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THE INTERVAL IN ALGEBRAIC TOPOLOGY

BY

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1. Introduction

An important property of the geometric realization functor (i.e. of the left Kan extension along the right Yoneda functor of the cosimplicial space of affine simplexes) is that it preserves finite products. In this paper it is shown that the category of all cosimplicial spaces that have such a Kan extension is equivalent to a category of intervals. Some properties of intervals are discussed and an explicit description is given of both the category of Hausdorff and the category of finite intervals. It should be noted that each interval X gives rise to an "algebraic topology" on spaces wherein each standard notion or construction (i.e. one that is based on the standard unit interval) is replaced by a corresponding X-notion or X-construction.

2. Categorical preliminaries

If V is a complete, symmetric, closed monoidal category and C is a small V-category [2, p. xiii] then the V-functor category $B = V^{C^{op}}$ also has the structure of a V-category [2, p. 150] and the right Yoneda functor $R: C \to B$ given by $R(c) = C(-, c)$ is a V-full and faithful V-functor [2, p. 152]. The image of C under R consists of the representable functors. If A is a tensored V-category [2, p. 20] and $T: C \to A$ is a V-functor then the left Kan extension of T along R, $\mathrm{Lan}_R\, T: B \to A$, is given in terms of coend and tensor by $L_T = \mathrm{Lan}_R\, T = \int^c B(R(c), -) \otimes_A T(c)$ ([2], dual of 1.43, p. 52). Since R is V-full, L_T may be assumed to satisfy $L_T R = T$ [2, dual of 1.4.5, p. 56]. If L_T is pointwise (e.g. if A is tensored and cotensored [2, dual 1.4.4 p. 55]) then for $F \in B$,

$$L_T(F) = \int^c B(R(c), F) \otimes_A Tc = \int^c Fc \otimes_A Tc$$

since $B(R(c), F) = Fc$ [2, IV.1.1 p. 152]. Hence if L_T is pointwise it is V-left adjoint to the singular V-functor $U_T: A \to B$ given by $U_T(a) = A(T(-), a)$ as the following calculation shows:

$$A(L_T(F), a) = A\left(\int^c Fc \otimes_A Tc, a\right) \approx \int_c A(Fc \otimes_A Tc, a)$$
$$\approx \int_c V(Fc, A(Tc, a)) = B(F, A(T(-), a)) = B(F, U_T(a)).$$

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The second equivalence follows from the general fact that

$$A\left(\int^{c} S(c, c), a\right) \approx \int_{c} A(S(c, c), a) \quad \text{for } S\colon C^{\mathrm{op}} \otimes C \to A^{\mathrm{op}}$$

(since end and coend are duals), the third equivalence follows from the definition of $\otimes$ [2, p. 19] and the fourth equivalence follows from the definition of V-structure of B [2, p. 150]. Thus if L_T is pointwise it is V-cocontinuous [2, dual III 1.4, p. 114].

2.1 PROPOSITION. *If A and B have "products"; i.e. a bifunctor*

$$x_A\colon A \otimes A \to A(x_B\colon B \otimes B \to B)$$

that is a V-cocontinuous V-functor in each variable and $L\colon B \to A$ is a V-cocontinuous V-functor, then L preserves the product if and only if L preserves the product of the representables.

Proof. If $F, G \in B$ then $F = \int^c R(c) \otimes Fc$ and $G = \int^d R(d) \otimes Gd$ [2, p. 57]. Since $Fx_B -$ and $- x_B R(d)$ are V-cocontinuous we have

$$\begin{aligned} Fx_B G &= Fx_B\left(\int^{d} R(d) \otimes Gd\right) \\ &\approx \int^{d} (Fx_B R(d)) \otimes Gd \\ &= \int^{d} \left(\left(\int^{c} R(c) \otimes Fc\right) x_B R(d)\right) \otimes Gd \\ &\approx \int^{d} \left(\int^{c} (R(c) x_B R(d)) \otimes Fc\right) \otimes Gd. \end{aligned}$$

Since L is V-cocontinuous, $LF \approx \int^c LR(c) \otimes Fc$, $LG \approx \int^d LR(d) \otimes Gd$ and a similar argument to the one above using now that $LFx_A -$ and $- x_A LR(d)$ are V-cocontinuous gives $LFx_A LG \approx \int^d (\int^c (LR(c) x_A LR(d)) \otimes Fc) \otimes Gd$. Finally, the application of the V-cocontinuous functor L to the above representation of $Fx_B G$ yields

$$L(Fx_B G) \approx \int^{d} \left(\int^{c} L(R(c) x_B R(d)) \otimes Fc\right) \otimes Gd.$$

Thus the equivalence of $L(R(c) x_B R(d))$ and $LR(c) x_A LR(d)$ clearly gives the equivalence of $L(Fx_B G)$ and $LFx_A LG$.

