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REPRESENTING PRODUCTS OF DISJOINT WORDS IN A FREE GROUP

BY

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1. Introduction

Let F be the free group freely generated by $x_1, x_2, \ldots$, and let G be any group. An *equation* over G is an expression $W = U$ where $W, U \in F * G$. A *solution* is a tuple of elements of G which, when substituted for the x_i's in W and U, make the equation true in G. A considerable amount of interest ([1]–[14]) has been shown in solving equations of the form

$$W = U \tag{1}$$

where $W \in F$ and $U \in G$. Ju. I. Hmelevskiĭ [7] calls these "equations with right-hand side". If such an equation has a solution in G it shows that U can be represented as a word of the same "form" as W. An early result of this type is the following theorem of H. B. Griffiths.

THEOREM [6]. *If G is a free product of groups G_i, $1 \neq g_i \in G_i$ $(1 \leq i \leq n)$,*

$$W = [x_1, x_2] \cdots [x_{2m-1}, x_{2m}]$$

and $U = g_1 \cdots g_n$, then equation (1) *has no solutions over G for $m < n$.*

(Here $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$.)

In the special case that G is a free group, say $G = \langle a_1, a_2, \ldots; \phi\rangle$, R. C. Lyndon and M. Newman [10] proved that with $W = x_1^2 \cdots x_m^2$ and $U = a_1^2 \cdots a_n^2$ equation (1) has no solutions over G for $m < n$. This was later extended by R. C. Lyndon, T. McDonough and M. Newman [9] to the case in which W and U are products of k^{th} powers $(k > 1)$.

Recently R. C. Lyndon [8] has generalized Griffiths' result to quadratic words (i.e. words in which each x_i occurs exactly twice with exponent $+1$ or -1).

THEOREM [8]. *If G is a free product of groups G_i, $1 \neq g_i \in G_i$ $(1 \leq i \leq n)$, $W(x_1, \ldots, x_m)$ is a quadratic word and $U = g_1 \cdots g_n$, then equation* (1) *has no*

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solutions over G for $m < n$. Furthermore, if $W \in F'$ (the commutator subgroup), there are no solutions over G for $m < 2n$.

Since a free group can be viewed as a free product, this gives the result of Lyndon and Newman as a corollary; however, since it requires that W be quadratic, it does not give the result about k^{th} powers. Using the methods developed in [5], we will obtain a result, in the special case where G is free, which removes the hypothesis that W is quadratic. An extension of this to free products appears to be difficult.

If the set $\{x_i : x_i \text{ occurs in } W^{\pm 1}\}$ has n elements we write $g(W) = n$. A word W in F is *universal* if it has every word of F as an endomorphic image. Two words in F are *disjoint* if they are words in disjoint subsets of the set $\{x_1, x_2, \ldots\}$. Our theorem can then be stated as follows.

THEOREM. *If W is a non-universal word in F and U is a cyclically reduced word in F which is a product of n disjoint words $U = U_1 \cdots U_n$, then equation* (1) *has no solutions over F for $g(W) < n$. Furthermore, if $W \in F'$, there are no solutions over F for $g(W) < 2n$.*

Since G is assumed to be free; it is convenient and no less general to let $G = F$. The condition that W be non-universal is clearly necessary but otherwise trivial since, by Corollary 2.3 of [5], the universal words are precisely those words W with g.c.d. $(|s_1|, \ldots, |s_k|) = 1$ where the s_i's are the exponent sums of the generators occurring in W.

2. Lemmas

We will assume that the reader is familiar with the terminology and results of [5]. As in [5] we assume, without loss of generality, that words are cyclic (i.e. written around a circle with the last letter preceding the first) and that subwords are ordinary (linear) words. Upper case letters represent words and subwords and lower case letters represent letters. The symbol "$\equiv$" denotes identical equality and "$=$" denotes equality in F.

