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THE STRUCTURE OF MINIMA FOR BINARY QUADRATIC FORMS

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Abstract

In this paper, we construct examples of binary quadratic forms with positive, unattained minimum. To this end, we investigate the structure of the set of values taken by a certain class of indefinite forms.

1. Introduction

The work in this paper grew out of an attempt to answer a question posed by Paul Bateman: construct a binary quadratic form with a non-zero, unattained minimum. Throughout this paper, f will represent the binary form with real coefficients given by

$$f(x, y) = (x - \alpha y)(x - \beta y) \tag{1}$$

where α and β are both real or are complex conjugate. The minimum of f is defined as $\mu(f) = \inf |f(x, y)|$ taken over non-zero integer points (x, y).

Examples of the forms sought have been given by Schur (reported by Remak in [5]), and, independently of the author, by Larry Pinzur. Their examples were constructed by choosing α and β to have particular bounded continued fraction expansions. In Remak's terminology, the forms sought are ones that are not unimodularly equivalent to minimal forms. The form f, defined in (1), is a minimal form if $|f(x, y)| \geq 1$ for all non-zero integer points (x, y). Now, α and β have bounded continued fraction expansions if they are quadratic, that is, each is an irrational solution of a quadratic equation with rational coefficients. The continued fraction expansions of such numbers are periodic, therefore, bounded, Again, see [4] for details. Pinzur's examples, as the ones presented here, involve only quadratic numbers. In Schur's example, α is chosen to be quadratic while β has a bounded but non-periodic expansion. Throughout the rest of this paper, we assume that both α and β are quadratic.

$\mu(f)$ is an attained minimum if $\mu(f) = |f(x_0, y_0)|$ for some non-zero integer point (x_0, y_0). A number of ways of choosing α and β can be immediately eliminated as not answering Bateman's question. If α and β are not real, then one easily checks that $\mu(f)$ is indeed attained. If α is real, but is "too well" approximable by rationals, then $\mu(f) = 0$. By "too well" approximable, we mean that for any $\varepsilon > 0$, $|y|\,|x - \alpha y| < \varepsilon$ has infinitely many integer

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solutions x, y. This is equivalent to the partial quotients of the continued fraction expansion being unbounded. In fact, it is true that almost all α are "too well" approximable (a consequence of Theorem 29, page 60 in [4]. Proofs of the other statements about continued fractions will be found there also). So, α and β must both be chosen from the set (of measure 0) of irrational real numbers with bounded continued fraction expansions.

We will show, for any f with α and β both quadratic, how to compute $\mu(f)$ and to decide whether $\mu(f)$ is attained. This depends on determining the structure of the set of values taken by $|f|$: $V=\{|f(x, y)|: (x, y)$ a non-zero, integer point$\}$. A key result concerns the set of accumulation points of V, V'. To describe V', we introduce two further forms related to F. If α is quadratic, the rational quadratic equation it satisfies is essentially unique. The other root is uniquely determined. This other root, called the conjugate of α, we denote α^*. Now, given α and β are quadratic, define

$$A(x, y)=(x-\alpha y)(x-\alpha^* y), \qquad B(x, y)=(x-\beta y)(x-\beta^* y).$$

These are quadratic forms with rational coefficients.

THEOREM 1. (a) *Suppose f is defined by* (1) *and that α and β are quadratic and that $\beta\neq\alpha, \alpha^*$. Then δ is a limit point of V if and only if*

$$\delta=\delta_A(x_0, y_0)=|A(x_0, y_0)(\alpha-\beta)(\alpha-\alpha^*)^{-1}|$$

or

$$\delta=\delta_B(x_0, y_0)=|B(x_0, y_0)(\alpha-\beta)(\beta-\beta^*)^{-1}|.$$

V' is a discrete set.

(b) *$\delta\in V$ is a limit from below if and only if one of the equations*

$$A(x, y)=\operatorname{sgn}(A(\beta, 1))\delta\,|(\alpha-\alpha^*)(\alpha-\beta)^{-1}|$$

or

$$B(x, y)=\operatorname{sgn}(B(\alpha, 1))\delta\,|(\beta-\beta^*)(\alpha-\beta)^{-1}|$$

is solvable in integers x, y.