If D: Set $\to V$ is a finite product preserving functor (monoidal), where Set is the category of sets and V is a cartesian closed category, then for any category (i.e. Set-category) C, setting $C(c, d) = D(\mathrm{Hom}\,(c, d)$ defines a V-structure on C for which the V-Yoneda functor $R; C \to V^{C^{\mathrm{op}}} = B$ is given by $R = D'R_0$ where $R_0\colon C \to (\mathrm{Set}^{C^{\mathrm{op}}}) = B_0$ is the Set-Yoneda functor and $D' = D^{C^{\mathrm{op}}}\colon B_0 \to B$.

Further, if D is exact (preserves equalizers, coequalizers, finite product and coproduct) then D' is also exact.

In the special case in which $C = \Delta$ is the skeletal category of finite, linearly ordered, non-empty sets and non-decreasing maps [3, p. 23] then $B_0(B)$ is the category of simplicial sets (simplicial objects in V [7, p. 4]). The category Δ is usually identified with the category generated by the objects $[n] = \{0, 1, \ldots, n\}$, $n = 0, 1, \ldots,$ the monos $\delta_i\colon [n-1] \to [n]$ (omit i), $i \in [n]$, and the epis $\sigma_j\colon [n+1] \to [n]$ (takes on j value twice), $j \in [n]$ [3, p. 23] and the relations $(*)$ [3, p. 24].

We now consider the relationship between linearly ordered sets and Δ. For a set X, let X^n denote the n-fold product, $n = 0, 1, \ldots$. Let $\sigma_i\colon X^n \to X^{n-1}$, $i = 0, \ldots, n-1$, and $\delta_i\colon X^{n-1} \to X^n$, $i = 1, \ldots, n-1$, be induced respectively by deleting, repeating the $(n-i)$-coordinate, and for fixed points $\underline{x}, \bar{x} \in X$ let δ_0, δ_n be given by $(x_1, \ldots, x_{n-1}) \to (x_1, \ldots, x_{n-1}, \bar{x}), (\underline{x}, x_1, \ldots, x_{n-1})$ respectively. It is readily checked that $[n] \mapsto X^n$, σ_i, $\delta_i \mapsto \sigma_i$, δ_i defines a functor $\Delta \to \text{Set}$; i.e. $X^* = \{X^n\}$ has the structure of a cosimplicial set. If X is linearly ordered with $\underline{x}$ $(\bar{x})$ as the minimum (maximum) point, then setting

$$X_0 = X^0, \quad X_n = \{(x_1, \ldots, x_n) \mid x_1 \leq \cdots \leq x_n\} \subset X^n,\ n = 1, 2, \ldots,$$

clearly defines a subcosimplicial set X_* of X^*. Further, if we use g to stand for an element of $S(n)$, the permutation group on $\{1, 2, \ldots, n\}$, and also for the corresponding map $(x_1, \ldots, x_n) \to (x_{g1}, \ldots, x_{gn})$ then

$$\text{(i)} \qquad \coprod_{g,h} gX_n \cap hX_n \rightrightarrows \coprod_g gX_n \to X^n$$

is a coequalizer, where the coproducts are taken over all $g, h \in S(n)$ and the maps are induced by the obvious inclusions. Also,

$$\text{(ii)} \qquad X_n \cap gX_n \subset X_n \underset{g}{\overset{e}{\rightrightarrows}} X^n$$

is an equalizer where e is the identity of $S(n)$. Furthermore, there exists an integer $n(g)$ and a map (a composite of certain δ_i's)

$$\delta_g\colon X^{n(g)} \longrightarrow X^n$$

that carries $X_{n(g)}$ isomorphically onto $X_n \cap gX_n$.

The explicit determination of $n(g)$ and δ_g can be made in terms of Δ as follows: In case $X = [1]$, the isomorphism

$$i \to (0, \ldots, 0, 1, \ldots, 1)(n - i \text{ 0's and } i \text{ 1's})\colon [n] \to [1]_n \subset [1]^n$$

identifies Δ, or more accurately the underlying cosimplicial set $U\colon \Delta \to \text{Set}$, with the subcosimplicial set $[1]_*$ of $[1]^*$. Under this identification, $[1]_n \cap g[1]_n \subset [1]_n$ is transformed into $\delta_g = \delta_{m_k} \cdots \delta_{m_1}\colon [n(g)] \to [n]$ where $m_1 < \cdots < m_k$ are the points of $[n]$ that correspond to the non-fixed points of g in $[1]_n$ and $n(g) =$

$n-k$. The corresponding map $\delta_g = \delta_{m_k} \cdots \delta_{m_1}\colon X^{n(g)} \to X^n$ then induces an isomorphism

$$\delta'_g\colon X_{n(g)} \to X_n \cap gX_n.$$

If we compose (ii) on the right with $h \in S(n)$ and replace (g, h) by $(g^{-1}h, g)$ and by $(h^{-1}g, h)$ we see that

$$X_n \cap g^{-1}hX_n = X_n \cap h^{-1}gX_n \subset X_n \underset{h}{\overset{g}{\rightrightarrows}} X^n$$

is an equalizer and that

$$\delta'_{g^{-1}h} = \delta'_{h^{-1}g}\colon X_{n(g^{-1}h)} = X_{n(h^{-1}g)} \approx X_n \cap g^{-1}hX_n.$$

