We will show in the following that the same end can be accomplished by replacing the object $\mathscr{E}$ by the extended object $\mathscr{E}^*$, which we claim is the correct generalization to type three of the ordinary jump.

$$\mathscr{E}^*(\dot{F}) \simeq \begin{cases} 0 & \text{if } \dot{F} \text{ is defined on its 1-section and } \exists\alpha\varepsilon \text{ 1-sc } \dot{F}[\dot{F}(\alpha) \simeq 0] \\ 1 & \text{if } \dot{F} \text{ is defined on its 1-section and } \forall\alpha\varepsilon \text{ 1-sc } \dot{F}[\dot{F}(\alpha) > 0] \\ & \text{undefined otherwise.} \end{cases}$$

Section 1

DEFINITION. For a possibly partial type two object $\dot{F}$, $\dot{F}$ is *acceptable* if $\dot{F}$ is defined on its 1-section.

Then, if recursion in $\mathscr{E}^*$ is defined à la Kleene, with the alteration of clause S8 to a clause which represents the application of $\mathscr{E}^*$; we will obtain:

(C) 1-env $\mathscr{E}^*$ is a Spector class Γ and Δ = 1-sc $\mathscr{E}^*$ = 1-sc $\mathscr{E}$ = 1-sc S = $L\rho_0 \cap 2^\omega$.

(D) 1-env $\mathscr{E}^* = \Sigma_1(L\rho_0) \cap 2^\omega$.

We emphasize that the proof of (C) and (D) above will consist in large part of a compendium of results from Aczel, Hinman, and Harrington, and that our own addition is basically a trick which elucidates the starred situation.

We will begin by recapitulating Harrington's definition of the hierarchy for S. A set of notations η, a set H_u for each $u \in \eta$, and a map $|\ \ |: \eta \to ON$ are defined.

(a) $1 \in \eta$; $|1| = 0$; $H_1 = \omega$.

(b) If $x \in \eta$ then $2^x \in \eta$; $|2^x| = |x| + 1$ and $H_2x = \{e \mid \{e\}^{H_x} \text{ is total}\}$.

(c) If $n \in \eta$ and for all $i \in \omega$, $\{e\}^{H_n}(i) \in \eta$, then $3^n \cdot 5^e \in \eta$; $|3^n \cdot 5^e|$ is the first limit ordinal greater than $|n|$ and $|\{e\}^{H_n}(i)|$ for all i, and

$$H_{3^n \cdot 5^e} = \{\langle m, 0\rangle \mid |m| \not< |3^n \cdot 5^e|\}\{\langle m, a+1\rangle \mid |m| < |3^n \cdot 5^e| \text{ and } a \in H_m\}.$$

(d) If σ is an ordinal and e is a recursive index such that

(i) σ is a limit,
(ii) $\sigma \neq 3^a \cdot 5^b$ for all a, b, and
(iii) for all n, if $|n| < \sigma$ then $|\{e\}(n)| < \sigma$,

then $7^e \in \eta$; $|7^e|$ is the least σ satisfying (i), (ii), (iii);

$$H_7e = \{\langle m, 0\rangle \mid |m| \not< |7^e|\} U \{\langle m, a+1\rangle \mid |m| < |7^e| \text{ and } a \in H_m\}.$$

For an ordinal σ, let $\eta_\sigma = \{n \in \eta \mid |n| < \sigma\}$. For $n \notin \eta$, let $|n| = \infty$, let $|\eta| = \sup_{n \in \eta} |\eta|$.

DEFINITION. An ordinal σ is *η-admissible* if $\sigma = |7^e|$ for some $7^e \in \eta$; σ is *η-inaccessible* if σ is η-admissible and a limit of η-admissibles.

Harrington makes some remarks which we reproduce below:

(1) For $n, m \in \eta$, if $|n| \leq |m|$, then η_n and H_n are both recursive in H_m uniformly in n, m; further, if $|n| < |m|$, then oj $(H_n) \leq_T H_m$, uniformly.
(2) For $n \in \eta$, if $|n|$ is η-admissible, then there is a real I_n, recursive in H_n, which codes $M_{|n|} = \langle L_{|n|}, \varepsilon \rangle$.
(3) For any ordinal σ, $\{\langle n, a \rangle \mid n \in \eta_\sigma$ and $a \in H_n\}$ is Σ_1 over M_σ.
(4) Then by (2) and (3) we have

$$\{X \mid X \leq_T H_n \text{ for some } n \in \eta\} = 2^\omega \cap L_{|\eta|}.$$