The proof of this theorem will be given in Section 2. However, note that, since A and B have rational coefficients, their values at integer points form discrete sets. So, given the form of the limit points in V', the discreteness of V' is immediate. One further remark is in order. If $\beta=\alpha^*$, then f would itself have rational coefficients, and so V would be discrete (and V' empty). If $\alpha=\beta$, then, as a consequence of a theorem of Weyl (p. 90 in [3]) V would be everywhere dense in the positive real axis. From now on, we will assume that α and β are quadratic and $\beta\neq\alpha$ or α^*. The smallest element of V' is denoted by γ. Evidently, γ depends on the smallest values taken by $|A|$ and $|B|$ at integer points. For given α and β there are standard methods for computing these values (and so γ). These depend mainly on expanding α and β as continued fractions. See also Section 7 of Chapter 2 in [1]. There is a case, however, in which these minimal values are easily computed. α is said to be quadratic integer if both $\alpha+\alpha^*$ and $\alpha\alpha^*$ are rational integers. If α is a quadratic integer, it is plain that the minimal value for $|A|$ is 1. Now, if γ is known, analysis of the small values of $|f|$ reduces to two considerations: from which direction is γ a limit; and, what anomalous minima are there? We say that $|f(x_1, y_1)|$ is an anomalous minimum of f if $|f(x_1, y_1)|<\gamma$ but

$$\gamma<\delta_A(x_1, y_1) \quad \text{and} \quad \gamma<\delta_B(x_1, y_1).$$

The remarks after the proof of Theorem 2, below, show that the number of anomalous minima is finite, and the anomalous minima can be effectively determined for a given form. Theorem 2 shows the general sort of results that can be proved.

Notation. f is said to have finitely many attained minima if only finitely many values of $|f(x, y)|$ at integer points (x, y) are less than γ. Otherwise, f has infinitely many attained minima.

THEOREM 2. *Let α, β be quadratic integers such that $\alpha^*<\alpha<\beta$ and $\beta^*<\alpha$. Then $f(x, y)$ has infinitely many attained minima in either of these two cases:*

(a) $\beta-\beta^*\leq\alpha-\alpha^*$
(b) $B(x, y)=-1$ *has an integral solution.*

If neither (a) *nor* (b) *is true, then $f(x, y)$ has only finitely many attained minima.*

Proof. The smallest value taken by $|A(x, y)|$ and $|B(x, y)|$ at non-zero integer points is 1. Thus

$$\gamma=|(\alpha-\beta)(\alpha-\alpha^*)^{-1}| \quad \text{or} \quad |(\alpha-\beta)(\beta-\beta^*)^{-1}|$$

as $\beta-\beta^*\leq\alpha-\alpha^*$ or $\alpha-\alpha^*\leq\beta-\beta^*$.

(a) If $\beta-\beta^*\leq\alpha-\alpha^*$ then $\gamma=|(\alpha-\beta)(\alpha-\alpha^*)^{-1}|$ Now,

$$A(\beta, 1)=(\beta-\alpha)(\beta-\alpha^*)>0$$

and $A(x, y)=1$ is solvable in integers, so, by Theorem 1(b), γ is a limit from below. That is, f has infinitely many attained minima.

(b) If $B(x, y)=-1$ is solvable and $\alpha-\alpha^*<\beta-\beta^*$, then

$$\gamma=|(\alpha-\beta)(\beta-\beta^*)^{-1}|$$

and is again a limit from below, by Theorem 1(b).

Now, if $\alpha-\alpha^*<\beta-\beta^*$ and $B(x, y)=-1$ is not solvable in integers, $\gamma=|(\alpha-\beta)(\beta-\beta^*)^{-1}|$. Now, $B(\alpha, 1)<0$ and $A(\beta, 1)>0$.

$$A(x, y)=(+1)(\beta-\alpha)(\beta-\beta^*)^{-1}(\alpha-\alpha^*)(\beta-\alpha)^{-1}$$

is not solvable in integers, since the right hand side is less than 1 in absolute value.

$$B(x, y)=(-1)(\beta-\alpha)(\beta-\beta^*)^{-1}(\beta-\beta^*)(\beta-\alpha)^{-1}$$

is not solvable by assumption. This shows that γ is not a limit from below. Since γ is the smallest accumulation point of V and is not a limit from below, there can only be finitely many elements of V less than γ. ∎