The map

$$\coprod_g g^{-1}\colon \coprod_g gX_n \to \coprod_g X_n$$

induces an isomorphism of (i) onto

$$\coprod_{g,h} X_n \cap g^{-1}\, hX_n \rightrightarrows \coprod_g X_n \to X^n$$

and thus, with the aid of the $\delta'_{n(g^{-1}h)}$'s, we obtain a coequalizer

$$\coprod_{g,h} X_{n(g^{-1}h)} \rightrightarrows \coprod_g X_n \to X^n.$$

2.2 PROPOSITION. *There is a bijection between linear orders on X with $\underline{x}$ $(\bar{x})$ as the minimum (maximum) element and subcosimplicial sets X_* of X^* for which*

$$\coprod_{g,h} X_{n(g^{-1}h)} \underset{v}{\overset{u}{\rightrightarrows}} \coprod_g X_n \xrightarrow{w} X^n$$

is a coequalizer, where the (g, h)-component of u (v) is induced by $\delta_{g^{-1}h}$ with codomain the g-th (h-th) copy of X_n in $\coprod_g X_n$, and the g-component of w is g. Furthermore, this bijection identifies X_n and $\{(x_1, \ldots, x_n) \mid x_1 \leq \cdots \leq x_n\}$.

Proof. In view of the above discussion, it is sufficient to show that the subcosimplicial sets mentioned in the proposition are defined by linear orders. Given X_*, $X_2 \subset X^2$ defines a relation $\leq$ on X satisfying:

(1) $x \leq y$ or $y \leq x$ (since w is onto for $n = 2$).

(2) $x \leq y$ and $y \leq x$ imply $x = y$. (Since the identity induced map $\coprod_g X_n \to X_n$ coequalizes u and v, there is an $r\colon X^n \to X_n$ such that $rg = \mathrm{id}$, $g \in S(n)$. If (x, y), $(y, x) \in X_2$, then $\tau(y, x) = (x, y)$ implies $(y, x) = r\tau(y, x) = r(x, y) = (x, y)$; i.e. $x = y$.)

(3) $x \leq y$ and $y \leq z$ imply $x \leq z$. (If $x < y$, $y < z$, and $z < x$ then since w is onto $(n = 3)$, some permutation of (x, y, z) is in X_3. But this contradicts the fact that σ_i (deletion of a coordinate) maps $X_3 \to X_2$.)

(4) $\underline{x}$ $(\bar{x})$ is the minimum (maximum) element (consider the form of $\delta_i\colon X \to X_2$, $i = 0, 2$).

Thus $(X, \leq)$ is a linearly ordered set. If $\bar{X}_*$ is the subcosimplicial set of X_* determined by the order then, by induction, $X_n \subset \bar{X}_n$. Since w is onto, some permutation $g(y)$ of any $y \in \bar{X}_n$ is in X_n. This implies $g(y) = y$ and thus $X_n = \bar{X}_n$. This shows 2.2.

For $X = [1]$, the coequalizer of 2.2 can be identified with the coequalizer (presentation of $[1]^n$)

$$\coprod_{g,h} [n(g^{-1}h)] \rightrightarrows \coprod_{g} [n] \to [1]^n,$$

which is essentially diagram $(*)$ of [3, p. 34] with $E = [1]^n$. The argument in [3, p. 35] (there $\Delta(n)$ denotes $R_0[n]$) gives the presentation

$$\coprod_{g,h} R_0[n(g^{-1}h)] \rightrightarrows \coprod_{g} R_0[n] \to (R_0[1])^n$$

in B_0. If D' is exact there is a corresponding presentation of $(R[1])^n$ in B.

2.3 Lemma. *Suppose θ: $F \to G$; $A \to B$ is a natural transformation of functors and α is a retract of β in A (there are maps*

$$\alpha \xrightarrow{i} \beta \xrightarrow{r} \alpha,$$

$ri = \mathrm{id}$). *Then if $\theta(\beta)$ is an isomorphism, so is $\theta(\alpha)$: $F(\alpha) \to G(\alpha)$.*

Proof. The inverse to $\theta(\alpha)$ is easily seen to be $F(r)\theta(\beta)^{-1}G(i)$.

A category A is said to have limits for all functors $J \to A$ if the diagonal functor Δ_A: $A \to A^J$ $(\Delta_A(a)(j) = a)$ has a right adjoint $\lim_A$: $A^J \to A$ [6, p. 229]. If the categories A and B have limits for functors on J we say a functor F: $A \to B$ preserves the limit of $\alpha \in A^J$ if $\theta(\alpha)$ is an isomorphism, where the natural transformation θ: $F \lim_A \to \lim_B F^J$ is the adjoint of

$$F^J\varepsilon\colon \Delta_B F \lim_A = F^J \Delta_A \lim_A \to F^J$$

for ε: $\Delta_A \lim_A \to I$, the counit of the adjunction. In view of 2.3, if α is a retract of β in A^J then F preserves $\lim_A \alpha$ if it preserves $\lim_A \beta$. If J is a set, then $\lim_A$ is product and thus F preserves $\prod_j a_j$ if it preserves $\prod_j b_j$, for a_j a retract of b_j, $j \in J$.