Harrington then makes the following definitions.
$[e]^{\eta_\beta}(x) \simeq y$ means $e = \langle e_0, e_1 \rangle$ and $\{e_0\}(x) \in \eta_\beta$ and
$\{e_1\}(x, H_{\{e_0\}(x)}) \simeq y$
$[e]^{\eta_\beta}(x)\downarrow$ means $[e]^{\eta_\beta}(x) \simeq y$ for some y.
If $[e]^{\eta_\beta}(x)\downarrow$, $|[e]^{\eta_\beta}(x)| = |\{e_0\}(x)|$; otherwise, $|\{e\}^{\eta_\beta}(x)| = \infty$.
A partial function ϕ is partial η_β-recursive if for some e, all x, $(x) \simeq [e]^{\eta_\beta}(x)$. In the above, β is either η-inaccessible or $|\eta|$. Harrington asserts that one can find f, g recursive such that for all e, x, y:

(a) $[e]^\eta(x) \simeq y \Leftrightarrow \{f(e)\}(x, \mathscr{E}) \simeq y$.
(b) $\{e\}(x, \mathscr{E}) \simeq y \Rightarrow [g(e)]^\eta(x) \simeq y$.

We will produce a g such that the implication in (b) can be made into an equivalence with $\mathscr{E}$ replaced by $\mathscr{E}^*$:

(b′) $\{e\}(x, \mathscr{E}^*) \simeq y \Leftrightarrow [g(e)]^\eta(x) \simeq y$.

This, together with the preceding remarks, will give the required characterization of 1-env $\mathscr{E}^*$. We will use the fact that $\rho_0 = |\eta|$, and other facts from [4], [8], [ ]. (b′) will be established in the customary manner by showing the existence of a recursive h, such that if $[e_0]^\eta$ is total, then for all x, y,

$$\{e_1\}(x, [e_0]^\eta, \mathscr{E}^*) \simeq y \Leftrightarrow [h(e)]^\eta(x) \simeq y.$$

We will define h by effective transfinite induction on the length of the left-hand side computation. Of course, there is no new case, except S8:

$$\{e\}(x, [j]^\eta, \mathscr{E}^*) \simeq \mathscr{E}^*(\lambda\alpha\{e_3\}(x, \alpha^\frown[j]^\eta, \mathscr{E}^*))$$

where $\alpha^\frown[j]^\eta$ is a primitive recursive join operation applied to the two functions α, $[j]^\eta$.

Let δ be primitive recursive such that for all a, b,

$$[\delta(a, b)]^\eta = [a]^{\eta\frown}[b]^\eta.$$

Let $\dot{F} = \lambda\alpha\{e_3\}(x, \alpha^\frown[j]^\eta, \mathscr{E}^*)$.

H1. It will be assumed that h is defined and correct so far.

We will produce an ordinal $\tau = |7^p|$, $7^p \in \eta$; such that if $\dot{F}$ is acceptable, then $\alpha \in$ 1-sc $\dot{F} \Rightarrow \alpha$ is η_τ-recursive. The index p will depend on e, x, j. This will be called "bounding $\dot{F}$".

To do this, it suffices to find a τ such that:

H2. If $\lambda x, y\phi(x, y)$ is η_τ-recursive and $\dot{F}$-recursive, then $\lambda \times \dot{F}(\lambda y\phi(x, y))$ is (uniformly) η_τ-recursive.

If this is done, then we can define by induction a recursive θ such that for all a, m, $\{a\}(m, \dot{F}) \simeq [\theta(a)]^{\eta_\tau}(m)$. The only nontrivial case follows:

$$\{a\}(m, \dot{F}) \simeq \dot{F}(\lambda p\{b\}(p, m, \dot{F})) \simeq \dot{F}(\lambda p\{\theta(b)\}^{\eta\tau}(p, m))$$

(by the induction hypothesis).

Since $\dot{F}$ is, by assumption, acceptable, $\dot{F}$ will be defined on the indicated argument exactly when $\lambda p[\theta(b)]^{\eta_\tau}(p, m)$ is total. Since (by hypothesis) $\lambda p, m[\theta(b)]^{\eta_\tau}(p, m)$ is $\dot{F}$-recursive and η_τ-recursive, then $\lambda m\dot{F}(\lambda p[\theta(b)]^{\eta_\tau}(p, m))$ is η_τ-recursive, by H2, say with index c; then we can let $\theta(a) = c$. So H2 will suffice to obtain the "bounding of $\dot{F}$." We now obtain τ as a witness to H2. Let Π be recursive such that for all a, m, $\{a\}(m, \dot{F}) \simeq [\Pi(a)]^\eta(m)$; this follows from the "master" induction hypothesis, H1, that $\dot{F}$ is η-effective on η-recursive functions. The purpose of τ is to bound 1-sc $\dot{F}$ *below* $|\eta|$. Informally, we will then be able to search (effectively) through 1-sc $\dot{F}$ for an α such that $\dot{F}(\alpha) \simeq 0$. If $\dot{F}$ is acceptable, then the entire search will be η-bounded (below τ); otherwise, the computation is allowed to diverge. Compare this with the intuitive situation in the case that $\mathscr{E}(F)\downarrow \Leftrightarrow F$ is defined on *all* of ω^ω. There is no possible way of bounding this search below a countable ordinal. Consider the following map. Given $d \in \omega$, let