Actually, a constructive proof can be given that there are only finitely many anomolous minima. We assume only that α and β are quadratic and $\beta \neq \alpha, \alpha^*$. Suppose

$$|f(x_1, y_1)|<\delta_B(x_0, y_0)<\delta_B(x_1, y_1).$$

Let $B_0=|B(x_2, y_2)|/|B(x_0, y_0)|$ where $|B(x_2, y_2)|$ is the next larger value of $|B|$ after $|B(x_0, y_0)|$. Then

$$|t_1-\beta|>\min\left(|\alpha-\beta|, |\beta^*-\beta|, \frac{|\alpha-\beta|\,|B_0-1|\,|\beta-\beta^*|}{|\beta^*-\alpha|+|B_0-1|\,|\beta-\beta^*|}\right)>0.$$

In other words, $|t_1-\beta|>c_1>0$ where c_1 depends only on (x_0, y_0) (and f, of course). Here, $t_1=x_1/y_1$. Similarly, if

$$|f(x_1, y_1)|<\delta_B(x_0, y_0)<\delta_A(x_1, y_1)$$

then $|t_1-\alpha|>c_2>0$, c_2 depending only on (x_0, y_0). But

$$\delta_B(x_0, y_0)>|f(x_1, y_1)|=y_1^2\,|t_1-\alpha|\,|t_1-\beta|>y_1^2c_1c_2$$

so $|y_1|$ is bounded. Thus, $|x_1|$ is also bounded and there can be, indeed, only finitely many anomalous minima.

Similar results to Theorem 2 can be obtained by reordering α, α^*, β, and β^*.

By examining the properties of quadratic integers more closely, the following stronger version of Theorem 2 can be obtained:

THEOREM 3. *Let α and β be quadratic integers such that $\alpha^*<\alpha<\beta$, $\beta^*<\alpha$, and $\alpha-\alpha^*<\beta-\beta^*$. Then, f has at most one anomalous minimum. If $|f(x_0, y_0)|$ is an anomalous minimum of f then $A(x_0, y_0)=1$ and $\beta-\beta^*<2(\alpha-\alpha^*)+1$.*

This theorem can be interpreted by using the representation of α and β obtained from the quadratic formula. That is, applying the quadratic formula to the equation satisfied by a quadratic integer α, then $\alpha=\frac{1}{2}(s+\sqrt{m})$ and $\alpha^*=\frac{1}{2}(s-\sqrt{m})$ (assuming that $\alpha^*<\alpha$) where s and m are rational integers. Further, m is positive but not a perfect square and $m\equiv s^2 \pmod 4$. Similarly, $\beta=\frac{1}{2}(r+\sqrt{n})$ and $\beta^*=\frac{1}{2}(r-\sqrt{n})$. The inequalities in the hypothesis of Theorem 3 translate to

$$s+\sqrt{m}-\sqrt{n}<r<s+\sqrt{m}+\sqrt{n} \quad \text{and} \quad \sqrt{m}<\sqrt{n}.$$

Now, if f has an anomalous minimum, then the conclusion of Theorem 3 is that $\sqrt{n} < 2\sqrt{m}+1$. Evidently, given a quadratic integer α, there are only finitely many quadratic integers β such that the hypotheses of Theorem 3 are satisfied and such that f has an anomalous minimum. The table was constructed to show all forms f with an anomalous minimum and such that α and β are quadratic integers, as above, and $s=0$ or 1 and $m \leq 24$. With f defined as in (1), define

$$f_k(x, y) = (x-(\alpha+k)y)(x-(\beta+k)y).$$

Then, if k is a rational integer, f_k and f have the same value set V. If α is a quadratic integer, then the effect of this translation by k is to change s to $s+2k$. That is, the structure of the minima of f can be studied assuming that $s=0$ or 1. This observation will be used in the proof of Theorem 3.

As an immediate corollary of Theorems 2 and 3, the following result shows how to construct numerous examples of forms f with positive unattained minimum.

COROLLARY. *Let α and β be quadratic integers such that $\alpha^* < \alpha < \beta$, $\beta^* < \alpha$ and $2(\alpha-\alpha^*)+1 < \beta-\beta^*$. Assume also that $B(x, y) = -1$ has no solution in integers. Then f, as defined by* (1), *has a positive, unattained minimum.*

Simple congruence arguments will show that $B(x, y) = -1$ has no integral solution if n is divisible by 8 or by a prime congruent to 3 (mod 4).