Recall [2, p. 7] that V-limits are exactly those Set-limits that are preserved by the representables. Thus in any V-category A that is tensored over V, V-limits and limits coincide since $A(a, -)$ has a left adjoint.

2.4 Theorem. *Under the previous assumptions on V, A and Δ, the left Kan extension L_T of T: $\Delta \to A$ along the right Yoneda functor preserves finite products if and only if*

$$(*) \qquad \coprod_{g,h} T[n(g^{-1}h)] \rightrightarrows \coprod_{g} T[n] \to (T[1])^n$$

(with obvious maps) is a coequalizer.

Proof. Since L_T is cocontinuous and $L_T R = T$,

$$L_T(\coprod R[n(g^{-1}h)] \rightrightarrows \coprod R[n] \to (R[1])^n)$$
$$= \coprod T[n(g^{-1}h)] \rightrightarrows \coprod T[n] \to L_T((R[1])^n)$$

is a coequalizer. Thus $(*)$ is a coequalizer if and only if

$$L_T((R[1])^n) \approx (L_T R[1])^n = (T[1])^n.$$

The argument given in the verification of (2) in the proof of 2.2, but with $X = R[1]$, $X_n = R[n]$, shows that $R[n]$ is a retract of $(R[1])^n$, $n = 1, 2, \ldots$. Since $R[0]$ is also a retract of $R[1]$, it follows from 2.3, as above, that L_T preserves the product of representables if and only if $(*)$ is a coequalizer. The result then follows from 2.1.

2.5 COROLLARY. *The full subcategory of the category of cosimplicial sets determined by those* $T\colon \Delta \to$ Set *for which* $\mathrm{Lan}_R\, T$ *is finite product preserving is equivalent to the category of linearly ordered, bounded sets in which the morphisms are the non-decreasing maps that preserve the endpoints.*

Proof. This follows from 2.2 and 2.4 with $V = A =$ Set.

3. Algebraic topologies

Let V be k-Top, the category of compactly generated spaces in the sense of Vogt [1, p. 229], [9]. k-Top contains the (colimit deficient) category $T_2 k$-Top of compactly generated Hausdorff spaces studied by Steenrod [8]. k-Top is small complete and cocomplete [1, Prop. 1.3, p. 229] and is cartesian closed [3, Prop. 1.9, p. 230]. We view Δ as a k-Top-category by means of the obviously exact functor D: Set $\to k$-Top (discrete topology). Define the category of algebraic topologies, Alg T, in k-Top to be the full subcategory of the category of cosimplicial spaces determined by those $T\colon \Delta \to k$-Top for which $\phi \neq T[1]$ is connected and $\mathrm{Lan}_R\, T$ preserves finite products. It is clear that the various notions and constructions discussed in Section 1 are present once T is given. For example, T-homotopy and T-path space are now defined in terms of $T[1]$ in place of the unit interval and the T-n-simplex is now $T[n]$ in place of the affine n-simplex. The first condition on T insures that T-homotopy is not degenerate and the second condition insures that T-topological realization $(= \mathrm{Lan}_R\, T)$ preserves finite products and consequently converts simplicial homotopy [3, p. 57] into T-homotopy. In particular, the T-n-simplex $T[n]$ is T-contractible, the contraction being induced by $(i, j) \to ij$: $[n]x[1] \to [n]$.

Each property P of spaces defines a category PAlg T: namely, the full subcategory of Alg T determined by those T for which $T[1]$ has property P. We thus have, for example, the categories T_0 Alg T, T_2 Alg T, and Finite T_0 Alg T of T_0, Hausdorff, and finite T_0 algebraic topologies respectively.

4. Intervals

By an *interval* X in k-Top we mean a linearly ordered, bounded, non-empty set X equipped with a compactly generated topology for which the n-fold product X^n (in k-Top), $n = 1, 2, \ldots$, has the weak (coinduced) topology relative to the family $\{gX_n\}$, $g \in S(n)$, of subsets of X^n. Let Int be the category of intervals in k-Top in which the morphisms are taken to be the continuous, non-decreasing, endpoint preserving maps. As above, each property P of spaces defines a full subcategory PInt of Int.