$$A_d = \{a \mid \forall m\{\Pi(a)_0\}(m) \in \eta_{|d|} \text{ and } \{\Pi(a)_1\}(m, H_{\{\Pi(a_0)\}(m)})\downarrow\}.$$

If $d \in \eta$, A_d is certainly H_{2^d}-recursive (uniformly). $A_d = \{a \mid a$ is an index of an $\dot{F}$-recursive (total) α such that α is $\eta_{|d|}$-recursive via $\Pi(a)\}$. As d runs through η, A_d runs through a complete set of indices for 1-sc $\dot{F}$.

For $a \in A_d$, $\alpha = [\Pi(a)]^{\eta|d|}$,

$$\begin{aligned} \dot{F}_x(a) &\simeq \{e_3\}(x, \alpha^\frown[j]^\eta, \mathscr{E}^*) \\ &\simeq \{e_3\}(x, [\Pi(a)]^{\eta|d|\frown}[j]^\eta, \mathscr{E}^*) \\ &\simeq [h(e_3, \delta(\Pi(a), j))]^\eta(x). \end{aligned}$$

Let $\{q\}^{H_{2}d}$ enumerate A_d. Then

$$T = \lambda s\{h(e_3, \delta(\Pi(\{q\}(s, H_2d), j)))_0\}(x)$$

enumerates the notations for ordinals of the computations of $\dot{F}_x(\alpha)$ for $\alpha = [\Pi(a)]^{\eta|d|}$ as a runs through A_d, that is, as α runs through 1-sc $\dot{F}$. T is an H_{2d}-(total) recursive function with index t, obtainable effectively from d, h, e, x, etc. Then $|3^{2^d} \cdot 5^t|$ is a bound for the ordinals of computations $\dot{F}_x(\alpha)$, for $\alpha = [\Pi(a)]^{\eta|d|}$ and $a \in A_d$. This map $d \mapsto 3^{2^d} \cdot 5^t$ is *recursive* with index w obtainable from x, h, e; therefore, $7^w \in \eta\tau = |7^w|$ is such that $|\{w\}(d)| < |7^w| = \tau$ for all $|d| < |7^w|$. Then, if $a \in A_7w$, the ordinal of the computation $\dot{F}(\alpha)$ is less then $\{w\}(d)$, where $\alpha = [\Pi(a)]^{\eta}|7^w|$; therefore, it is less than $|7^w|$. This is the necessary τ. (τ is η-admissible; this fact is of some importance in clearing up details.) Now we can see that if $\alpha \in$ 1-sc F with index a, then $\Pi(a) \in A_7w$. So we need only to search through A_7w for the value of $\mathscr{E}^*(\dot{F})$:

$$\mathscr{E}^*(\dot{F}) \simeq 0 \Leftrightarrow \exists a[\Pi(a) \in A_7w \text{ and } \dot{F}([\Pi(a)]^{\eta}) \simeq 0]$$

$$\mathscr{E}^*(\dot{F}) \simeq 1 \Leftrightarrow \forall a[\Pi(a) \notin A_7w \text{ or } \dot{F}([\Pi(a)]^{\eta}) > 1]$$

where $\dot{F}([\Pi(a)]^{\eta})$ is η_7d-computable. So, $h(e, j)$ can be defined accordingly; $[h(e, j)]^{\eta}(x)$ will diverge just in case $\dot{F}_x$ is *unacceptable*; in this case, either for some s; $|T(s)| = \infty$ or $T(s) \in \eta$ and $\{k\}^{H_{T(s)}}(x)$ diverges, where

$$k = h(e_3, \delta(\Pi(\{q\}(s, H_2d), j)))_1$$

(either case can be made to force $[h(e, j)]^{\eta}(x)$ to diverge).

