ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 3, Fall 1981

ON THE QUASI-SIMILARITY OF HYPONORMAL CONTRACTIONS

BY

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The main purpose of this paper is to show that if T and S are hyponormal contractions with finite defect indices, then T is quasi-similar to S if and only if their unitary parts are unitarily equivalent and their completely non-unitary (c.n.u.) parts are quasi-similar. As it turns out the proof depends on hyponormality only through the fact that c.n.u. hyponormal contractions are of class $C_{\cdot 0}$. This is where the contraction theory of Sz.-Nagy and Foiaş comes into play. Along the way we also obtain other results for C_{10} contractions which are interesting on their own. The main result in this paper partially generalizes a result of Hastings [4] for the quasi-similarity of subnormal contractions. The research here is also inspired by the recent work of Conway [1].

Only bounded linear operators on complex, separable Hilbert spaces will be considered in this paper. For operators T_1 and T_2 on H_1 and H_2, respectively, $T_1 \prec_i T_2$ denotes that T_1 is *injected into* T_2, that is, there is an injection $X: H_1 \to H_2$ which intertwines T_1 and T_2. If X also has dense range, then we call X a *quasi-affinity* and say T_1 is a *quasi-affine transform* of T_2 (denoted by $T_1 \prec T_2$). $T_1 \prec_{ci} T_2$ denotes that T_1 is *completely injected into* T_2, that is, there is a family $\{X_\alpha\}$ of injections $X_\alpha: H_1 \to H_2$ intertwining T_1 and T_2 such that $H_2 = \bigvee_\alpha X_\alpha H_1$. T_1 and T_2 are *quasi-similar* ($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$. A contraction T ($\|T\| \leq 1$) is *of class* $C_{\cdot 0}$ (resp. $C_{0\cdot}$) if $T^{*n}x \to 0$ (resp. $T^n x \to 0$) for all x. T is of class $C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^n x \not\to 0$ (resp. $T^{*n}x \not\to 0$) for all $x \neq 0$. $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$ for $\alpha, \beta = 0, 1$. The *defect indices* of a contraction T are, by definition, $d_T = \text{rank } (I - T^*T)^{1/2}$ and $d_{T^*} = \text{rank } (I - TT^*)^{1/2}$. For an arbitrary operator T on H, let μ_T denote its *multiplicity*, that is, the least cardinal number of a set K of vectors in H such that $H = \bigvee_{n=0}^{\infty} T^n K$. We use S_α to denote the unilateral shift with multiplicity α, $\alpha = 1, 2, \ldots, \infty$. For properties of various classes of contractions, the readers are referred to [10]

We start with the following lemma.

LEMMA 1. *Let T be a contraction on H.*

(1) *$T \prec V$ for some isometry V if and only if T is of class $C_{1\cdot}$.*
(2) *If $T \prec S_\alpha$ for some cardinal number α, then T is of class C_{10}.*

Received October 15, 1979.

[1] This research was partially supported by the National Science Council of the Republic of China while the author was visiting Indiana University during the year 1979–1980.

Proof. (1) Assume that V is acting on K and $X: H \to K$ is a quasi-affinity intertwining T and V. For any $x \in H$ with $T^n x \to 0$ as $n \to \infty$, we have $V^n X x = X T^n x \to 0$. Since V is of class $C_{1\cdot}$, this implies that $Xx = 0$, whence $x = 0$. Thus T is of class $C_{1\cdot}$. The converse was proved by Sz.-Nagy and Foiaş [10, pp. 71–72] and also by Douglas [3].

(2) Assume that S_α is acting on K and $Y: H \to K$ is a quasi-affinity intertwining T and S_α. For any $x \in K$, we have $T^{*n} Y^* x = Y^* S_\alpha^{*n} x \to 0$. Since Y^* has dense range in H, for any $\varepsilon > 0$ and $y \in H$ there is some $x \in K$ such that $\|y - Y^* x\| < \varepsilon$. Hence

$$\|T^{*n} y\| - \|T^{*n} Y^* x\| \leq \|T^{*n} y - T^{*n} Y^* x\| \leq \|y - Y^* x\| < \varepsilon \quad \text{for all } n.$$

As $n \to \infty$, we have $\|T^{*n} Y^* x\| \to 0$ and therefore $\|T^{*n} y\|$ can be made arbitrarily small for n sufficiently large. This, together with (1), shows that T is of class C_{10}.

The converse of (2) in the preceding lemma is false, that is, there are C_{10} contractions which are not the quasi-affine transform of any unilateral shift. Indeed, if C is the Cesaro operator defined on l^2 then $I - C$ is such a contraction (cf. [7, Theorem 1 and p. 212]).