If $X \in$ Int then

$$\coprod_{g,h} X_n \cap g^{-1}hX_n \rightrightarrows \coprod_{g} X_n \to X^n$$

is clearly a coequalizer in Top (topological spaces) where $X_n \cap g^{-1}X_n \subset X_n \subset X^n$ have the induced topology. Again, as in the verification of (2) in the proof of 2.2, X_n is a retract of X^n and consequently the induced and coinduced (by the retract) topologies on X_n coincide; i.e. $X_n \in k$-Top [1, Cor. 1.4, p. 229], $n = 1, 2, \ldots$. Since the map $\delta_g = \delta_{m_k} \cdots \delta_{m_1}$: $X^{n(g)} \to X^n$ has $\sigma_{m_1} \cdots \sigma_{m_k}$ as a left inverse, $\delta_{g^{-1}h}$ induces a homomorphism of $X_{n(g^{-1}h)}$ onto $X_n \cap g^{-1}hX_n$. Thus

$$\coprod_{g,h} X_{n(g^{-1}h)} \rightrightarrows \coprod_{g} X_n \to X^n$$

is a coequalizer in Top that lies in k-Top, and therefore [1, Prop. 1.3, p. 229] is a coequalizer in k-Top. In view of 2.4, then, $X \mapsto T_X: \Delta \to k$-Top with $T_X[n] = X_n$ with the induced topology from $X^n \in k$-Top clearly defines a functor Int $\to$ Alg T. On the other hand, since the underlying functor U: k-Top $\to$ Set is exact (it has both a left (discrete topology) and a right (indiscrete topology) adjoint) if $T \in$ Alg T then $UT[1]$ is a linearly ordered, bounded set by 2.4 and 2.2. Further, since the inclusion k-Top $\subset$ Top is cocontinuous and since $T[n]$ is a retract of $(T[1] = X)^n$ (as above), it follows that

$$\coprod_{g} g(X_n) \approx \coprod_{g} X_n \approx \coprod_{g} T[n] \to X^n$$

is a quotient map; i.e. $X \in$ Int. It is easily seen, then, that $T \mapsto T[1]$ defines a functor Alg $T \to$ Int that together with the functor $X \mapsto T_X$, determines an equivalence of Alg T and Int. More generally:

4.1 THEOREM. *The functor $T \mapsto T[1]$ induces an equivalence of the categories* PAlg T *and* PInt *for any property* P *of spaces.*

We conclude this section with a lemma upon which most of the remaining results depend.

4.2 LEMMA. (a) *A space Y has the weak topology relative to a family $\{K_g\}$, $g \in G$, of subsets if for all $x \in Y$ there exist $g(i) \in G$, $i = 1, \ldots, n$ and a neighborhood N of x such that $x \in \bigcap_{i=1}^n K_{g(i)}$ and $N \subset \bigcap_{i=1}^n K_{g(i)}$.*

(b) *Let Y have the weak topology relative to a cover $\{gK\}$, $g \in G$, where $K \subset Y$ and G is a group of continuous automorphisms of Y such that $x, g(x) \in K$ implies $g(x) = x$. If, for N a neighborhood of $a \in K$, there are $g_0 \in G$ and $b \in N$ with $g_0(b) \in K - N$ then there is a neighborhood of a missing b.*

(c) *If $q: X \to Y$ is an onto, non-decreasing map between linearly ordered sets then Y, with the q-coinduced topology, is an interval if X is. (We call such a Y an order quotient of X.)*

Proof. (a) Suppose that $W \cap K_g$ is open in K_g for all $g \in G$. If, for $x \in W$, N and $g(i)$ are as in (a) then $x \in W \cap K_{g(i)} = V_i \cap K_{g(i)} \subset W$ for V_i an open set in Y, $i = 1, \ldots, n$. Hence

$$x \in \left(\bigcap_{i=1}^{n} V_1\right) \cap \left(\bigcup_{i=1}^{n} K_{g(i)}\right) \subset \bigcup_{i=1}^{n} (V_i \cap K_{g(i)}) \subset W$$

and consequently $x \in V = (\bigcap_{i=1}^{n} V_i) \cap N \subset W \cap N \subset W$. Since V is open the result follows.

(b) Since Y has the weak topology relative to $\{gK\}$ and

$$\left[\bigcup_h h(N \cap K)\right] \cap gK = \bigcup_h (hN \cap hK \cap gK) = \bigcup_h (gN \cap hK \cap gK)$$
$$= gN \cap gK \cap \left(\bigcup_h hK\right) = gN \cap gK,$$

$W = \bigcup_h h(N \cap K)$ is a neighborhood of a. If $b \in W$ then $b = h(z)$ for $z \in N \cap K$. Hence $g_0(b) = (g_0 h)(z) \in K$ and consequently $g_o(b) = z \in N$, a contradiction.

(c) Since $q^n: X^n \to Y^n$ is a quotient map [1], Cor. 1.11, p. 230], Y^n has the weak topology relative to the sets $q^n(gX_n) = g(q^n X_n) = gY_n$.

5. T_0-intervals

The first result of this section shows the relationship between $\mathrm{PT_0Int}$ and PInt where P is a divisible (preserved by quotient [5, p. 133]) property.