From the properties of η-recursion derived in [4] we can conclude that the $\mathscr{E}^*$-recursion theory inherits selection operators from η-recursion theory, and 1-env $\mathscr{E}^*$ is a Spector 1-point class. It is also clear that 1-env $\mathscr{E}^*$ is *Mahlo*. It is in fact the *smallest* Mahlo Spector 1-point class. Of course, we have $\omega_{,}^{\mathscr{E}^*} = \omega_{,}^{S} = \rho_0 < |\Pi_1^0| < |\Delta_1^1|$ and since Gandy has shown $|\Delta_1^1| \leq |\Sigma_{,}' - \text{mon}|$, we have $\omega_{,}^{\mathscr{E}^*} < \omega_{,}^{E\#} = |\Sigma_{,}' - \text{mon}|$ (Aczel). See [8] for terminology and a proof of the first inequality.

Section 2

In [4], the hierarchy is related to an arbitrary ${}^2F$ by amending the definition of H_2x (in clause b) to read

$$H^F_{\alpha x} = \{\langle e, n\rangle \mid \{e\}^{H_x F} \text{ is a total function } \alpha \text{ and } {}^2F(\alpha) = n\}$$

The resulting system is labeled η^F, H^F_a, $|\ \ |^F$; and yields a hierarchy for 1-sc S, F. Indeed, the proof in Section 1, upon suitable modification, yields the fact that 1-env $\mathscr{E}^*$, $F = \Sigma_1(L_{\rho_0 F}[F]) \cap 2^{\omega}$, where $\rho_0^F = |\eta^F|$ is the first F-recursively Mahlo ordinal.

Upon inspection of the relevant clause, we remark that the value of F is required only at functions α which arise as $\alpha = \lambda x\{e\}^{H_a F}(x)$ for some e and

$a \in \eta^F$; i.e., $\alpha \in$ 1-sc $\mathscr{E}^*, F$. Call a partial $\dot{F}$ $\mathscr{E}^*$*-acceptable* if 1-sc $\dot{F}$, $\mathscr{E}^* \subseteq$ domain $\dot{F}$.

The diagonal of $\mathscr{E}^*$ is

$$D(\mathscr{E}^*)(e, {}^2F) = \begin{cases} 0 & \text{if } \{e\}(e, {}^2F, \mathscr{E}^*)| \\ 1 & \text{otherwise.} \end{cases}$$

Then to compute $D(\mathscr{E}^*)(e, {}^2F)$, we need only develop η^F and check

$$\exists_y([g(e)]^{\eta^F}(e) \simeq y)$$

where g is as in Section 1, modified to (uniformly) accommodate the relativization to F.

By the above remark we can extend $D(\mathscr{E}^*)$ to $\mathscr{E}_1^*$, defined at (possibly partial) $\dot{F}$ by

$$\mathscr{E}_1^*(e, \dot{F}) \simeq \begin{cases} 0 & \text{if } \dot{F} \text{ is } \mathscr{E}^*\text{-acceptable and } \exists_y([g(e)]^{\eta^{\dot{F}}}(e) \simeq y) \\ 1 & \text{if } \dot{F} \text{ is } \mathscr{E}^*\text{-acceptable and } \neg\, \exists_y([g(e)]^{\eta^{\dot{F}}}(e) \simeq y) \\ & \text{undefined if } \dot{F} \text{ is not } \mathscr{E}^*\text{-acceptable.} \end{cases}$$

Let ρ_1 be the first ordinal k such that

(1) k is admissible and

(2) for every $f: k \to k$ such that f is Δ_1 over L_K there is $\delta < k$, δ recursively Mahlo with $f''\delta - \delta$.

Then we can show that

(E) 1-sc $\mathscr{E}_1^* =$ 1-sc $D(\mathscr{E}^*) = L_{\rho_1} \cap 2^\omega$ and

(F) 1-env $\mathscr{E}_1^* = \Sigma_1(L_{\rho_1}) \cap 2^\omega$.

The key step is again to show that if F is $\mathscr{E}^*$-acceptable and arises as a λ-term; $\dot{F} = \lambda\alpha\{e\}(\alpha, \text{—}, \mathscr{E}_1^*)$; then $|\eta^{\dot{F}}| = \delta < \rho_1$ and δ is uniform in $(e, \text{—})$. δ will be the (recursively Mahlo) fixed point of a function which takes $\gamma < \rho_1$ into the supremum (over all $\alpha \in L_\gamma$; $\alpha: \omega \to \omega$) of the ordinals required to compute $\{e\}(\alpha, \text{—}, \mathscr{E}_1^*)$. (This is best done using notations for ordinals $< \rho_1$.)

Because δ is recursively Mahlo, we have no trouble with clause d of $\eta^{\dot{F}}$, and because it is a fixed point of the above function the $\dot{F}$-applications of clause b are possible.

We can now diagonalize again and extend again to get $\mathscr{E}_2^*$, etc. In each case, selection operators and hierarchies will follow from the "countable computation" feature of the extended object.

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