The following examples will illustrate some of the possibilities:

(1) $\alpha = \sqrt{2}$, $\beta = \sqrt{12}$. $B(x, y) = -1$ has no solution integers (cf. previous remark). So, by the Corollary, f has the unattained minimum

$$(\sqrt{12}-\sqrt{2})/2\sqrt{12} \approx .2959.$$

(2) $\alpha = \sqrt{2}$, $\beta = \sqrt{3}$. $B(x, y) = -1$ has no integer solution. But, as will be found in the table. $|f(3, 2)| \approx .0796$ is an anomalous minima.

(3) $\alpha = \sqrt{3}$, $\beta = 1+\sqrt{5}$. $B(3, 1) = -1$, so f has infinitely many attained minima. But, as will be seen from the table, $|f(2, 1)| \approx .3312$ is an anomalous minimum. In this case,

$$\gamma = (1+\sqrt{5}-\sqrt{3})/2\sqrt{5} \approx .3363.$$

The computations for the table were done on the IBM 370 at the Computer Center at University of Illinois, Chicago Circle.

2. Proofs of Theorems 1 and 3

Proof of Theorem 1. (a) Suppose $\delta = \lim_n |f(x_n, y_n)|$ where the $|f(x_n, y_n)|$ are all distinct. Assume that $y_n \geq 0$ for all n. Since $|f(x_n, y_n)|$ is bounded for all n, on some subsequence, $y_n > 0$ and $\lim_n y_n = \infty$. Replace (x_n, y_n) by this subsequence. Setting $t_n = x_n/y_n$, we have

(2) $$f(x_n, y_n)| = |A(x_n, y_n)(t_n - \beta)(t_n - \alpha^*)^{-1}|.$$

If $|A(x_n, y_n)|$ does not remain bounded, then, replacing (x_n, y_n) by a subsequence, if necessary, $\lim_n t_n = \beta$. Similarly, if $|B(x_n, y_n)|$ does not remain bounded, $\lim_n t_n = \alpha$. Now $\alpha \neq \beta$ so one of $|A(x_n, y_n)|$ or $|B(x_n, y_n)|$ must remain bounded. Suppose $|A(x_n, y_n)|$ is bounded. Since the values taken by $|A(x, y)|$ at integer points form a discrete set, replacing (x_n, y_n) by a subsequence, we may assume $A(x_n, y_n) = A_0$ for all n. Then

$$|t_n - \alpha|\,|t_n - \alpha^*| = |A_0|\, y_n^{-2} \quad \text{for all } n.$$

But, from (2), $|t_n - \alpha^*|$ is bounded away from 0, since $\beta \neq \alpha^*$. Thus, $\lim_n t_n = \alpha$. Finally, $\delta = \delta_A(x_1, y_1)$, using (2). If $|B(x_n, y_n)|$ remains bounded, then $\delta = \delta_B(x_r, y_r)$ for some r.

Conversely, suppose $\delta = \delta_A(x_0, y_0)$. Then, by the Corollary to Lemma 3 on page 23 in [2], there is an infinite sequence (x_n, y_n) of integer points such that $A(x_n, y_n) = A(x_0, y_0)$ and $\lim_n (x_n/y_n) = \alpha$. Using (2) again, for given (x_n, y_n), there are at most three other (x_r, y_r) such that $|f(x_n, y_n)| = |f(x_r, y_r)|$. Going to a subsequence of (x_n, y_n), we may assume that all $|f(x_n, y_n)|$ are distinct. But, by (2), $\lim_n |f(x_n, y_n)| = \delta_A(x_0, y_0)$ as desired. The case $\delta = \delta_B(x_0, y_0)$ is similar.

(b) Suppose

(3) $$|f(x_0, y_0)| < \delta_A(x_0, y_0)$$

and

(4) $$A(\beta, 1)A(x_0, y_0) < 0.$$

We show that this leads to a contradiction. First, assume $y_0 \neq 0$. Then, from (3),

(5) $$|(t_0 - \beta)(\alpha - \alpha^*)| < |(\alpha - \beta)(t_0 - \alpha^*)|$$

where $t_0 = x_0/y_0$. But, from (4),

$$(t_0 - \beta)(\alpha - \alpha^*)(\alpha - \beta)(t_0 - \alpha^*) > 0,$$

so, (5) implies that one of these two inequalities is true:

(6) $$0 > (t_0 - \beta)(\alpha - \alpha^*) > (\alpha - \beta)(t_0 - \alpha^*),$$

(7) $$0 < (t_0 - \beta)(\alpha - \alpha^*) < (\alpha - \beta)(t_0 - \alpha^*).$$

The right hand inequality in each can be simplified to

(8) $$0 > (t_0 - \alpha)(\alpha^* - \beta),$$

(9) $$0 < (t_0 - \alpha)(\alpha^* - \beta)$$

respectively. But (6) and (8) together and (7) and (9) together each imply

$$0<(t_0-\alpha)(t_0-\alpha^*)(\beta-\alpha)(\beta-\alpha^*)y_0^2=A(\beta,1)A(x_0,y_0)$$

which contradicts (4). If $y_0=0$, then from (3),

$$1\leq x_0^2<|x_0^2(\alpha-\beta)(\alpha-\alpha^*)^{-1}|$$

so $|\alpha-\alpha^*|<|\alpha-\beta|$. This implies that β cannot be between α and α^*, so $A(\beta,1)A(x_0,0)=(\beta-\alpha)(\beta-\alpha^*)x_0^2>0$, again contradicting (4). From this, and the similar result for B, it is plain that if $\delta\in V'$ is a limit from below, then either $\delta=\delta_A(x_0,y_0)$ and $A(\beta,1)A(x_0,y_0)>0$ or $\delta=\delta_B(x_0,y_0)$ and $B(\alpha,1)B(x_0,y_0)>0$ which shows necessity.

Now although $|f(x_0,y_0)|<\delta_A(x_0,y_0)$ implies $A(\beta,1)A(x_0,y_0)>0$, the converse is not necessarily true (see the remarks below). However, assuming $|f(x_0,y_0)|>\delta_A(x_0,y_0)$ and that $t_0=x_0/y_0$ is defined and sufficiently near α, one can prove

$$A(\beta,1)A(x_0,y_0)<0.$$

The idea is this: from $|f(x_0,y_0)|>\delta_A(x_0,y_0)$, derive (5), but with the inequality reversed. If t_0 is sufficiently near α then

$$(t_0-\beta)(\alpha-\beta)(t_0-\alpha^*)(\alpha-\alpha^*)>0$$

after which the derivation proceeds as before, but with appropriate changes in the direction of the inequalities. To show sufficiency, suppose that $A(\beta,1)A(x_0,y_0)>0$ for some integer point (x_0,y_0). As in (a), construct a sequence (x_n,y_n) of integer points such that $\lim_n (x_n/y_n)=\alpha$, $\lim_n |f(x_n,y_n)|=\delta_A(x_0,y_0)$, and $A(x_n,y_n)=A(x_0,y_0)$ for all n. For large enough n, the above considerations apply to show that $|f(x_n,y_n)|<\delta_A(x_0,y_0)$. That is. $\delta_A(x_0,y_0)$ is a limit from below. Similarly $\delta_B(x_0,y_0)$ is a limit from below if $B(x_0,y_0)B(\alpha,1)>0$. ■

Remark. A sequence (x_n,y_n) can be constructed so that $A(x_n,y_n)=A(x_1,y_1)$ for all n and so that $\lim_n (x_n/y_n)=\alpha^*$. But then, from (2), $\lim_n |f(x_n,y_n)|=\infty$. So, even if

$$A(x_n,y_n)A(\beta,1)>0,$$

it is evident that $|f(x_n,y_n)|>\delta_A(x_n,y_n)$ for sufficiently large n.

Proof of Theorem 3. As in the remarks following the statement of Theorem 3, let $\alpha=\frac{1}{2}(s+\sqrt{m})$ and $\beta=\frac{1}{2}(r+\sqrt{n})$. Let $|f(x_0,y_0)|$ be an anomalous minimum of f. Then, since $B(\alpha,1)<0$, $B(x_0,y_0)=-b<0$ by Theorem 1. Indeed, $b\geq 2$ since $|f(x_0,y_0)|$ is anomalous. Since $A(\beta,1)>0$, then $A(x_0,y_0)=a\geq 1$. Since $|f(x_0,y_0)|<\gamma<1$, then $y_0\neq 0$ so $t_0=x_0/y_0$ is defined. We will assume that $y_0>0$. Now if $t_0<\alpha^*$ were true, then

$$|f(x_0,y_0)|=|a(t_0-\beta)(t_0-\alpha^*)^{-1}|>1\geq\gamma,$$

a contradiction, so $\alpha^* < t_0$. Since $A(x_0, y_0) > 0$ and $B(x_0, y_0) < 0$, then $\alpha < t_0 < \beta$.