5.1 THEOREM. *$\mathrm{PT_0Int}$ is a reflective* [6, p. 89] *subcategory of* PInt *for any divisible property* P *of spaces.*

Proof. If $q: X \to Q(X)$ is the quotient map determined by the equivalence relation R on $X \in \mathrm{PInt}$ given by $x \mathrel{R} y$ if x and y have exactly the same neighborhoods, then $Q(X)$ is obviously T_0. If the equivalence classes of R are convex ($x < y < z$ and $x \mathrel{R} z$ then $x \mathrel{R} y$) then it is readily seen that $q^2(X_2) \subset Q(X)^2$ is a linear order on $Q(X)$ relative to which q is non-decreasing, and consequently, by 4.2(c) and the fact that P is divisible, $Q(X) \in \mathrm{PT_0Int}$. However, if $x \mathrel{R} z$ but not $x \mathrel{R} y$ for $x < y < z$ then there is either a neighborhood N_x of x missing y or a neighborhood N_y of y missing x. In the first (second) case there is, by 4.2(b) with $K = X_2 \subset X^2 = Y$, $G = S(2)$, $a = (y, z) \in N = X \times N_x$ $(N_y \times X)$, $b = (y, x) \in N$ and $g_0(b) = (x, y) \in X_2 - N$, a neighborhood U of (y, z) missing (y, x) and consequently a neighborhood $f_y^{-1}(U)$ of z missing x, where $f_y\colon p \mapsto (y, p)\colon X \to X^2$. This contradicts $x \mathrm{R} z$. Finally, since the fibers of q are indiscrete, any continuous function f of X into a T_0 space uniquely factors through q, and in particular this factorization is in $\mathrm{PT_0Int}$ if $f \in \mathrm{PInt}$. This gives the result.

A point x in a linearly ordered, bounded, connected k-space X is called a *cut point* if at least one segment (denoted by $[0, x\}$, $\{x, 1]$ respectively) in each pair $([0, x), [0, x])$ and $((x, 1], [x, 1])$ of segments is non-empty and open, where 0 (1) is the minimum (maximum) endpoint of X. Recall [4, p. 150] that the unit interval can be characterized in terms of cut points. The next result shows that a cut point property determines some, but not all (see 6.3), of the objects of $\mathrm{T_0Int}$.

5.2 THEOREM. *If each non-endpoint of a linearly ordered, bounded, non-empty, connected k-space X is a cut point then* (a) $X \in \mathrm{T_0Int}$ *and* (b) *X is T_2 if it is T_1.*

Proof. Define neighborhoods N_x, N_y of x, y respectively, for $x < y$, as follows: If there exists z, $x < z < y$, $N_x = [0, z\}$, $N_y = \{z, 1]$, otherwise $N_x = [0, x]$, $N_y = \{x, 1]$ if $[0, x]$ is open and $N_x = [0, y\}$, $N_y = (x, 1]$ if $[0, x)$ is open (Note that $(x, 1]$ is open if $[0, x)$ is open, otherwise $[0, x)$ and $[x, 1]$ would separate X). Since $y \notin N_x$ in the first two cases and $x \notin N_y$ in the last case, X is T_0. If X is T_1 then $[0, x)$ and $(x, 1]$ are open and the topology on X contains the order topology [5, p. 57]. Thus X is T_2 and (b) follows. To show $X \in \mathrm{Int}$ we verify the conditions of 4.2(a) for $Y = X^n$, $K_g = g(K)$, $G = S(n)$, $K = X_n$, $n = 1, 2, \ldots$. If $x \in X_n$ then

$$x = (a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k)$$

where there are m_i $(m_i > 0)$ copies of a_i, $i = 1, \ldots, k$, and $a_1 < \cdots < a_k$. Pick neighborhoods N_i of a_i so that $N_i x N_{i+1} \subset X_2$ (note that the neighborhoods N_x, N_y defined above satisfy $N_x x N_y \subset X_2$). Any y in the neighborhood $N = N_1^{m_1} x \cdots x N_k^{m_k}$ of x has the form

$$y = (y_1^1, \ldots, y_{m_1}^1, \ldots, y_1^k, \ldots, y_{m_k}^k) \quad \text{with } y_j^i \in N_i \text{ and } y_{m_i}^i \leq y_1^{i+1}.$$

If $g_i \in S(m_i)$ is such that $(y_{g_i(1)}^i, \ldots, y_{g_i(m_i)}^i) \in X_{m_i}$ then

$$g = (g_1, \ldots, g_k) \in S(m_1) \times \cdots \times S(m_k) = H(n) \subset S(n)$$

satisfies $g(y) \in X_n$. Hence $N \subset \bigcup_{g \in H(n)} K_g$ and, since $g(x) = x$ for $g \in H(n)$, $x \in \bigcap_{g \in H(n)} K_g$. Thus 4.2(a) holds for $x \in X_n$, which is clearly sufficient to give (a).