LEMMA 2. (1) *If $S_\alpha \prec S_\beta$, where $1 \leq \alpha, \beta \leq \infty$, then $\alpha = \beta$.*
(2) *If $S_\alpha \prec_i S_\beta$, where $1 \leq \alpha, \beta \leq \infty$, then $\alpha \leq \beta$.*

Proof. (1) $S_\alpha \prec S_\beta$ implies that $\beta = \mu_{S_\beta} \leq \mu_{S_\alpha} = \alpha$. If $\beta = \infty$, then $\alpha = \beta = \infty$. Hence we may assume that $\beta < \infty$. Since S_β is of class $C_{\cdot 0}$ and $S_\alpha \prec_i S_\beta$, a result of Sz.-Nagy and Foiaş [11, Theorem 5] implies that $\alpha \leq \beta$. Hence $\alpha = \beta$.

(2) Assume that S_α and S_β act on K_1 and K_2, respectively, and let X: $K_1 \to K_2$ be an injection intertwining S_α and S_β. Then $S_\alpha \prec S_\beta | (\operatorname{ran} X)^-$. Since $S_\beta | (\operatorname{ran} X)^-$ is also a unilateral shift, the multiplicity of $S_\beta | (\operatorname{ran} X)^-$ is α by (1). But it must also not exceed β, the multiplicity of S_β (cf. [10, p. 198]). This completes the proof.

It was shown by Sz.-Nagy [9, Theorem 3] that if T is a C_{10} contraction with $d_T < \infty$ then $S_\alpha \prec_{ci} T \prec S_\alpha$, where $\alpha = d_{T^*} - d_T$. (Note that for C_{10} contractions we always have $d_T \leq d_{T^*}$.) The next lemma says that S_α is the only unilateral shift of which T is a quasi-affine transform.

LEMMA 3. *Let T be a C_{10} contraction with $d_T < \infty$. Assume that $T \prec S_\alpha$ for some cardinal number α. Then $\alpha = d_{T^*} - d_T$.*

Proof. Let $m = d_T$ and $n = d_{T^*}$. Since $S_{n-m} \prec_i T$, we have $S_{n-m} \prec_i S_\alpha$. It follows from Lemma 2 that $n - m \leq \alpha$. If $n = \infty$, then $\alpha = n - m = \infty$. Hence we may assume that $n < \infty$. Consider the functional model of T, that is, consider T being defined on $H \equiv H_n^2 \ominus \Theta_T H_m^2$ by $Tf = P(e^{it} f)$, where for any positive integer k, H_k^2 denotes the usual Hardy space of $\mathbf{C}^k$-valued functions defined on the unit circle of $\mathbf{C}$, Θ_T denotes the characteristic function of T and P denotes the (orthogonal) projection onto H (cf. [10]). Let $X: H_n^2 \to H$ be the contraction defined by $Xf = Pf$ for $f \in H_n^2$ and let $Y: H \to H_\alpha^2$ be the quasi-affinity intertwining T and S_α. We may assume that Y is a contraction. It is easily seen that X intertwines S_n and T and has dense range in H. Hence YX intertwines S_n and S_α, has dense range in H_α^2 and $\ker YX = \Theta_T H_m^2$. Therefore, there exists a contractive outer function $\{\mathbf{C}^n, \mathbf{C}^\alpha, \Phi(\lambda)\}$ such that $YXf = \Phi f$ for $f \in H_n^2$ (cf. [10, p. 195]). Since for almost all t, the operator $\Phi(e^{it}): \mathbf{C}^n \to \mathbf{C}^\alpha$ has dense range in $\mathbf{C}^\alpha$ (cf. [10, p. 191]), we infer that $\alpha < \infty$ and $\dim \ker \Phi(e^{it}) = n - \alpha$. On the other hand, that Θ_T is an inner function implies that $\Theta_T(e^{it})$: $\mathbf{C}^m \to \mathbf{C}^n$ is an isometry for almost all t (cf. [10, p. 190]). Hence for a fixed t, the subspace $K \equiv \{g(e^{it}) \in \mathbf{C}^n: g \in \Theta_T H_m^2\}$ has dimension m. But $K = \{g(e^{it}) \in \mathbf{C}^n: g \in \ker YX\} \subseteq \ker \Phi(e^{it})$. It follows that $m \le n - \alpha$ and hence $\alpha = n - m$.

The next lemma is a generalization of Lemma 2, (1) and Lemma 3.

LEMMA 4. *Let T_1 and T_2 be C_{10} contractions with $d_{T_1}, d_{T_2} < \infty$. Assume that $T_1 \prec T_2$. Then*

$$d_{T_1*} - d_{T_1} = d_{T_2*} - d_{T_2}.$$

Proof. Let $m = d_{T_2}$ and $n = d_{T_2*}$. By [9], Theorem 3, we have $T_2 \prec S_{n-m}$. Hence $T_1 \prec S_{n-m}$ and Lemma 3 implies that $n - m = d_{T_1*} - d_{T_1}$, completing the proof.