Now,

$$|f(x_0, y_0)| = a(\beta - t_0)(t_0 - \alpha^*)^{-1} < \gamma \quad \text{and} \quad \gamma = (\beta - \alpha)(\beta - \beta^*)^{-1}$$

so, solving for t_0, we have

$$t_0 \geq \frac{a(\beta^2 - \beta\beta^*) + \alpha^*\beta - \alpha\alpha^*}{a(\beta - \beta^*) + (\beta - \alpha)}. \tag{10}$$

Similarly

$$t_0 \leq \frac{b(\alpha\beta - \alpha\beta^*) - \beta\beta^* + \alpha\beta^*}{b(\beta - \beta^*) - \beta + \alpha}. \tag{11}$$

Eliminating t_0 from (10) and (11), then solving for a,

$$a \leq \frac{b}{b-1}\left(\frac{\alpha - \alpha^*}{\beta - \beta^*}\right) + \frac{(\beta - \alpha)(\alpha^* - \beta^*)}{(\beta - \beta^*)^2(b-1)}. \tag{12}$$

If $\alpha^* < \beta^*$, then (12) implies that $a < b(b-1)^{-1} \leq 2$ since $b \geq 2$. So, $a = 1$. But, (12) can be rewritten as

$$a \leq \frac{\alpha - \alpha^*}{\beta - \beta^*} + \frac{(\beta - \alpha^*)(\alpha - \beta^*)}{(\beta - \beta^*)^2(b-1)}.$$

From this, if $\beta^* < \alpha^*$, then $a < 1 + 1$, so $a = 1$ in any case, as desired. $A(x_0, y_0) = 1$ implies that

$$t_0 - \alpha = y_0^{-2}(t_0 - \alpha^*)^{-1}.$$

But, $t_0 > \alpha$ so

$$0 < t_0 - \alpha < (y_0^2\sqrt{m})^{-1}. \tag{13}$$

Now, $|f(x_0, y_0)| < \gamma$ can be rewritten as

$$\beta - \beta^* < t_0 - \alpha^* + (t_0 - \alpha)(t_0 - \alpha^*)(\beta - t_0)^{-1}.$$

Using $A(x_0, y_0) = 1$ and $B(x_0, y_0) = -b$, we have

$$\beta - \beta^* < t_0 - \alpha^* + b^{-1}(t_0 - \beta^*).$$

Since $b \geq 2$ and $t_0 < \beta$, we then have

$$\tfrac{1}{2}(\beta - \beta^*) < t_0 - \alpha^* < \alpha - \alpha^* + (y_0^2\sqrt{m})^{-1}$$

by (13). Since $m \geq 5$, we finally have $\frac{1}{2}\sqrt{n} < \sqrt{m} + \frac{1}{2}$ as desired.

Enough has been proved at this point to allow, for a given quadratic integer α, the determination of all quadratic integers β such that f has an anomalous minimum. The table records all such f for $s = 0$ or 1 and $m \leq 24$. Indeed, by the remarks following Theorem 3, this table is enough to verify

Theorem 3 for all $m \leq 24$ since all that remains is to show that f has at most one anomalous minimum. We will, therefore, assume that $m \geq 28$ for the remainder of this proof.

Suppose that $|f(x_1, y_1)|$ is another anomalous minimum of f. Then the previous discussion is valid for (x_1, y_1) as well. We suppose that $t_1 = x_1/y_1 > t_0$ and that $y_1 > 0$. Now, let

$$x_2 = x_0x_1 + \alpha\alpha^* y_0y_1 - (\alpha + \alpha^*)x_0y_1 \quad \text{and} \quad y_2 = y_0x_1 - x_0y_1.$$