In the proof of 5.2(b) the topology on X was seen to contain the order topology. More generally:

5.3 THEOREM. *If $X \neq \phi$ is a linearly ordered, bounded, connected k-space then $X \in \mathrm{T_2Int}$ if and only if the topology of X contains the order topology.*

Proof. If X contains the order topology, then X satisfies the conditions of 5.2 and thus $X \in \mathrm{T_2Int}$. On the other hand, the equations

$$\mathring{X}_2 = \{(x, y) \mid x < y\} = \mathring{X}_2 \cap X_2 = (X^2\text{-diagonal}) \cap X_2$$

and

$$\mathring{X}_2 \cap \tau X_2 = \phi,$$

where $\tau(x, y) = (y, x)$, show that $\mathring{X}_2$ is open if X is T_2(X^2-diagonal is then open) and if X^2 has the weak topology relative to $\{X_2, \tau X_2\}$. The result then follows since $[0, x) = f_x^{-1}(\mathring{X}_2)$ and $(x, 1] = f_x^{-1}(\tau\mathring{X}_2)$ are open, where f_x: $y \to (y, x)$: $X \to X^2$.

6. Finite intervals

Since the property F of being finite is divisible, $\mathrm{FT_0Int}$ is a reflective subcategory of PInt by 5.1. Intuitively, any finite interval can be obtained from a finite T_0 interval by replacing each point by a non-empty, finite, linearly ordered indiscrete space. We now completely determine $\mathrm{FT_0Int}$. To this end let Δ_* be the extension of Δ obtained by adding $[-1] = \phi$. Recall [6, p. 47] that the comma category $U_* \downarrow \{0, 1\}$, where U_*: $\Delta_* \to$ Set is the underlying functor, has as objects all functions f: $[n] \to \{0, 1\}$ and as maps $f \to g$ all h: $[n] \to [m]$ in Δ_* such that $gh = f$.

6.1 THEOREM. $\mathrm{FT_0Int}$ *is equivalent to* $(U_* \downarrow \{0, 1\})^{\mathrm{op}}$.

Proof. We begin with a preliminary result.

6.2 LEMMA. *The following statements are equivalent:*

(a) *X is a finite order quotient of the unit interval.*
(b) $X \in \mathrm{FT_0Int}$.
(c) *X is a finite, non-empty, linearly ordered, locally convex k-space such that if M_i are the minimum neighborhoods of x_i for $x_1 < x_2 < x_3$ then $M_1 \cap M_3 \subset M_2$.*

Proof. Essentially, 4.2(c) shows that (a) implies (b). Given (b), an argument analogous to the one in the proof of 5.1 showing that the equivalence classes of R are convex shows any minimum open set in X is convex. Further, if $y \in M_1 \cap M_3 - M_2$ and $y < x_2$ then by 4.2(b), with $K = X_3 \subset X^3 = Y$, $G = S(3)$,

$$a = (x_2, x_2, x_3) \in N = M_2 \times M_2 \times M_3, \quad b = (x_2, x_2, y) \in N$$

and $g_0(b) = (y, x_2, x_2) \in X_3 - N$, N is not the minimum neighborhood of a, a contradiction. Since $x_2 < y$ leads similarly to a contradiction, it follows that (b) implies (c). To show (c) implies (a) we proceed by induction on the number of points (length) in X. Suppose (c) implies (a) for all X of length $\leq n$ (trivial if $n = 1$). If $X = \{x_0, \ldots, x_n\}$ satisfies (c) then by local convexity, the minimum neighborhood M of x_n contains x_{n-1} unless $M = \{x_n\}$. By induction hypothesis there is a non-decreasing quotient map

$$q': [0, 1] \to X' = \{x_0, \ldots, x_{n-1}\}.$$

Define a continuous, non-decreasing, onto map q'': $[0, 2] \to X$ by $q'' = q'$ on $[0, 1]$, $q''(1, \frac{3}{2}) = x_{n-1}$, $q''(\frac{3}{2}, 2] = x_n$ and $q''(\frac{3}{2}) = x_{n-1}, x_n$ if $x_{n-1} \in M, \notin M$ respectively. To show q'' is a quotient map it is sufficient to show that W is open whenever $(q'')^{-1}(W)$ is open. If $(q'')^{-1}(W)$ is open then $W' = W \cap X'$ is open in X'. Let U be the minimum open set in X such that $W' = U \cap X'$. Clearly W is open if $W = U$. If $W \neq U$ then either (1) $W = W'$ and $U = W' \cup \{x_n\}$ or (2) $W = W' \cup \{x_n\}$ and $U = W'$. If (1) holds and X' is open in X then W is open. If X' is not open in X then necessarily $\{x_n\}$ is open and, since $(q'')^{-1}(W)$ is open, $x_{n-1} \notin W$. This, together with local convexity, contradicts the minimality of U. If (2) holds then W is open if $\{x_n\}$ is open, otherwise $\{x_n\}$ is necessarily closed and, since $(q'')^{-1}(W)$ is open, $x_{n-1} \in W$ and thus W contains M_1, the minimum neighborhood of x_{n-1}. If W contains M then clearly W is open. However, if $M \subsetneq W$ and M_2 is the minimum neighborhood of some $y \in M - W$ then $y \leq x_{n-1} < x_n$ and consequently $y \in M \cap M_2 \subset M_1 \subset W$, a contradiction. Setting $q(t) = q''(2t)$ then gives the desired quotient map q: $[0, 1] \to X$.