THEOREM 5. *Let T be a C_{10} contraction on H with $d_T, d_{T*} < \infty$. Assume that $X \in \{T\}'$ has dense range. Then X must be an injection.*

Recall that for an operator T, we use $\{T\}'$ to denote its *commutant*.

Proof. Let $m = d_T$ and $n = d_{T*}$. Let $\Theta_T = \Theta_2 \Theta_1$ be the regular factorization of the characteristic function $\{\mathscr{H}_1, \mathscr{H}_2, \Theta_T(\lambda)\}$ of T into the product of $\{\mathscr{H}_1, \mathscr{F}, \Theta_1(\lambda)\}$ and $\{\mathscr{F}, \mathscr{H}_2, \Theta_2(\lambda)\}$ which corresponds to the triangulation

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$$

on $H = \ker X \oplus (\operatorname{ran} X^*)^-$ (cf. [10, p. 288]). Assume that the intermediate space $\mathscr{F}$ of $\Theta_T = \Theta_2 \Theta_1$ has dimension l. Since the characteristic function of T_2 is the purely contractive part $\{\mathscr{F}^0, \mathscr{H}_2^0, \Theta_2^0(\lambda)\}$ of $\{\mathscr{F}, \mathscr{H}_2, \Theta_2(\lambda)\}$, we deduce that

$$d_{T_2} = \dim \mathscr{F}^0 = \dim \mathscr{F} - k = l - k$$

and

$$d_{T_2*} = \dim \mathscr{H}_2^0 = \dim \mathscr{H}_2 - k = n - k,$$

where $k = \dim(\mathscr{F} \ominus \mathscr{F}^0) = \dim(\mathscr{H}_2 \ominus \mathscr{H}_2^0)$ (cf. [10, p. 289]). Note also that $T_2 \prec T$. Indeed, the operator $X \mid (\operatorname{ran} X^*)^-: (\operatorname{ran} X^*)^- \to H$ furnishes the quasi-affinity intertwining T_2 and T. Hence $T_2 \prec T \prec S_{n-m}$ and we conclude from Lemma 1 that T_2 is of class C_{10}. By Lemma 4, we have $n - m = d_{T_2*} - d_{T_2} = n - l$. Therefore $m = l$. Θ_T is inner implies that Θ_1 is also inner (cf. [10, p. 299]). From $m = l$ we infer that Θ_1 is inner from both sides (cf. [10, p. 190]), whence T_1 is of class C_{00}. But T is of class $C_{1\cdot}$ implies that T_1 is also of class $C_{1\cdot}$. Thus the only possibility is that T_1 is acting on $\ker X = \{0\}$. This shows that X is an injection as asserted.

In the preceding theorem, if $X \in \{T\}''$, the double commutant of T, then $X = \phi(T)$ for some $\phi \in H^\infty$ (by [12, Theorem 2]) and the conclusion follows from Lemmas 1 and 2 of [12]. Also note that essentially the same arguments as above can be used to show that if T_1 and T_2 are C_{10} contractions on H_1 and H_2, respectively, with finite defect indices satisfying $d_{T_1*} - d_{T_1} = d_{T_2*} - d_{T_2}$ and X is an operator intertwining T_1 and T_2 with dense range in H_2, then X is an injection. However Theorem 5 is in general not true in case $d_{T*} = \infty$ as the following example shows: Let T be the unilateral shift with infinite multiplicity defined as

$$T(f_1 \oplus f_2 \oplus \cdots) = e^{it}f_1 \oplus e^{it}f_2 \oplus \cdots \quad \text{for} \quad f_1 \oplus f_2 \oplus \cdots \in H \equiv H^2 \oplus H^2 \oplus \cdots.$$

Let X in $\{T\}'$ be defined by $X(f_1 \oplus f_2 \oplus f_3 \oplus \cdots) = f_2 \oplus f_3 \oplus \cdots$ on H. Then $d_T = 0$, $d_{T*} = \infty$ and X has dense range in H without being injective.

Now we are ready to prove our main result.