It is straightforward to check that

$$(x_0 - \alpha y_0) = (x_1 - \alpha y_1)(x_2 - \alpha y_2),$$

using $A(x_1, y_1) = 1$. Evidently x_2 and y_2 are integers, and $y_2 = y_0y_1(t_1 - t_0) > 0$, so x_2 is also positive and $t_2 = x_2/y_2 > \alpha$. The formulae for x_2 and y_2 can be inverted to give

$$y_0 = x_1y_2 + x_2y_1 - (\alpha + \alpha^*)y_1y_2$$

so

$$y_0(y_1y_2)^{-1} = t_1 + t_2 - \alpha - \alpha^* > \sqrt{m}$$

so

$$y_0 > \sqrt{m}y_1. \tag{14}$$

We wish to show next that

$$y_1^2\sqrt{m} > \tfrac{1}{4}y_0^2 \tag{15}$$

which contradicts (14), since $m \geq 28$. The starting point is the inequality obtained from (10) (with $a = 1$) and (13), namely

$$y_0^2\sqrt{m}(\beta - \alpha)(\beta - \alpha - \beta^* + \alpha^*) \leq 2\beta - \alpha - \beta^*$$

This may be rewritten as

$$(\beta - \alpha - (1/y_0^2\sqrt{m}))(\beta - \beta^* - \alpha + \alpha^* - (1/y_0^2\sqrt{m})) \leq y_0^{-2} + (y_0^4\sqrt{m})^{-1} \tag{16}$$

We claim that $\beta - \alpha < 5/(2y_0^2)$. Indeed, since $n > m \geq 28$,

$$\beta - \beta^* - \alpha + \alpha^* = \sqrt{n} - \sqrt{m} > 9/(19\sqrt{m}) > (y_0^2\sqrt{m})^{-1}.$$

Substituting into (16), we have $\beta - \alpha < 2.2/\sqrt{m} < .41$. But

$$r - s = 2(\beta - \alpha) - (\beta - \beta^* - \alpha + \alpha^*) < .82$$

is an integer, so $r - s \leq 0$. If $r = s$ then, by (16),

$$\sqrt{n} - \sqrt{m} = \beta - \alpha < 1.5/y_0 < 1.5/\sqrt{m},$$

which implies that $m < n \leq m + 3$. However, this contradicts $n \equiv r^2 = s^2 \equiv m \pmod 4$. So, $r - s \leq -1$, that is, $\beta - \beta^* - \alpha + \alpha^* \geq 1$. Finally, using (16) again, we have

$$\beta-\alpha<5/2y_0^2 \tag{17}$$

Similarly to (13), $0<t_1-\alpha<(y_1^2\sqrt{m})^{-1}$ so

$$t_1-\alpha=(y_1^2(t_1-\alpha^*))^{-1}>(y_1^2(\sqrt{m})^{-1})^{-1}>2/(3y_1^2\sqrt{m}).$$

But, $\beta>t_1$, so, by (17), $5/(2y_0^2)>2/(3y_1^2\sqrt{m})$ which implies (15).

As remarked before, (14) and (15) are contradictory for $m\geq 28$, so the proof is complete.

Table of Forms with Anomalous Minima

s	m	r	n	x_0	y_0	$-B_0$
1	5	4	8	2	1	2
1	5	0	12	5	3	2
1	5	2	12	2	1	2
1	5	1	17	2	1	2
0	8	0	12	3	2	3
0	12	3	13	2	1	3
0	12	5	13	2	1	3
0	12	1	17	2	1	2
0	12	3	17	2	1	4
0	12	2	20	2	1	4
0	12	1	21	2	1	3
0	12	0	24	2	1	2
0	12	−2	44	2	1	2
0	20	1	21	9	4	35
1	21	2	24	3	1	2
1	21	4	24	3	1	5
1	21	6	24	3	1	6
1	21	8	24	3	1	5
1	21	10	24	3	1	2
1	21	2	28	3	1	3
1	21	4	28	3	1	6
1	21	3	29	3	1	5
1	21	0	32	14	5	4
1	21	2	32	3	1	4
1	21	1	33	3	1	2
1	21	1	37	3	1	3
1	21	0	44	3	1	2
1	21	0	48	3	1	3
1	21	−2	72	3	1	2
0	24	0	28	5	2	3
0	24	0	32	5	2	7
0	24	−2	52	5	2	3

All forms f (defined by (1)) having an anomalous minimum are given, such that $\alpha=\frac{1}{2}(s+\sqrt{m})$ and $\beta=\frac{1}{2}(r+\sqrt{n})$ are quadratic integers, $s=0$ or 1,

$m \leq 24$, and the conditions $\alpha^* < \alpha < \beta$, $\beta^* < \alpha$, and $\alpha - \alpha^* < \beta - \beta^*$ are satisfied. Each such form actually has only one anomalous minimum, $|f(x_0, y_0)|$. $B(x_0, y_0) = B_0$ in each case.

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