The image category Δ' of the functor $\mathrm{FT_0Int} \to \Delta_*$ induced by $X \mapsto [n]$ for X of length $n + 1$ has all of the objects of Δ_* and all of the maps of Δ_* that preserve endpoints (delete $\delta_0, \delta_n : [n-1] \to [n]$). For h: $[m] \to [n]$ in Δ', $h^*(i) = \max\{h^{-1}[0, i]\}$ clearly defines a map h^*: $[n-1] \to [m-1]$ in Δ_* for which $h^{-1}[0, i] = [0, h^*(i)]$ and $h^{-1}(i, n] = (h^*(i), n]$, $i \in [n]$. Further, for g: $[n-1] \to [m-1]$ in Δ_*, $g'(j) = \min\{g^{-1}[j, m-1]\}$ $(= n$ if $g^{-1}[j, m-1] = \phi)$, $j \in [m-1]$ and $g'(m) = n$ defines a map $[m] \to [n]$ in Δ' such that $h = (h^*)'$ and $g = (g')^*$. In fact, $[n] \mapsto [n-1]$, $h \mapsto h^*$ defines a contravariant functor $\Delta' \to \Delta_*$ that gives an isomorphism of Δ' onto Δ_*^{op}. In view of 6.2, the topologies on $[n]$ making it a T_0 interval are precisely those coinduced by the maps $q(f)$: $[0, 1] \to [n]$, where $q(f)(x_{i-1}, x_i) = i$, $i \in [n]$ for $x_i = (i + 1)/(n + 1) \in [0, 1]$, $i = -1, 0, \ldots, n$ and $q(f)(x_i) = i + f(i)$, $i \in [n-1]$, $q(f)(x_{-1}) = 0$, $q(f)(x_n) = n$ for f: $[n-1] \to \{0, 1\}$. It is readily seen that the $q(f)$-coinduced topology on $[n]$ (denote $[n]$ with this topology by $[n]_f$) has

$$\{[0, i], (j, n] \mid f(i) = 1, f(j) = 0, i, j \in [n-1]\}$$

as a subbase. Further, h: $[m] \to [n]$ in Δ' determines a continuous map $[m]_g \to [n]_f$ if and only if $h^{-1}[0, i] = [0, h^*(i)]$ is open (i.e. $gh^*(i) = 1$) when $[0, i]$ is open (i.e. $f(i) = 1$), and $h^{-1}(j, n] = (h^*(j), m]$ is open $(gh^*(j) = 0)$ when $(j, n]$ is open $(f(j) = 0)$; that is, if and only if $gh^* = f$. Thus the contravariant functor $[n]_f \mapsto f$, $h \mapsto h^*$ define an isomorphism from the skeletal subcategory of $\mathrm{FT_0Int}$

determined by the intervals $[n]_f$ to the category $(U_* \downarrow \{0, 1\})^{\text{op}}$ and the result follows.

Various numerical results can be obtained from 6.1 and 6.2.

6.3 COROLLARY. *The number of non-isomorphic* (T_0) *intervals of length* n *is* $3^{n-1}(2^{n-1})$, $n = 1, 2, \ldots$. *Of those* T_0 *intervals*, $2\binom{n-2}{k}$ *have exactly* k *cutpoints for* $0 \leq k \leq n-2$. ($\binom{a}{b}$ *denotes the binomial coefficient.*) *Hence there are exactly two* T_0 *intervals of each length* $n \geq 2$ *satisfying the cutpoint condition* $(k = n-2)$ *of* 5.2.

Proof. The number of distinct T_0 intervals of length n is clearly 2^{n-1} since, by 6.1, 6.2, this number coincides with the number of functions $[n-2] \to \{0, 1\}$. Further, the distinct intervals X of length n for which $Q(X)$ (see proof of 5.1) is a fixed T_0 interval of length m are easily seen to be obtained from $Q(X)$ by replacing each of its m points by a non-empty indiscrete space. The number of such replacements is the number of ways one can write n as a sum of m positive integers (order counts). Since this number is $\binom{n-1}{m-1}$ it follows that there are $\binom{n-1}{m-1}2^{m-1}$ intervals X for which $Q(X)$ has length m and consequently there are $\sum_{m=1}^{n} \binom{n-1}{m-1}2^{m-1} = 3^{n-1}$ intervals of length n. Finally, a point $i \in [n]_f$ is a cut point if and only if $f(i-1) + f(i) = 1$ and $0 < i < n$, for $f\colon [n-1] \to \{0, 1\}$. Thus $[n]_f$ has k cut points if and only if the sequence f "alternates" k times. The number of such sequences that begin with 0 is $\binom{n-1}{k}$ and thus the total number of such sequences is $2\binom{n-1}{k}$. Since $[n]_f$ has length $n+1$, it follows that there are $2\binom{n-2}{k}$ distinct T_0 intervals of length n with exactly k cut points. The intervals satisfying the cut point condition of 5.2, then, are exactly those determined by the alternating sequences.

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