THEOREM 6. *Let $T = T_1 \oplus T_2$ and $S = S_1 \oplus S_2$ be contractions, where T_1 and S_1 are of class C_{11} and T_2 and S_2 are of class $C_{\cdot 0}$. Assume that the defect indices of T_2 and S_2 are finite. Then the following statements are equivalent:*

(1) *T is quasi-similar to S;*
(2) *T_1 and T_2 are quasi-similar to S_1 and S_2, respectively.*

Proof. To prove (1)$\Rightarrow$(2), assume that $T = T_1 \oplus T_2$ and $S = S_1 \oplus S_2$ are acting on $H = H_1 \oplus H_2$ and $K = K_1 \oplus K_2$, respectively. Let $X\colon H \to K$ and $Y\colon K \to H$ be quasi-affinities intertwining T and S. For any $x \in K_2$, we have $T^{*n}X^*x = X^*S^{*n}x \to 0$ as $n \to \infty$. It follows that $X^*x \in H_2$ and hence $X^*K_2 \subseteq H_2$. Considering the adjoint, we have $XH_1 \subseteq K_1$. Since the C_{11} contractions T_1 and S_1 are quasi-similar to unitary operators, say, T_1' and S_1' (cf. [10, p. 72]), we infer that T_1' is unitarily equivalent to a direct summand of S_1' (cf. [2, Lemma 4.1]). Similarly, S_1' is unitarily equivalent to a direct summand of T_1'. By the third test problem in [6], we conclude that T_1' and S_1' are unitarily equivalent to each other, whence $T_1 \sim S_1$.

Next we show the quasi-similarity of T_2 and S_2. As shown above, we have $XH_1 \subseteq K_1$ and $YK_1 \subseteq H_1$. Let

$$X = \begin{bmatrix} X_1 & * \\ 0 & X_2 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_1 & * \\ 0 & Y_2 \end{bmatrix}$$

be the triangulations with respect to $H = H_1 \oplus H_2$ and $K = K_1 \oplus K_2$ and let $Z = YX$. Then $Z \in \{T\}'$. Let

$$T_2 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$$

be the triangulation of type

$$\begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix}$$

on $H_2 = H_3 \oplus H_4$ (cf. [10, p. 75]). For any $x \in H_3$, we have

$$T^n Zx = ZT^n x = ZT_3^n x \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that $Zx \in H_3$, whence H_3 is invariant for Z. Let

$$Z = \begin{bmatrix} Z_1 & 0 & * \\ 0 & Z_3 & * \\ 0 & 0 & Z_4 \end{bmatrix}$$

be the triangulation on $H = H_1 \oplus H_3 \oplus H_4$. Since both X and Y have dense ranges, so does Z. This implies that Z_4 has dense range. But T_4 is a C_{10} contraction with finite defect indices and $Z_4 \in \{T_4\}'$. By Theorem 5, Z_4 is an injection. The injectivity of X and Y implies that of Z, hence of Z_3. We conclude that

$$Z_2 \equiv \begin{bmatrix} Z_3 & * \\ 0 & Z_4 \end{bmatrix}$$

is an injection. But $Z_2 = Y_2 X_2$. Hence X_2 is an injection. Since X_2 also has dense range, we have a quasi-affinity X_2 which intertwines T_2 and S_2, that is, $T_2 \prec S_2$. In a similar fashion, we can prove $S_2 \prec T_2$. Thus $T_2 \sim S_2$, completing the proof.

COROLLARY 7. *Let T and S be hyponormal contractions with finite defect indices. Then T is quasi-similar to S if and only if their unitary parts are unitarily equivalent and their c.n.u. parts are quasi-similar.*

Proof. The conclusion follows immediately from Theorem 6 and the fact that c.n.u. hyponormal contractions are of class $C_{\cdot 0}$ (cf. [8]).

Corollary 7 partially generalizes a result of Hastings [4] that if $T = T_1 \oplus T_2$ is an isometry, where T_1 is unitary and T_2 is a unilateral shift of finite multiplicity and $S = S_1 \oplus S_2$ is a subnormal contraction, where S_1 is unitary and S_2 is c.n.u., then T is quasi-similar to S if and only if T_1 is unitarily equivalent to S_1 and T_2 is quasi-similar to S_2. However our result is not strong enough to cover his case. Also note that Corollary 7 may not hold if there is no finiteness assumption on the defect indices. A counterexample was given in [5].

In the case of similarity, we have the corresponding result even without the assumption on defect indices. The proof is essentially the same as the one given for [1, Proposition 2.6]. We only give the statements below and leave the details to the readers.

THEOREM 8. *Let $T = T_1 \oplus T_2$ and $S = S_1 \oplus S_2$ be contractions, where T_1 and S_1 are unitary operators and T_2 and S_2 are of class $C_{\cdot 0}$. Then the following statements are equivalent:*

(1) *T is similar to S:*
(2) *T_1 is unitarily equivalent to S_1 and T_2 is similar to S_2.*

COROLLARY 9. *Let T and S be hyponormal contractions. Then T is similar to S if and only if their unitary parts are unitarily equivalent and their c.n.u. parts are similar.*

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