LEMMA 1. *If $U \equiv U_1 U_2 \cdots U_n$ is a cyclically reduced disjoint product of the U_i's, then there is an automorphism of F sending U to $U' \equiv U'_1 U'_2 \cdots U'_n$ where each U_i cyclically reduces to U'_i.*

Proof. If $U_i \equiv A^{-1} U'_i A$ let $\alpha_i \colon x_j \to A x_j A^{-1}$ for every generator x_j occurring in U_i and $\alpha_i \colon x_k \to x_k$ otherwise. Clearly $U_i \alpha_i = U'_i$ and $U_k \alpha_i = U_k$ for $k \neq i$. Thus the automorphism $\alpha_1 \alpha_2 \cdots \alpha_n$ sends U to U'.

LEMMA 2. *If $U \equiv U_1 U_2 \cdots U_n$ is a disjoint product of cyclically reduced subwords and if V is a non-universal word with U as a c-free image under an endomorphism ϕ, then V is a disjoint product $V_1 V_2 \cdots V_n$ where each V_i is a cyclically reduced subword of V and $V_i \phi \equiv U_i$ for each i.*

Proof. If $n = 1$ the result is trivial.

Assuming $n > 1$, we suppose first that x is a letter occurring in V which ϕ sends to a subword of U containing the last letter, y, of some U_i and the first letter, z, of U_{i+1} (where we take $i + 1$ as 1 if $i = n$). Thus $x\phi \equiv AyzB$. Since V is non-universal, x occurs again in V as x or x^{-1}. But U cannot take the form $LyzMyzR$, since y and z are letters from disjoint subwords of U. Therefore V must contain one occurrence each of x and x^{-1} and, thus, $U \equiv LyzMz^{-1}y^{-1}R$. Since y and z are from disjoint subwords, it follows that the z^{-1} is the last letter in U_{i+1}. This, however, is a contradiction to the hypothesis that U_{i+1} is cyclically reduced. Thus there is no such letter x in V.

It follows that ϕ sends each letter x in V to a subword of some U_i. Thus V is a disjoint product $V_1 V_2 \cdots V_n$ where $V_i\phi = U_i$. If any V_i were not cyclically reduced, U_i would not be cyclically reduced, contrary to hypothesis.

3. Proof of the theorem

If $U \equiv U_1 U_2 \cdots U_n$ is an endomorphic image of $W(\neq 1)$ in F, then by Lemma 1 we may assume without loss of generality that each U_i is cyclically reduced. By Theorem 2.1 of [5] there is a word $V \in D_W$ having U as a c-free image under some endomorphism ϕ. It then follows from Lemma 2 that V is a disjoint product, $V_1 V_2 \cdots V_n$, of cyclically reduced subwords with $V_i\phi = U_i$ for each i. Using Lemmas 3.3 and 5.2 of [5] we can assume without loss of generality that W is irredundant. Thus by Proposition 3.8 of [5] it follows that $\Delta(W) \geq \Delta(V)$. A simple induction on n using Lemma 5.8 of [5] then shows that $\Delta(V) = (\sum_i \Delta(V_i)) + (n - 1)$. Thus we have

$$g(W) - c(W) - \Delta(W) \geq \Delta(V) = \left(\sum_i \Delta(V_i)\right) + (n - 1),$$

and hence

$$g(W) \geq \left(\sum_i \Delta(V_i)\right) + (n - 1) + c(W). \qquad (*)$$

Since each V_i is cyclically reduced we have $\Delta(V_i) \geq 0$. We also have $c(W) \geq 1$. Using these inequalities in (*) then yields $g(W) \geq n$.

If $W \in F'$, it follows that $V \in F'$ and, thus, that the exponent sum on each generator occurring in V is zero. But this is impossible unless, for each i, $g(V_i) \geq 2$. If $\Delta(V_i) = 0$ for some i, then $g(V_i) = c(V_i)$; thus all components of $\Gamma(V_i)$ are of order two. Since $V \in D_W$ it is irredundant and therefore V_i can have at most one component of order two by Lemma 3.3 of [5]. Thus $c(V_i) = 1$ and $g(V_i) = 1$; a contradiction. It follows that $\Delta(V_i) > 0$ for each i. Using this in (*) yields $g(W) \geq 2n$.

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